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# NUMERICAL SIMULATIONS OF FLOWS THROUGH A SIMPLIFIED MODEL OF A HEART VALVE

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***Abstract.*** *In this work, the classical Galerkin finite element method with the Characteristic-based Split scheme is applied for numerical simulation of incompressible flow through a simplified model of a heart valve. The Characteristic-based Split scheme is a technique for stabilization of the Galerkin finite element method in order to eliminate oscillations when this method is applied to solve convective dominant problems. Numerical simulations of unsteady flows have been carried out on unstructured meshes of triangular finite elements. The results of some test cases show that the classical Galerkin finite element method with the Characteristic-based Split scheme predicts the expected behavior of the flow.*

***Keywords:*** *finite element, Characteristic-based Split (CBS) scheme, incompressible flows.*

## 1. INTRODUCTION

The Galerkin finite element method combined with the Characteristic-based Split scheme is a stabilized method and doesn´t present spurious oscillations when applied for solution of convective dominant flows. The classical Galerkin finite element method in its original formulation may produce solutions without physical significance for flows with high Reynolds numbers.

The Characteristic-based Split scheme for both incompressible and compressible flows was first presented by Zienkiewicz and Codina (1995) and has been extended to investigate other applications: solid dynamics, shallow water flows, thermal and porous medium flows, for example. Several other authors have presented contributions and improved the method. The Characteristic-based Split scheme has been combined with the standard Artificial Compressibility method to obtain an efficient and accurate explicit matrix free procedure, Liu (2005). In this work a semi-implicit Characteristic-based Split scheme, in which a matrix solution procedure is required for the implicit solution of a pressure Poisson equation, has been applied for computational solution of the Navier-Stokes equations in two-dimensional domains, Octaviani (2013).

The Characteristic-based Split scheme based on in a first step by removing all pressure gradient terms from the Navier-Stokes equations leads to a non-singular solution for any shape functions used for velocity and pressure. The velocities obtained in this first step don´t satisfy the mass conservation. In a second step, the pressure is obtained from the continuity equation and finally in a third step the intermediate velocities obtained from the first step are corrected to get the final velocity values that now satisfy the continuity equation. In previous works, Pereira (2010), Octaviani, Pereira and Campos-Silva (2011) presented applications for simulations of flows with heat transfer in tube bundles. In this work, simulations of isothermal flows in a model of heart valve are presented. The expected behavior of the flows was well predicted using the Galerkin finite element method together the Characteristic-based Split scheme.

## 2. THE GOVERNING EQAUTIONS AND THE CHARACTERISTIC BS SCHEME

In this section the governing equations for incompressible viscous flows are presented in dimensionless form considering a length $L$, a velocity $u_0$ and a time $L/u_0$ as reference parameters. The original equations are the continuity and the momentum equations. Also, the main aspects of the Characteristic-based Split scheme are described for completeness.

### 2.1 Governing equations

The governing equations are presented in a non-dimensional form as shown in Octaviani (2013) and this form is:

$$\frac{\partial U_i}{\partial X_i} = 0 \tag{1}$$

$$\rho\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial X_j}\right)+\frac{\partial P}{\partial X_i}-\frac{1}{Re}\frac{\partial \tau_{ji}}{\partial X_j}=S_i \qquad (2a)$$

$$\tau_{ji}=\mu\left(\frac{\partial U_i}{\partial X_j}+\frac{\partial U_j}{\partial X_i}\right) \qquad (2b)$$

where $X_i = x_i / L$ ; $U_i = u_i / u_0$ ; $P = p / \rho_0 u_0^2$ ; $\tau_{ji} = \tau_{ji} / \rho_0 u_0^2$ and all physical properties in Eqs. (1) – (2) are non-dimensionalized in relation to reference properties at $T_0$. $\rho = \rho^* / \rho_0$ ; $\mu = \mu^* / \mu_0$ are the density and the dynamic viscosity. The Reynolds number is defined as $Re = \rho_0 u_0 L / \mu_0$. An asterisk indicates dimensional values. $S_i$ is a source term.

### 2.2 The Characteristic-based Split scheme

Details of the derivation of the Characteristic-based Split scheme, for the Navier-Stokes equations, can be found in Lewis, Nithiarasu and Seetharamu (2004) and it's an extension of the Characteristic Galerkin method of Zienkiewicz and Taylor (2000). It's based on evaluation of the time derivative along the characteristic. This procedure eliminates the convective term in a transport equation in a first moment but introduce an inconvenience of a non-inertial system of reference. So, a pure diffusion equation is obtained and after an expansion in Taylor series of the terms of that equation prevents the needing to follow the flow and introduces new terms that work as stabilizing in the numerical solution. Details can be found in Pereira (2010) and also in Octaviani (2013). The basic equation after a time discretization of a transport equation is of the form

$$\begin{aligned}\Delta\phi=\phi(x_i,t_{n+1})-\phi(x_i,t_n)=\Delta t\left[-u_j(x_j,t_n)\frac{\partial\phi(x_j,t_n)}{\partial x_j}+\frac{\partial}{\partial x_i}\left(\alpha\frac{\partial\phi(x_i,t_n)}{\partial x_i}\right)-Q(x_i,t_n)\right]+\\ +\frac{(\Delta t)^2}{2}\left\{u_i(x_i,t_n)\frac{\partial}{\partial x_i}\left(u_j(x_j,t_n)\frac{\partial\phi(x_j,t_n)}{\partial x_j}\right)\right\}+\\ \frac{(\Delta t)^2}{2}\left\{-u_j(x_j,t_n)\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_i}\left(\alpha\frac{\partial\phi(x_i,t_n)}{\partial x_i}\right)\right]+u_j(x_j,t_n)\frac{\partial Q(x_j,t_n)}{\partial x_j}\right\}\end{aligned} \qquad (3)$$

