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ENHANCED LUMPED FORMULATIONS FOR THE GRAETZ PROBLEM IN INFINITE DOMAINS

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Abstract. *The objective of this paper is to propose analytical approximations for solving an extended version of the Graetz problem with axial diffusion in an infinite domain. The adopted methodology consists of transforming the original convection-diffusion partial-differential equations into a simpler one-dimensional form, using approximation rules provided by the Coupled Integral Equations Approach (CIEA). This technique is employed for calculating the mean stream temperature in thermally developing fluid flow, and different levels of approximations are analyzed. The results are compared with an exact analytical solution to the problem, and a solution using the Classical Lumped System Analysis (CLSA). The proposed solutions are simple and show very good agreement with the exact solution*

Keywords: *lumped-capacitance, convection-diffusion, analytical solution, mathematical modeling*

1. INTRODUCTION

When a fluid is flowing, laminar and fully developed, in a cylindrical tube or within a parallel plates channel with heating or cooling applied at the solid walls, the solution of the energy equation depends on the value of a single dimensionless group, the well-knonwn Péclet number. In classical Gratez problems, this parameter is usually assumed large such that the heat transport equation is simplified as axial diffusion becomes negligible. When axial diffusion is taken account, an extended Graetz problem is obtained. An exact solution for this problem was presented by Acrivos (1980), Vick and Özisik (1981) and Ebadian and Zhang (1989) for low Péclet numbers and different boundary conditions. Approximate solutions were used also (Villadsen and Michelson, 1976), (Laohakul *et al.*, 1985) and (Barros and Sphaier, 2012). The purpose of this paper, is to present simple approximate solutions for the problem in an infinite domain with a discontinuous boundary condition (wall temperature given by a step-function), including the effects of axial diffusion.

Approximating an integral by a linear combination of the integrand values and its derivatives at the integration limits was an idea originally developed by Hermite (1878) and first presented by Mennig *et al.* (1983), the first ones to use this two-point approach, deriving it in a fully differential form called $H_{\alpha,\beta}$. Using the Hermite formulas in improved-lumped formulations is known as the Coupled Integral Equation Approach (CIEA), which can be found in a variety of heat transfer studies. Recent applications include ablation (Ruperti *et al.*, 2004), drying (Dantas *et al.*, 2007), heat conduction with temperature-dependent conductivity (Su *et al.*, 2009) and adsorbed gas storage (Sphaier and Jurumenha, 2012). In this study, the CIEA is employed for the problem of thermally developing fluid flow within a parallel-plates duct. With this approach, enhanced lumped-differential formulations for representing the problem are obtained. The formulations are naturally simpler the the original equation since they involve simple ODEs for determining the mean stream temperature, while the original problem requires the solution of a PDE for calculating the temperature field, and from this result calculating the same averaged mean stream temperature.

2. PROBLEM FORMULATION AND HERMITE APPROXIMATION

In order to illustrate the proposed methodology, a general problem of flow within parallel plates is considered, which written in dimensionless form is given by:

$$u^* \frac{\partial \theta}{\partial \xi} = \mathrm{Pe}^{-2} \frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2}, \qquad \theta(0,\eta) = 0, \qquad \left|\frac{\partial \theta}{\partial \xi}\right|_{\xi\to\pm\infty} < \infty, \tag{1a}$$

$$\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0} = 0, \qquad \theta(\xi,1) = U_1(\xi), \tag{1b}$$

where $U_1(\xi)$ is the Heaviside function, i.e., $U_1 = 1$, for $\xi \geq 0$ and $U_1 = 0$ otherwise; the employed dimensionless parameters and variables are defined as:

$$\mathrm{Pe} = \frac{\bar{u}\,H}{\alpha} \qquad \eta = \frac{y}{H/2}, \qquad \xi = \frac{x}{L}, \qquad L = \frac{H}{2}\,\mathrm{Pe}, \qquad \theta = \frac{T - T_{\min}}{T_{\max} - T_{\min}}, \tag{2}$$

