ISSN 2176-5480

22nd International Congress of Mechanical Engineering (COBEM 2013)
November 3-7, 2013, Ribeirão Preto, SP, Brazil

ISOGEOMETRIC ANALYSIS OF FREE VIBRATION OF BARS

Mateus Rauen
Roberto Dalledone Machado
Marcos Arndt
Universidade Federal do Paraná, Programa de Pós-Graduação em Métodos Numéricos em Engenharia, Caixa Postal 19011 – CEP 81531-990 Curitiba, PR
arndt.marcos@gmail.com, mateus.rauen@yahoo.de, rdm@ufpr.br

Abstract. *The isogeometric analysis is presented as a similar tool to the finite elements method, with the advantage to describe a mesh as the exact geometry as real, using a connection between computer aided design (CAD) and non-uniform rational b-splines (NURBS) shape functions. The application of the method to the analysis of free vibration of bars is proposed and the equations are deducted. The convergence of the method is checked using numerical examples. The efficiency of the method according the applications is verified, comparing the results with the hierarchical finite elements method.*

Keywords: *Free Vibration, Isogeometric Analysis, Finite Elements Method*

1. INTRODUCTION

With the advent of advanced computers, new and more efficient kinds of numerical methods arise to supply the needs of the modern engineering to solve its problems more accurately. The isogeometric analysis, introduced by Hughes *et al.* (2005) and developed by Cottrell *et al.* (2009) is a new method to solve partial differential equations numerically. This is similar to the Finite Elements Method but with some more advantages. One advantage is the object under analysis has its geometry exactly described within the connection between CAD (Computer Aided Design) and the finite elements analysis through NURBS (Non Rational B-Splines) shape functions, even with a coarse mesh. A second advantage is related with the CAD connection during a mesh refinement: one time built the mesh there's no need to access again the CAD model to rebuild the refined mesh. The shape functions used to describe the object geometry are the same used to approach the differential equations, for this reason the method is called Isogeometric.

Based in numerical experiments, this work tests the efficiency of the NURBS as approach shape functions to the problem of free vibration of bars, compares the results with the Finite Elements Method to a two edge fixed bar. To the case of a one edge fixed bar, the eigenvalue errors and the convergence rate of the proposed method is compared with the finite elements method.

2. FREE VIBRATION PROBLEM

In a multi-degree of freedom system, the structural free vibration problem is to solve an generalized eigenvalue problem: find a pair (λ, μ) so that:

$$K.\mu = \lambda.M.\mu \tag{1}$$

where K is the stiffness matrix, M the mass matrix, λ a eigenvalue which is related with the natural vibration frequency (eq. 2) and μ is the eigenvector which represents the natural vibration modes associated with λ. If a system has N degrees of freedom, there'll be N vibration frequencies and N eigenvectors related with λ (Chopra, 1995).

$$\omega = \sqrt{\lambda} \tag{2}$$

In finite elements and also the isogeometric analysis, the eigenvalue problem changes to: find a pair (λ^h, μ^h) so that:

$$K.\mu^h = \lambda^h.M.\mu^h \tag{3}$$

where K is the stiffness matrix, M the mass matrix and (λ^h, μ^h) are the approximate results to the eigenvalue problem. To the problem of bars, in a domain $\Omega(0, L)$, the variational form of K and M are defined by:

$$K = \int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx \tag{4}$$

$$M = \int_0^L \rho Auwdx \tag{5}$$

where E is the Young Modulus, A the cross section area and ρ the specific mass, u is the axial displacement and w the weighting function.

ISSN 2176-5480

3. ISOGEOMETRIC ANALYSIS

The main proposal of the Isogeometric Analysis is to increase the accuracy of a Finite Elements analysis using data from a CAD Model. This goal are related not only with the final results, but also with the object geometry. The NURBS shape functions describes exactly the object geometry from a CAD model regardless the amount of nodes (Hughes *et al.*, 2005).