Note that the convection terms reappear in the time discretized equation and also higher order terms appeared in the Eq. (3). These high order terms work as stabilizing of the solution.

### 2.3 Time discretization of the Navier-Stokes equations

The time discretization of the continuity and the momentum equations is done in three steps. In the first step the pressure terms are dropped from the momentum equations and an intermediate velocity that doesn´t satisfy the continuity is obtained as

**Step 1: Intermediate velocity**

$$\Delta U_i^*=U_i^*-U_i^n=\Delta t\left[-\frac{\partial}{\partial X_k}(U_kU_i)+\frac{1}{\mathrm{Re}}\frac{\partial\tau_{ij}}{\partial X_j}\right]^n+\frac{(\Delta t)^2}{2}\left\{U_m\frac{\partial}{\partial X_m}\left[\frac{\partial}{\partial X_k}(U_kU_i)-\frac{1}{\mathrm{Re}}\frac{\partial\tau_{ij}}{\partial X_j}\right]\right\}^n \qquad (4)$$

In the second step, a Poisson equation is solved for the pressure field considering the continuity equation. This Poisson equation is of the form

**Step 2: Pressure calculation**

$$\frac{\partial^2 P^n}{\partial X_j \partial X_j}=\frac{1}{\Delta t}\frac{\partial U_i^*}{\partial X_i} \qquad (5)$$

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In the third step the final velocity is obtained by correcting the intermediate velocity
**Step 3: Corrected velocity**

$$U_i^{n+1} = U_i^* - \Delta t \frac{\partial P^n}{\partial X_i} \tag{6}$$

If a transport equation has to be solved for any additional scalar variable it's similar to Eq. (3)

### 2.4 Spatial discretization and matrix form

Now the standard Galerkin approximation with the divergence theorem is applied to the time discretized Eqs. (4) to (6). In the Galerkin finite element method the following set of interpolations are used for the variables:

$$U_i = N_u \tilde{U}_i;\ P = N_p \tilde{P}; \tag{7}$$

where $\tilde{U}_i = \begin{bmatrix} U_i^1 & U_i^2 & \cdots & U_i^k & \cdots & U_i^l \end{bmatrix}^T$ are the nodal variables and $N = \begin{bmatrix} N^1 & N^2 & \cdots & N^k & \cdots & N^l \end{bmatrix}$ are the interpolation functions. The integration of the equations and the use of the divergence theorem result in the weak formulation:

**Step 1: Weak form of intermediate momentum**

$$\int_\Omega N_u^T \Delta U_i^* d\Omega = \Delta t \left[ -\int_\Omega N_u^T \frac{\partial}{\partial X_k}\left(U_k U_i\right) d\Omega - \frac{1}{\mathrm{Re}} \int_\Omega \frac{\partial N_u^T}{\partial X_j} \tau_{ij} d\Omega \right]^n + \\ + \frac{\Delta t^2}{2} \left[ \int_\Omega \frac{\partial}{\partial X_m}\left(U_m N_u^T\right) \left( -\frac{\partial}{\partial X_k}\left(U_k U_i\right) \right) d\Omega \right]^n + \Delta t \left[ \int_\Gamma N_u^T t_d d\Gamma \right]^n \tag{8}$$

**Step 2: Weak form of pressure equation (semi-implicit Characteristic-based Split scheme)**

$$\int_\Gamma N_p^T \frac{\partial P^{n+1}}{\partial X_j} n_j d\Gamma - \int_\Gamma \frac{\partial N_p^T}{\partial X_j} \frac{\partial P^{n+1}}{\partial X_j} d\Gamma = \frac{1}{\Delta t} \int_\Omega N_p^T \frac{\partial U_i^*}{\partial X_i} d\Omega \tag{9}$$

**Step 3: Weak form of momentum correction**

$$\int_\Omega N_u^T \Delta U_i d\Omega = \int_\Omega N_u^T \Delta U_i^* d\Omega + \Delta t \int_\Omega \frac{\partial N_u^T}{\partial x_i} P^{n+1} d\Omega - \Delta t \int_\Gamma N_u^T t_p d\Gamma \tag{10}$$

The final matrix form of the weak formulation (8)-(10) is
**Step 1: Intermediate momentum**

$$\Delta \tilde{U}^* = -M_u^{-1} \Delta t \left[ \left( C_u \tilde{U} + K_\tau \tilde{U} - f_u \right) - \Delta t \left( K_u \tilde{U} \right) \right]^n \tag{11}$$