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The basis for the Coupled Integral Equations Approach (CIEA) is the Hermite approximation of an integral, denoted, $H_{\alpha,\beta}$, which is given by the general expression:

$$\int_{x_{i-1}}^{x_i} f(x)dx = \sum_{\nu=0}^{\alpha} c_\nu(\alpha,\beta) h_i^{\nu+1} f^{(\nu)}(x_{i-1}) + \sum_{\nu=0}^{\beta} c_\nu(\beta,\alpha)(-1)^\nu h_i^{\nu+1} f^{(\nu)}(x_i) + E_{\alpha,\beta} \tag{3a}$$

where,

$$h_i = x_i - x_{i-1}, \qquad c_\nu(\alpha,\beta) = \frac{(\alpha+1)!(\alpha+\beta-\nu+1)!}{(\nu+1)!(\alpha-\nu)!(\alpha+\beta+2)!} \tag{3b}$$

and $f(x)$ and its derivatives $f^{(\nu)}(x)$ are defined for all $x \in [x_{i-1}, x_i]$. $E_{\alpha,\beta}$ is the error in the approximation. It is assumed that $f^{(\nu)}(x_{i-1}) = f_{i-1}^{(\nu)}$ for $\nu = 0, 1, 2, \dots, \alpha$ and $f^{(\nu)}(x_i) = f_i^{(\nu)}$ for $\nu = 0, 1, 2, \dots, \beta$.

The Hermite integration formula can provide different approximation levels, starting from the classical lumped system analysis towards improved lumped-differential formulations. A detailed error analysis of the application of the CIEA to diffusion problems using $H_{0,0}$, $H_{0,1}$, $H_{1,0}$, and $H_{1,1}$ Hermite approximations was carried out in (Alves *et al.*, 2000). Since approximations of order higher than $H_{1,1}$ involve derivatives of order higher than one, these are avoided for the sake of simplicity of the methodology. Hence, only the two different approximations below are considered:

$$H_{0,0} \Rightarrow \int_0^h f(x)\,\mathrm{d}x \approx \frac{1}{2}h(f(0)+f(h)), \tag{4a}$$

$$H_{1,1} \Rightarrow \int_0^h f(x)\,\mathrm{d}x \approx \frac{1}{2}h(f(0)+f(h)) + \frac{1}{12}h^2(f'(0)-f'(h)), \tag{4b}$$

which correspond to the well-known trapezoidal and corrected trapezoidal integration rules, respectively.

3. PLUG-FLOW ANALYSIS

For the simplified plug-flow case, $u^* = 1$ and the mean stream temperature equals the average temperature definition. Integrating equations (1a) and applying the average definition leads to the following ODE system:

$$\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}, \qquad \theta(\xi,1) = U_1(\xi), \qquad \left|\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi}\right|_{\xi\to\pm\infty} < \infty, \tag{5}$$

The Classical Lumped-System Analysis (CLSA) consists in approximating the averages directly by boundary values, which corresponds to applying the rectangular integration approximation rule.

In order to avoid a constant mean stream temperature, and in order to obtain a relation that leads to $\theta(\xi,0)$ different than $\theta(\xi,1)$, the following integral approximations are used:

$$\overline{\theta}(\xi) = \int_0^1 \theta\,\mathrm{d}\eta \approx \theta(\xi,0), \qquad \int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \approx \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \tag{6}$$

which leads to the following relation for the wall derivative:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx \theta(\xi,1) - \overline{\theta}(\xi) = 0 - \overline{\theta}(\xi), \text{ for } \xi < 0 \text{ and} \tag{7}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx \theta(\xi,1) - \overline{\theta}(\xi) = 1 - \overline{\theta}(\xi), \text{ for } \xi > 0. \tag{8}$$