3.1 NURBS Shape Functions

The Isogeometric Analysis uses NURBS shape functions as described in Hughes *et al.* (2005) and Cottrell *et al.* (2009). These functions are built from a knot vector (eq. 6): a non-decreasing set of parametric coordinates, called knots, which could be repeated respecting the ascending sequence. Given a n number of shape functions and the polynomial order p, when the edge knots appear $p+1$ times, the knot vector is said to be a open knot vector. A knot vector, defined by eq. 6 must have $n+p+1$ coordinates to describe the NURBS functions on the order p.

$$\Xi = \{\xi_1, \xi_2, \xi_3, ..., \xi_{n+p+1}\} \tag{6}$$

Defined the knot vector, the shape functions are build recursively starting with (7) for $p = 0$ and (8) for $p = 1, 2, 3, ...$

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$N_{i,p} = \frac{\xi - \xi_i}{\xi_{i+p} - \xi i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi i + 1} N_{i+1,p-1}(\xi) \tag{8}$$

The functions defined by (7) and (8) follow a partition of unity and are non-negative, which means that all terms in the mass matrix are positive. Figure 1 shows an example of NURBS shape functions.

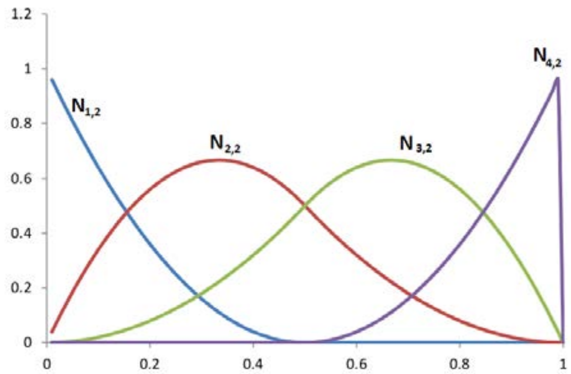

Figure 1. Example of NURBS shape functions generated from a open knot vector $\Xi = \{0, 0, 0, 0.5, 1, 1, 1\}$ and polynomial order p=2

3.2 Isogeometric Refinement

The refinement in isogeometric analysis (Cottrell *et al.*, 2007) consists to change the basic parameters of the shape functions and rewrite them.

The h refinement is related with knot insertion without increase the polynomial order. The shape functions are built from a new set of parametric coordinates $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+m+p+1}\}$ where m new control point are added. Figure 2 shows an example of h refinement from and open knot vector $\Xi = \{0, 0, 0, 0.5, 1, 1, 1\}$ to new knot vector $\Xi = \{0, 0, 0, 1/3, 2/3, 1, 1, 1\}$.

The p refinement is done increasing the polynomial order and also the number of shape functions in each element. In p refinement every distinct knot are repeated one time and the number of shape functions increase from n to $n+e$ where e is the total number of elements. In isogeometric analysis an element is the spam between two distinct knots. Figure 3 shows an example of p refinement.

The polynomial increasing are not directly related with the knot vector, but the number of control points in the knot vector must be $n+p+1$. If the polynomial order is increased by 1, the number of shape functions also increase by 1 and the number of knots will be $(n+1)+(p+1)+1$ and the shape functions will now applied along all elements. This procedure is a new refinement called k refinement.