**Step 2: Pressure**

$$H \Delta \tilde{P} = -\frac{1}{\Delta t} G \tilde{U}^* + f_p \tag{12}$$

**Step 3: Momentum correction**

$$\Delta \tilde{U} = \Delta \tilde{U}^* - M_u^{-1} \Delta t \left[ G^T \left( \tilde{P}^n + \Delta \tilde{P} \right) \right] \tag{13}$$

In the Equations (11) – (13) the matrices and vectors are defined as

$$M_u = \int_\Omega N_u^T N_u d\Omega \; ; \; C_u = \int_\Omega N_u^T \left(\nabla^T \left(UN_u\right)\right) d\Omega \; ; \; K_u = -\frac{1}{2}\int_\Omega \left(\nabla^T \left(UN_u\right)\right)^T \left(\nabla^T \left(UN_u\right)\right) d\Omega \; ; \qquad (14a, b, c)$$

$$K_\tau = \int_\Omega B^T \frac{1}{\mathrm{Re}}\left(I_o - \frac{2}{3} mm^T\right) B d\Omega \; ; \; f_u = \int_\Gamma N_u^T t_d d\Gamma \qquad (15a, b)$$

$$H = \int_\Omega \left(\nabla N_p\right)^T \nabla N_p d\Omega \; ; \; G = \int_\Omega \left(\nabla N_p\right)^T N_u d\Omega \; ; \; f_p = \int_\Gamma N_p^T \nabla p^{n+1} n^T d\Gamma \; ; \qquad (16a, b, c)$$

Where $B$ is a matrix defined by

$$B = SN_u \qquad (17)$$

and $S$ is a differential operator that for two-dimensional problems has the form:

$$S^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \qquad (18)$$

And the vectors $m$ and $I_o$ are

$$m = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T ; \quad I_o = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix} \qquad (19a, b)$$

In the literature it has been proved that with the use of Characteristic-based Split scheme, the interpolations functions don't require the Ladyzhenskaya-Babuska-Brezzi (LBB) condition be satisfied. Velocity and pressure fields can be interpolated by functions of the same order without any spurious oscillation in the pressure field. For the semi-implicit scheme used in this work, the mass matrices in Eqs. (11) and (13) are the lumped matrices.

## 3. RESULTS AND DISCUSSIONS

The application considered in this work is the flow in a channel with a constriction followed by an obstruction as illustrated in Figure 1. This configuration is supposed to represent a heart valve in position totally open that is the most critical case for the flow, i.e., the maximum mass flow rate. The channel has a dimensionless length 6.5 in the main flow direction and dimensionless height 2 in the transverse direction. This geometry was presented by Mueller (1978) for a finite difference solution to the problem with vorticity and stream function as dependent variables.

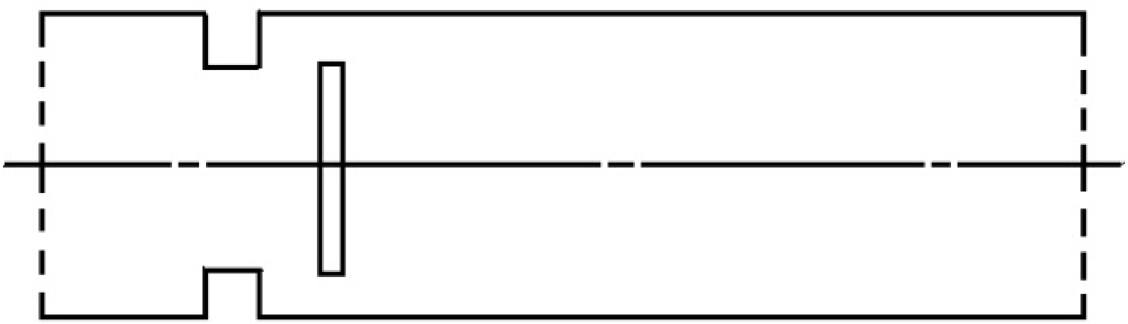

Figure 1. Flow in a channel with an obstruction (model a simplified heart valve).

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The obstruction has a height of 1.4 and its left face is at position X=2. Its width is 0.2. The reentrances in the channel are at position X=1.4 and they have width 0.4 a height 0.4. Only half of the channel is considered for spatial discretization as shown in Figure 2. There are 2792 nodes and 4673 elements in the mesh.