In order to guarantee the continuity at $\xi = 0$, the following coupling conditions are employed:

$$\lim_{\xi\to 0^-} \overline{\theta}(\xi) = \lim_{\xi\to 0^+} \overline{\theta}(\xi) \quad \text{and} \tag{9}$$

$$\lim_{\xi\to 0^-}\left(\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi}\right) = \lim_{\xi\to 0^+}\left(\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi}\right), \tag{10}$$

such that the solution of the averaged system (5) is given by:

$$\overline{\theta}(\xi) = \begin{cases} \exp\left(\mathrm{Pe}^2 + \kappa\,\mathrm{Pe}\,\xi/2\right)\cdot\left(\frac{1}{2} - \frac{\mathrm{Pe}}{2\,\kappa}\right), & \text{if } \xi \le 0 \\ 1 - \exp\left(\mathrm{Pe}^2 - \kappa\,\mathrm{Pe}\,\xi/2\right)\cdot\left(\frac{1}{2} + \frac{\mathrm{Pe}}{2\,\kappa}\right), & \text{if } \xi > 0 \end{cases} \tag{11}$$

where $\kappa = \sqrt{4 + \mathrm{Pe}^2}$.

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3.1 Improved Lumped-System Analysis

3.1.1 $H_{0,0}/H_{0,0}$ formulation

Using the $H_{0,0}$ scheme for approximating the integrals of θ and its derivative yields:

$$\int_0^1 \theta \,\mathrm{d}\eta \approx \frac{1}{2}\Big(\theta(\xi,0) + \theta(\xi,1)\Big), \qquad \int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \approx \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right). \tag{12}$$

Using the boundary conditions and solving for the wall derivative gives:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx -4\,\overline{\theta}(\xi) \text{ for } \xi < 0 \text{ and} \tag{13}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx 4\left(1 - \overline{\theta}(\xi)\right) \text{ for } \xi > 0, \tag{14}$$

such that the solution of the averaged system (5) is given by:

$$\overline{\theta}(\xi) = \begin{cases} \exp\Big(\mathrm{Pe}^2 + \kappa\,\mathrm{Pe}\,\xi/2\Big)\cdot\Big(\frac{1}{2} - \frac{\mathrm{Pe}}{2\,\kappa}\Big), & \text{if } \xi \le 0 \\ 1 - \exp\Big(\mathrm{Pe}^2 - \kappa\,\mathrm{Pe}\,\xi/2\Big)\cdot\Big(\frac{1}{2} + \frac{\mathrm{Pe}}{2\,\kappa}\Big), & \text{if } \xi > 0 \end{cases} \tag{15}$$

where $\kappa = \sqrt{16 + \mathrm{Pe}^2}$.

3.1.2 $H_{1,1}/H_{0,0}$ formulation

This scheme s based on using $H_{1,1}$ approximation for the temperature integral:

$$\int_0^1 \theta \,\mathrm{d}\eta \approx \frac{1}{2}\Big(\theta(\xi,0) + \theta(\xi,1)\Big) + \frac{1}{12}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} - \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) \tag{16}$$

and the same $H_{0,0}$ approximation for its derivative integral. Applying boundary conditions and solving for the wall derivative yields:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx -3\,\overline{\theta}(\xi) \text{ for } \xi < 0 \text{ and} \tag{17}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx 3\left(1 - \overline{\theta}(\xi)\right) \text{ for } \xi > 0, \tag{18}$$

such that the solution of the averaged system (5) is given by:

$$\overline{\theta}(\xi) = \begin{cases} \exp\Big(\mathrm{Pe}^2 + \kappa\,\mathrm{Pe}\,\xi/2\Big)\cdot\Big(\frac{1}{2} - \frac{\mathrm{Pe}}{2\,\kappa}\Big), & \text{if } \xi \le 0 \\ 1 - \exp\Big(\mathrm{Pe}^2 - \kappa\,\mathrm{Pe}\,\xi/2\Big)\cdot\Big(\frac{1}{2} + \frac{\mathrm{Pe}}{2\,\kappa}\Big), & \text{if } \xi > 0 \end{cases} \tag{19}$$

where $\kappa = \sqrt{12 + \mathrm{Pe}^2}$.