ISSN 2176-5480

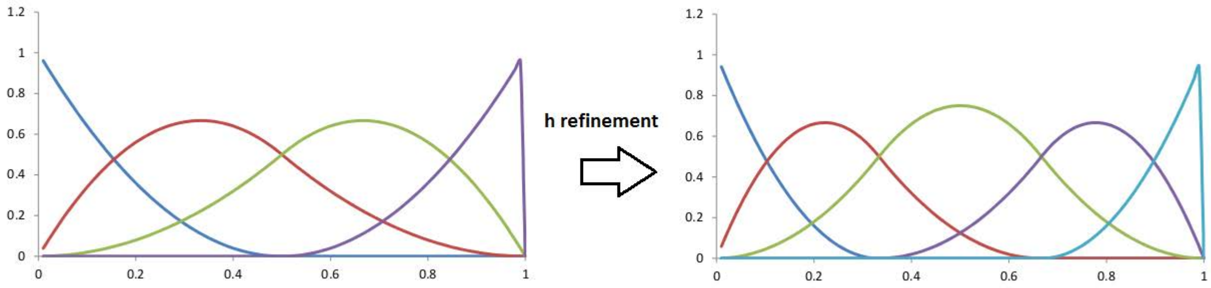

Figure 2. Example of Isogeometric h refinement

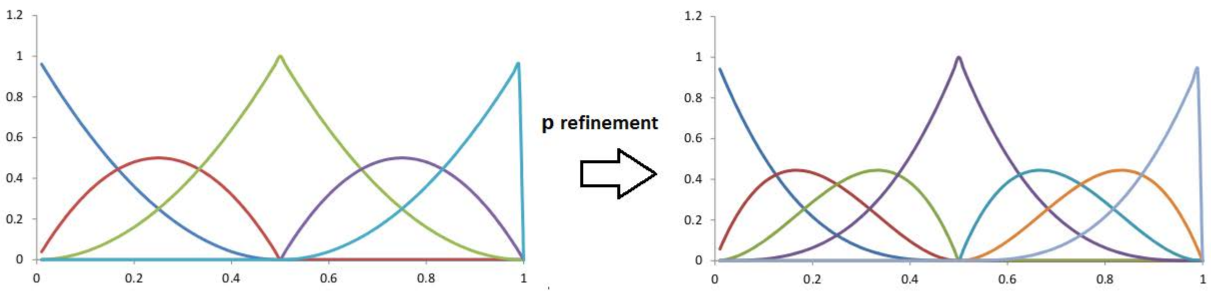

Figure 3. Example of Isogeometric p refinement

Compared if the p refinement, the k refinement takes advantage one time that the smoothness is increased and the number of shape functions are smaller. If the polynomial order is increased by r, the number of shape functions in p refinement increases from n to $n + r.e$ and in k refinement it increases to $n + r$.

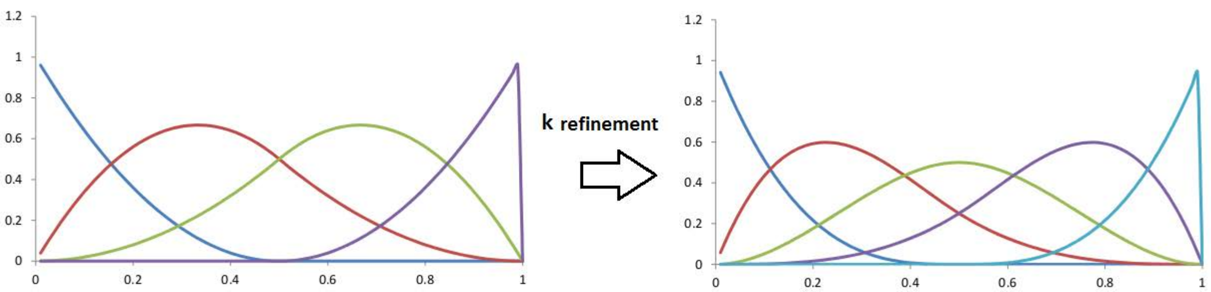

Figure 4. Example of Isogeometric k refinement

4. NUMERICAL EXPERIMENTS

4.1 Bar Fixed in Two Edges

The problem of a bar fixed in two edges (figure 5) (also called fixed-fixed bar) was developed in quadratic isogeometric analysis and quadratic finite elements method. With the goal to simplify the problem all bar properties was considered unitary, this means $L = 1m$, $E = 1N/m^2$, $\rho = 1kg/m^3$ and $A = 1m^2$. The analytical solution to this problem (eq. 9) are presented in Cottrell *et al.* (2006).

$$\omega = n.\pi \quad \text{with } n = 1, 2, 3... \tag{9}$$

Where n now is mode number of each frequency. The analysis results are shown in figure 6 where the natural vibration frequencies ω^h are normalized with the exact solution ω (eq. 9) versus the mode number (n) normalized with the total number of degrees of freedom (N).

The finite elements analysis presented smaller errors to $n/N < 0.5$. After a optical branch the errors at $n/N > 0.5$ are higher where the poorest result occurs next to $n/N = 0.9$ where $\frac{\omega^h}{\omega} \approx 1.30$.

The isogeometric analysis presents a better behavior along the whole bar, where the highest error occurs also closer to $n/N = 0.9$ where $\frac{\omega^h}{\omega} \approx 1.06$.

The results were plotted to a $N = 200$ degrees of freedom, but both curves behaviors don't depend of N.