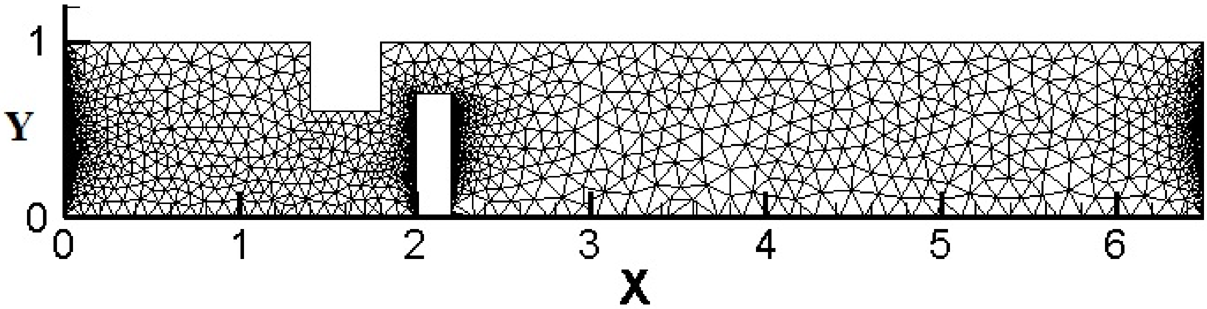


Figure 2. Mesh of three node triangular elements in half of the channel.

The equations of the flow were solved for several numbers of Reynolds. The imposed boundary conditions were uniform velocity at the inlet section, no slip at the walls, symmetry at the central line of the channel and null pressure field at outlet section. Totally explicit schemes were used for the intermediate and corrected velocities. The only algebraic system solved was that one for the pressure field in step 2. As the linear system resulting in step 2 is symmetrical and positive defined, the preconditioned conjugate gradient method is employed for solution of that system. This procedure is named semi-implicit scheme. Other schemes such as matrix free have also been employed in the literature, Liu (2005). In the solution the velocity and pressure fields are obtained. As a post processing we obtain the stream functions that are shown in the figures that follow.

The Characteristic-based Split scheme has also been validated and verified in previous works by several authors, mainly of Swansea University. In context of our research group, see the works of Pereira (2010), Octaviani, Pereira and Campos-Silva (2011) and Octaviani (2013) for other applications.

Obviously, the geometry considered in this work is very simple compared with the geometry of the real flow. In real flow the walls are non rigid, the flow is oscillatory and pulsatile. However, the problem considered here, is a preliminary introduction for considering most realistic cases in future works.

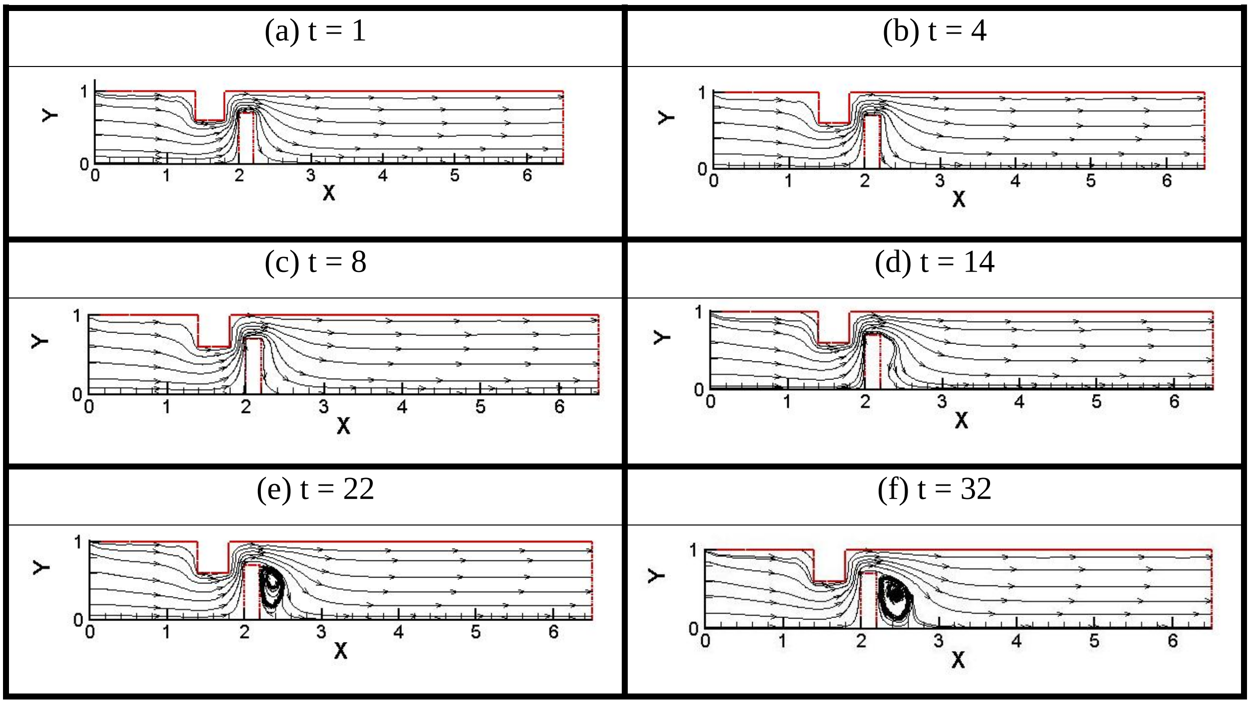


Figure 3. Stream functions for Re = 20 at some instants of time.