3.1.3 $H_{1,1}/H_{1,1}$ formulation

This approximation scheme relies on using the $H_{1,1}$ for approximating the integral of θ and its derivative, the later being given by:

$$\int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \approx \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) + \frac{1}{12}\left(\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0} - \left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=1}\right) \tag{20}$$

$$U_1(\xi) - \theta(\xi,0) \approx \frac{1}{2}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} + \frac{1}{12}\left(\left(\frac{\partial\theta}{\partial\xi}\right)_{\eta=0} - \mathrm{Pe}^{-2}\left(\frac{\partial^2\theta}{\partial\xi^2}\right)_{\eta=0}\right) \tag{21}$$

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Eliminating the wall derivative from the previous equation and the $H_{1,1}$ temperature integral, and substituting in the integrated energy balance leads to:

$$\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + 6\,\theta_0 - 12\,\overline{\theta}(\xi) \quad \text{and} \quad \mathrm{Pe}^2\left(48\,\theta_0 - 72\,\overline{\theta}(\xi) + \frac{\mathrm{d}\theta_0}{\mathrm{d}\xi}\right) = \frac{\mathrm{d}^2\theta_0}{\mathrm{d}\xi^2}, \text{ for } \xi < 0, \text{ and} \tag{22}$$

$$\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + 6 + 6\,\theta_0 - 12\,\overline{\theta}(\xi) \quad \text{and} \quad \mathrm{Pe}^2\left(24 + 48\,\theta_0 - 72\,\overline{\theta}(\xi) + \frac{\mathrm{d}\theta_0}{\mathrm{d}\xi}\right) = \frac{\mathrm{d}^2\theta_0}{\mathrm{d}\xi^2}, \text{ for } \xi \geq 0. \tag{23}$$

where $\theta_0 = \theta(\xi, 0)$. This coupled ODE system can be solved directly for $\overline{\theta}$; however the solution is not presented due to space limitations.

4. LAMINAR-FLOW ANALYSIS

This section presents the methodology for laminar flow (Hagen-Poiseuille profile), $u^* = (3/2)\,\bar{u}\,\left(1 - \eta^2\right)$, in which the average is given by $\overline{\theta}(\xi) = \int_0^1 \theta(\xi, \eta)\,\mathrm{d}\eta$ and the meam stream temperature is defined by $\theta_m(\xi) = \int_0^1 u^*\theta(\xi, \eta)\,\mathrm{d}\eta$. Averaging the transport equations, and substituting the mean stream temperature definition and boundary conditions lead to:

$$\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}, \qquad \theta(\xi, 1) = U_1(\xi), \qquad \left|\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi}\right|_{\xi\to\pm\infty} < \infty. \tag{24}$$

For isothermal wall different levels of approximation can lead to different lumped formulations, as described next.

4.1 Improved Lumped-System Analysis

4.1.1 $H_{0,0}/H_{0,0}/H_{0,0}$ formulation

This scheme is based on using $H_{0,0}$ approximation for the misture temperature integral:

$$\int_0^1 u^*\theta\,\mathrm{d}\eta \approx \frac{1}{2}\Big(u^*(0)\,\theta(\xi, 0) + u^*(1)\,\theta(\xi, 1)\Big) \tag{25}$$

and the same $H_{0,0}$ approximation for the temperature and for its derivative integral. Applying boundary conditions and solving for the wall derivatives yields:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx 2\,U_1(\xi) - \frac{8}{3}\,\theta_m(\xi), \qquad \frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} \approx \frac{2}{3}\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2} \tag{26}$$

such that the solution of the averaged system (24) is given by:

$$\theta_m(\xi) = \begin{cases} \exp\Big(3\,\mathrm{Pe}^2 + \kappa\,\mathrm{Pe}\,\xi/4\Big) \cdot \Big(\frac{3}{8} - \frac{9\,\mathrm{Pe}}{8\,\kappa}\Big), & \text{if } \xi \leq 0 \\ \frac{3}{4} - 3\,\exp\Big(3\,\mathrm{Pe}^2 - \kappa\,\mathrm{Pe}\,\xi/4\Big) \cdot \Big(\frac{1}{8} + \frac{3\,\mathrm{Pe}}{8\,\kappa}\Big), & \text{if } \xi > 0 \end{cases} \tag{27}$$

where $\kappa = \sqrt{64 + 9\,\mathrm{Pe}^2}$.