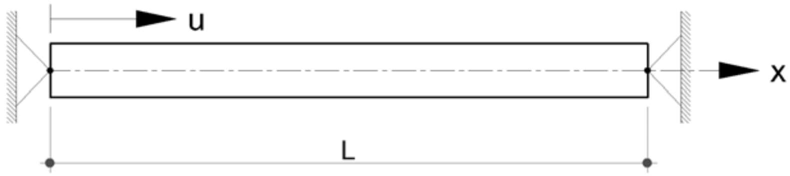

Figure 5. Bar Fixed in two edges

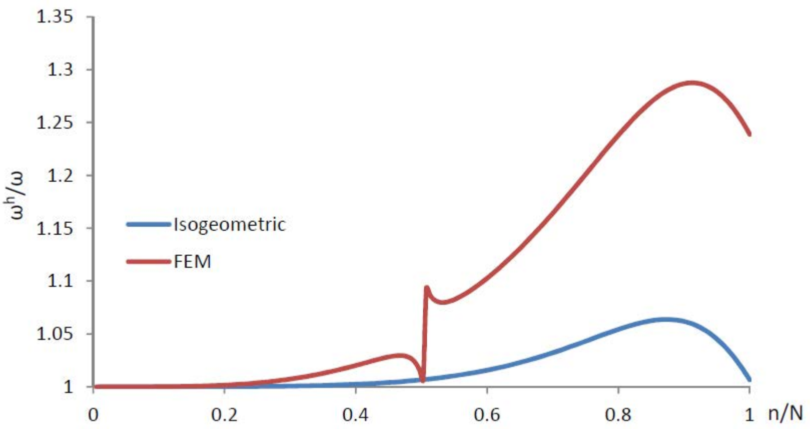

Figure 6. Numerical results of a two edge fixed bar

4.2 Bar Fixed in One Edge

In this problem five analysis were developed: FEM Linear Element, FEM Cubic Element, FEM p refinement, Isogeometric p refinement and Isogeometric k refinement. The analytical solution (eq. 10) of the fixed-free bar is presented in Arndt *et al.* (2011). Figure 7 shows the scheme of a fix-free bar.

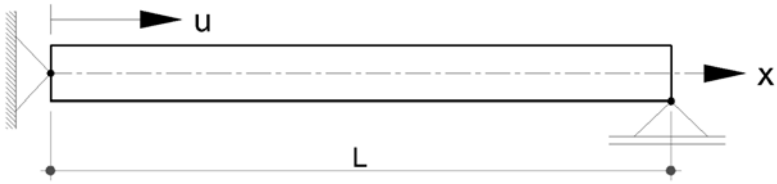

Figure 7. Bar fixed in one edge

$$\lambda = ((2n-1).\frac{\pi}{2})^2 \quad \text{with } n = 1, 2, 3... \tag{10}$$

The percentage error are defined by (11).

$$ERR = \frac{\lambda^h - \lambda}{\lambda}.100 \tag{11}$$

The figure 8 shows the percentage eigenvalue errors versus the total number of degrees of freedom to the first six eigenvalues.

Analyzing the results, the convergence rate of the p FEM, Isogeometric p and k are greater than the h refinements of finite elements. When the eigenvalues increases the p refinement of FEM presents the same precision as the isogeometric k refinement. As expected the convergence rate of isogeometric k refinement are grater than isogeometric p refinement.