In Figure 3 it´s showed the stream functions from a simulation for a low Reynolds numbers equal to 20. In lower times the flow doesn´t recirculate behind the obstacle. The recirculation appears in dimensionless time next to 20. At

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time, t=22, we can see a recirculation behind the obstacle and after time t=32, the recirculation seems to be already established indicating a possible steady state. In next figures are shown results for other Reynolds numbers. In Figure 4, the Reynolds numbers is equal to 40. We can notice a very small vortex at time, t=8. The vortex grows up and at time 32, similar the case of Fig. 3, the flow seems also be in steady state.

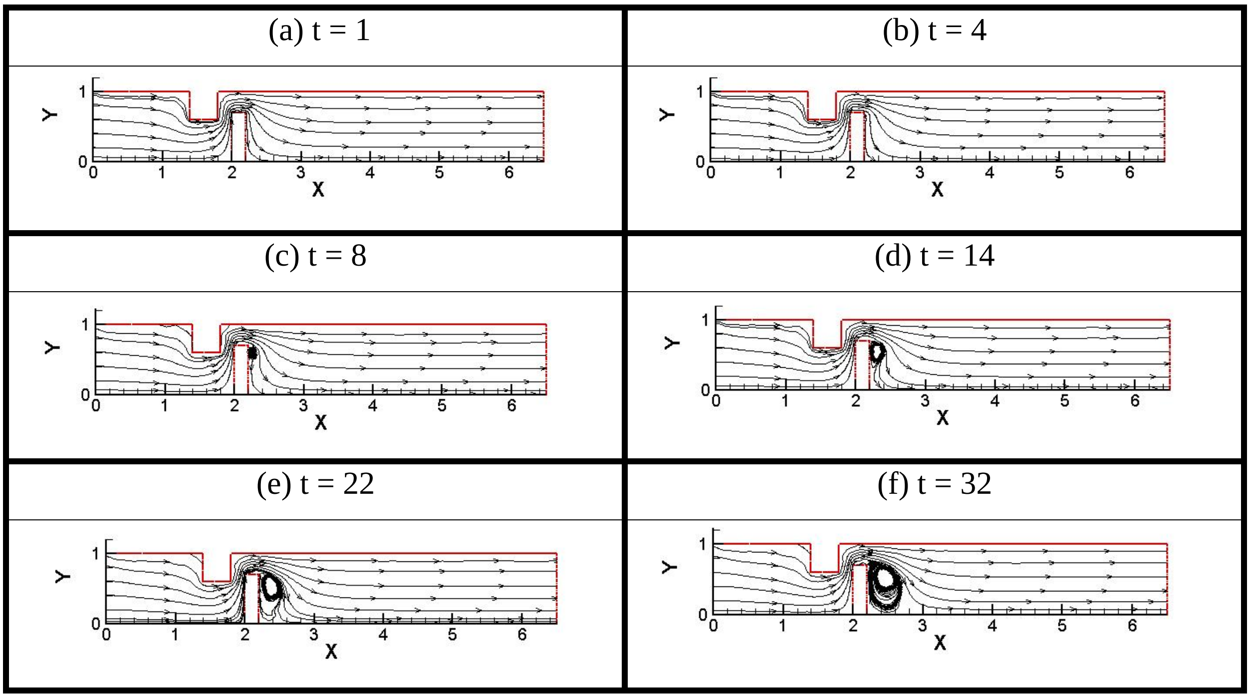


Figure 4. Stream functions for Re = 40 at some instants of time.

In next figures, Fig. 5 to 8, the simulation were for Reynolds numbers, Re=60, 80, 200 and 800 respectively. We can notice that the first visible vortex, in all cases simulated, appears about time, t=8. Until the Reynolds numbers of 800, the length of the recirculation zone is approximate the same in all cases. No significant change occurs in that length.