4.1.2 $H_{1,1}/H_{0,0}/H_{1,1}$ formulation

This scheme is based on using $H_{1,1}$ approximation for the misture temperature integral and for the averaged temperature integral:

$$\int_0^1 u^*\theta\,\mathrm{d}\eta \approx \frac{1}{2}\Big(u^*(0)\,\theta(\xi, 0) + u^*(1)\,\theta(\xi, 1)\Big) + \frac{1}{12}\left(\left(\frac{\partial(u^*\theta)}{\partial\eta}\right)_{\eta=0} - \left(\frac{\partial(u^*\theta)}{\partial\eta}\right)_{\eta=1}\right) \tag{28}$$

$$\int_0^1 \theta\,\mathrm{d}\eta \approx \Big(\theta(\xi, 0) + \theta(\xi, 1)\Big) + \frac{1}{12}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} - \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) \tag{29}$$

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and the same $H_{0,0}$approximation for its derivative integral. Applying boundary conditions and solving for the wall derivatives yields:

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \approx \frac{8}{3}\Big(U_1(\xi)\, - \theta_m(\xi)\Big), \qquad \frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} \approx \frac{8}{9}\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2} \tag{30}$$

such that the solution of the averaged system (24) is given by:

$$\theta_m(\xi) = \begin{cases} \exp\Big(9\,\mathrm{Pe}^2 + \kappa\,\mathrm{Pe}\,\xi/16\Big)\cdot\Big(\frac{1}{2} - \frac{9\,\mathrm{Pe}}{2\,\kappa}\Big), & \text{if } \xi \le 0 \\ 1\, - \exp\Big(9\,\mathrm{Pe}^2 - \kappa\,\mathrm{Pe}\,\xi/16\Big)\cdot\Big(\frac{1}{2} + \frac{9\,\mathrm{Pe}}{2\,\kappa}\Big), & \text{if } \xi > 0 \end{cases} \tag{31}$$

where $\kappa = \sqrt{768 + 81\,\mathrm{Pe}^2}$.

4.1.3 $H_{1,1}/H_{1,1}/H_{1,1}$ formulation

This scheme is based on using $H_{1,1}$ approximation for the derivative temperature integral:

$$\int_0^1 \frac{\partial\theta}{\partial\eta}\,\mathrm{d}\eta \approx \frac{1}{2}\left(\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} + \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}\right) + \frac{1}{12}\left(\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0} - \left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=1}\right) \tag{32}$$

and the same $H_{1,1}$approximation for the averaged and misture temperature integral. Using equation (1a), we have the below relations:

$$\left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=0} = \frac{3}{2}\frac{\mathrm{d}\theta_0}{\mathrm{d}\xi} - \mathrm{Pe}^{-2}\frac{\mathrm{d}^2\theta_0}{\mathrm{d}\xi^2} \quad \text{e} \quad \left(\frac{\partial^2\theta}{\partial\eta^2}\right)_{\eta=1} = 0 \tag{33}$$

Substituting boundary conditions leads to:

$$\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} = \mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\xi^2} + \frac{1}{9}\Big(24\,U_1(\xi) - 24\,\theta_m(\xi)\, - 3\,\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} + 2\,\mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2}\Big) \quad \text{e} \tag{34}$$

$$\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\xi} = \frac{1}{108}\left(12\,U_1(\xi) + 96\,\theta_m + 3\,\frac{\mathrm{d}\theta_m}{\mathrm{d}\xi} - 2\,\mathrm{Pe}^{-2}\,\frac{\mathrm{d}^2\theta_m}{\mathrm{d}\xi^2}\right) \tag{35}$$

This coupled ODE system can be solved directly for θ_m; however the solution is not presented due to space limitations.