ISSN 2176-5480

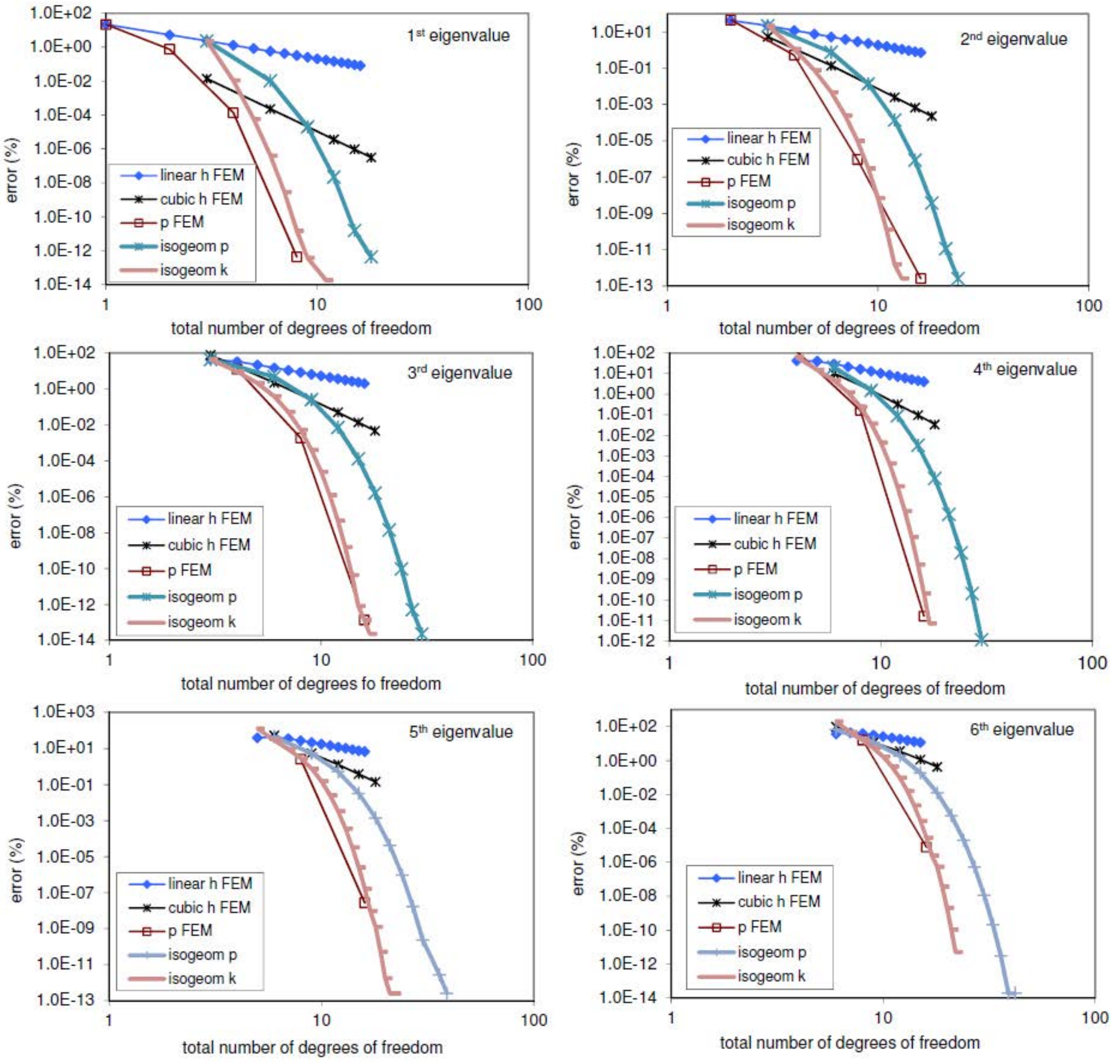

Figure 8. Numerical results of a one edge fixed bar

5. CONCLUSION

This work presents the free vibration problem and the variational form to straight bars and the basic concepts of isogeometric analysis. The NURBS functions were tested as approach shape functions to the problem of free vibration of bars and compared with the finite elements method.

To the problem of fixed-fixed bar the isogeometric analysis presents better results than the finite elements to all vibration modes. The k and p refinement of the isogeometric analysis to fix-free bar had their convergence tested and the convergence rate was grater than the finite elements method, mostly in the higher eigenvalues. As expected, the precision of k refinement was grater.

6. REFERENCES

Arndt, M., Machado, R.D. and Scremin, A., 2011. "Enriched methods for vibration analysis of framed structures". In *Proceedings of the 21st International Congress of Mechanical Engineering - COBEM2011*. Natal, Brazil.

Chopra, A.K., 1995. *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. Prentice Hall, New Jersey, 3rd edition.

Cottrell, J.A., Bazilevs, Y. and Hughes, T.J.R., 2009. *Isogeometric Analysis: Toward Integration of CAD and FEA*. John Wiley & Sons, USA, 1st edition.

Cottrell, J.A., Hughes, T.J.R. and Reali, A., 2007. "Studies of refinement and continuity in isogeometric structural analysis". *Computer Methods in Applied Mechanics and Engineering*, Vol. 196, pp. 4160–4183.

Cottrell, J.A., Reali, A., Bazilevs, Y. and Hughes, T.J.R., 2006. "Isogeometric analysis of structural vibrations". *Computer Methods in Applied Mechanics and Engineering*, Vol. 195, pp. 5257–5196.

Hughes, T.J.R., Cottrell, J.A. and Bazilevs, Y., 2005. "Isogeometric analysis: Cad, finite elements, nurbs, exact geometry and mesh refinement". *Computer Methods in Applied Mechanics and Engineering*, Vol. 194, pp. 4135–4195.

7. RESPONSIBILITY NOTICE

The author(s) is (are) the only responsible for the printed material included in this paper.