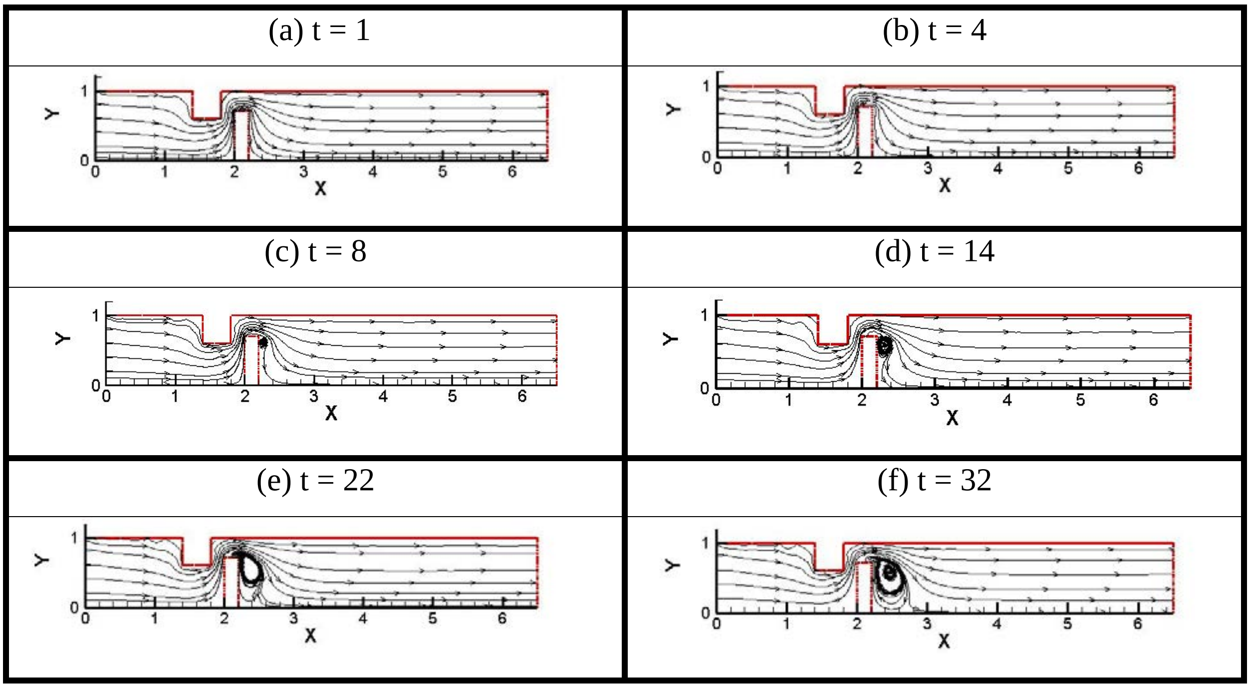


Figure 5. Stream functions for Re = 60 at some instants of time.

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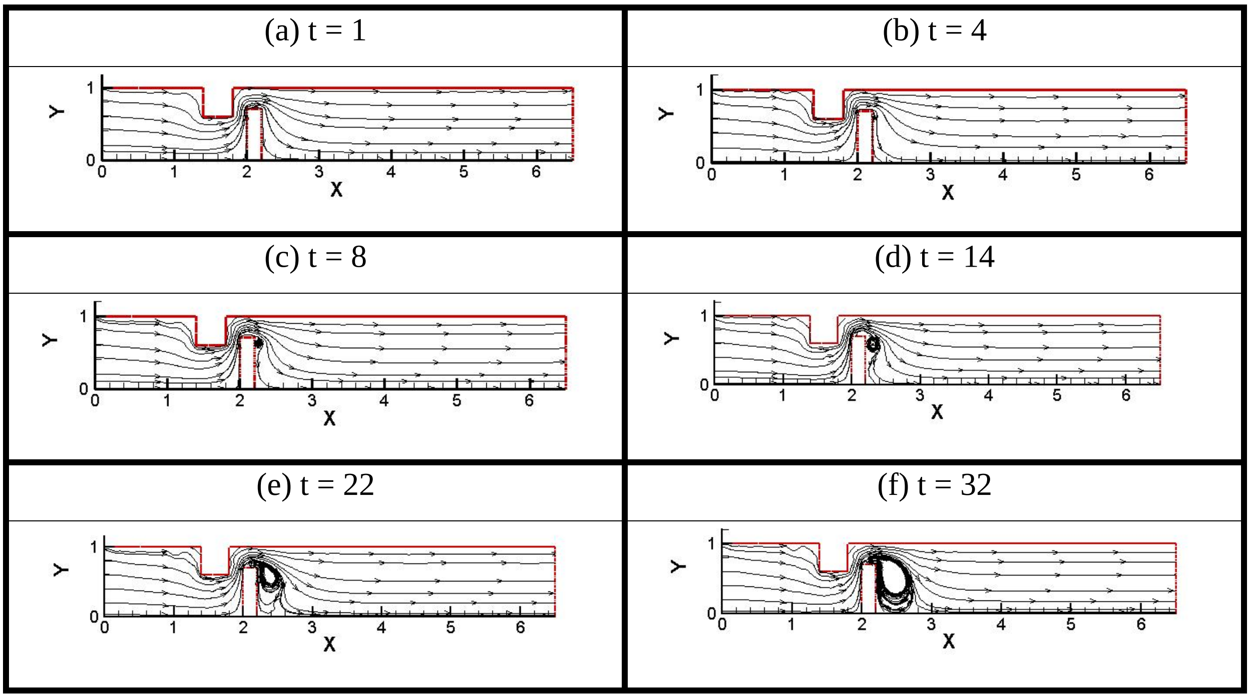


Figure 6. Stream functions for Re = 80 at some instants of time.

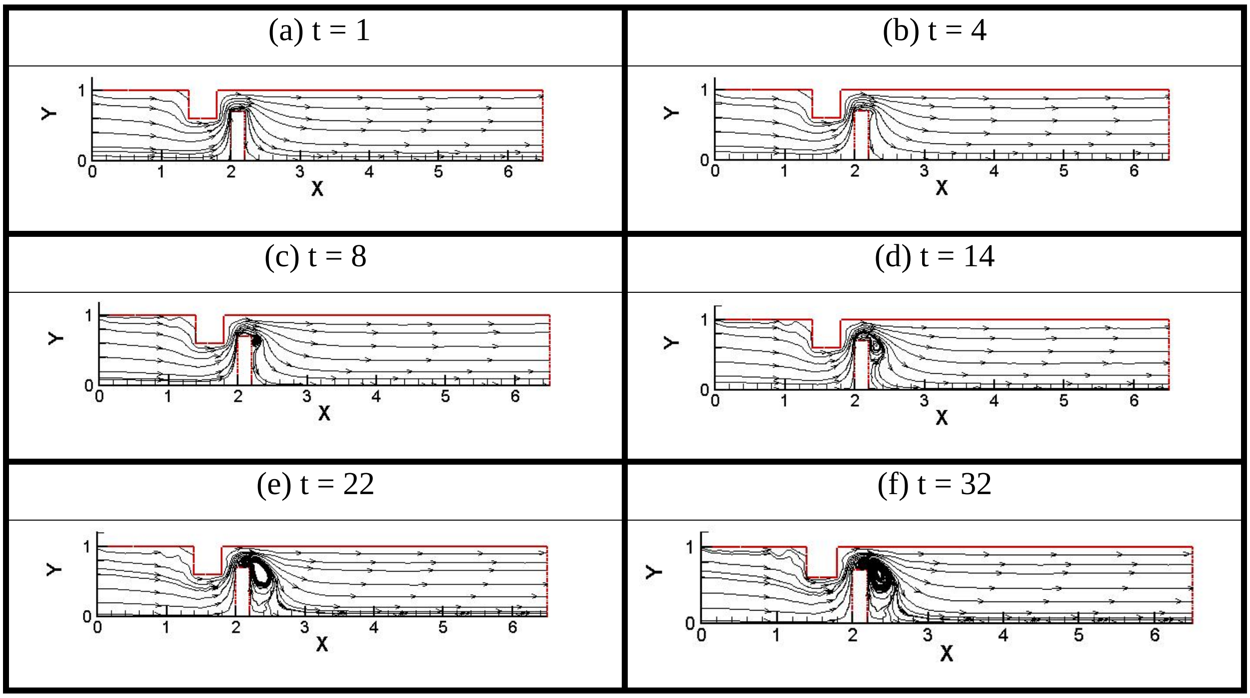


Figure 7. Stream functions for Re = 200 at some instants of time.

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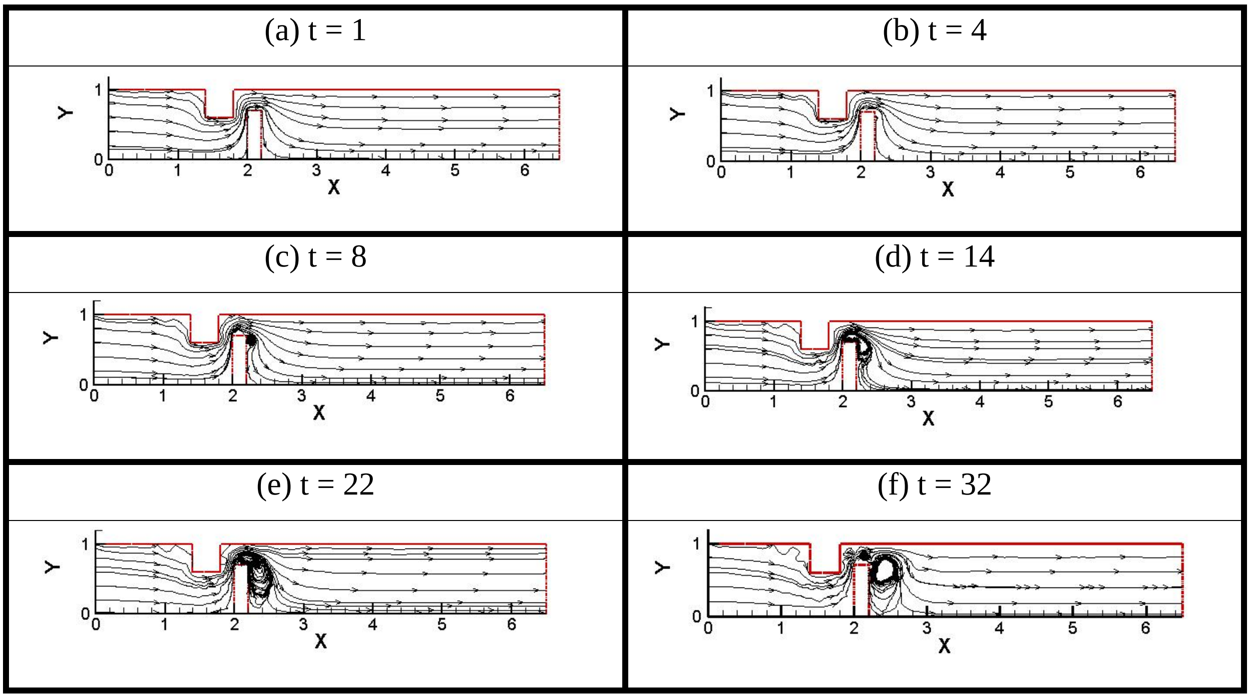


Figure 8. Stream functions for Re = 800 at some instants of time.

In Figure 9 we present results for simulations considering the Reynolds number of 1,500, 2,500 and 10,000. In the case of Re = 1,500, the plotted results are for t = 32. In the case of Re = 2,500 and 10,000 the results are for the instant of time, t = 22. For these more high numbers of Reynolds we can notice a small vortex in the corner before lower obstacle.

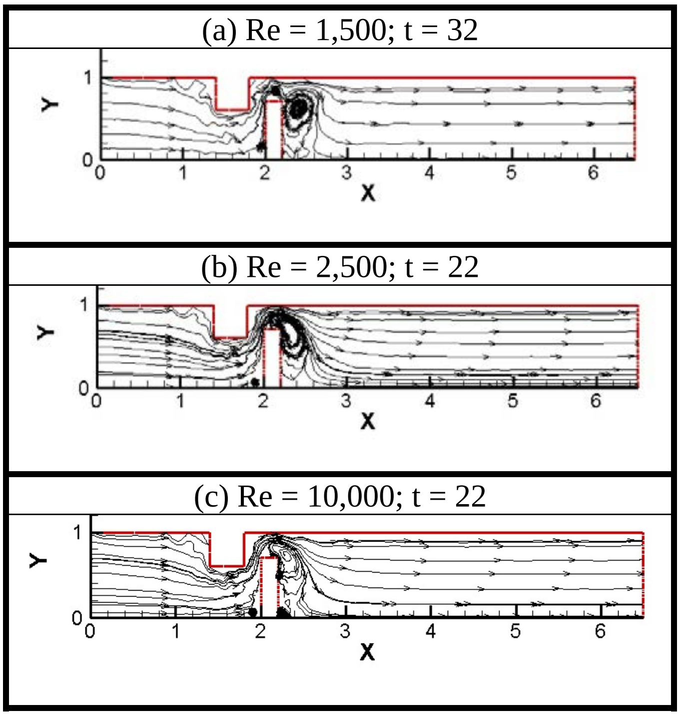


Figure 9. Stream functions for more higher Reynolds numbers.

A zoom of the region next the restriction and obstacle is shown in Figure 10, for Re = 1,500. In that figure we can see the small vortex

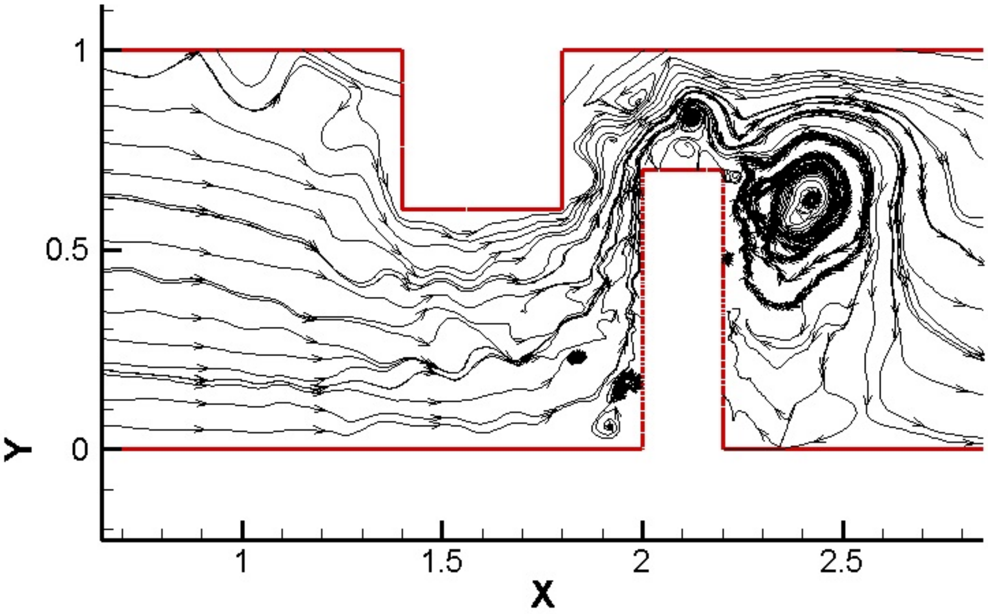


Figure 10. An ampliation of the flow around the restriction and obstacle; Re =1500, t = 22.

Mueller (1978) presented results for other configurations: axisymmetric flows with model disc valve, aortic-shaped channel with model disc valve and planar flows similar the case considered in this work. In future works we intend to aboard those cases of axisymmetric and other flows. In the present work, the application was utilized for learning and test of the stabilized Galerkin finite element method with the Characteristic-based Split scheme. Although, we don´t show the results from Mueler (1978), the qualitatively expected behavior of the flow was predicted with the present method.

## 4. CONCLUSIONS

An application of the Galerkin finite element method with the Characteristic-based Split scheme was realized in this work. Until the Reynolds numbers considered no spurious oscillations appeared in the results. Higher Reynolds numbers and more complex cases have to be considered. In literature several cases of flows including turbulent and 3D flows have already been simulated by the Galerkin finite element method and the use of Characteristic-based Split scheme as stabilization technique.

The obtained results encourage the sequence of studies on the numerical method applied in the present work.

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## 7. RESPONSIBILITY NOTICE

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