5. RESULTS AND DISCUSSION

The previous solutions are compared with the exact solution of the Graetz problem. Considering a plug-flow profile, the exact solution is given by:

$$\overline{\theta}(\xi) = \begin{cases} \sum_{n=1}^{\infty} A_n\,\exp(\beta_n\,\xi), & \text{if } \xi \le 0 \\ \sum_{n=1}^{\infty} B_n\,\exp(\gamma_n\,\xi), & \text{if } \xi > 0 \end{cases} \tag{36}$$

where,

$$\beta_n = \frac{\mathrm{Pe}}{2}\left(\mathrm{Pe} + \sqrt{\mathrm{Pe}^2 + (2n-1)^2\,\pi^2}\right), \qquad \gamma_n = \frac{\mathrm{Pe}}{2}\left(\mathrm{Pe} - \sqrt{\mathrm{Pe}^2 + (2n-1)^2\,\pi^2}\right), \tag{37}$$

$$b_n = \frac{2\,(-1)^n}{\pi(1-2\,n)}, \tag{38}$$

$$A_n = \frac{2\,\gamma_n\,b_n^2}{\gamma_n\, - \beta_n}, \text{ and } \qquad B_n = \frac{2\,\beta_n\,b_n^2}{\gamma_n\, - \beta_n} \tag{39}$$

And for a laminar profile, the exact solution is given by:

$$\theta_m(\xi) = \begin{cases} \sum_{n=1}^{\infty} A_n^+\,\Gamma_n\,(\xi), & \text{se } \xi \le 0 \\ \sum_{n=1}^{\infty} B_n^+\,\Phi_n\,(\xi), & \text{se } \xi > 0 \end{cases} \tag{40}$$

where the functions Γ e Φ are linear combinations of exponentials functions, as described in details in (Sphaier, 2012).

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Figure 1 shows a comparison of all employed approximate schemes and the exact solution. As can be seen, the CLSA solution underestimates the mean temperature by a significant amount, while the $H_{0,0}/H_{0,0}$ scheme overestimate it. However, both $H_{1,1}/H_{0,0}$ and $H_{1,1}/H_{1,1}$ show a very good agreement with the exact solution. When looking into the effect of axial diffusion, it is cleat that this phenomena has a negligible effect the on mean temperature distribution for $\mathrm{Pe} \gg 1$.

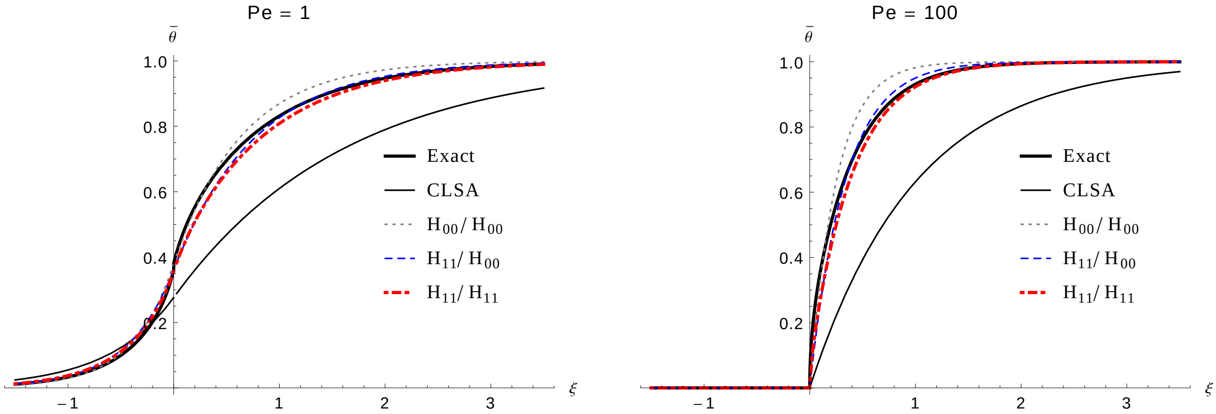

Figure 1. Comparison between different lumped approximation schemes and exact solution for different Péclet numbers for a plug-flow.

Next, in fig. 2, a similar comparison with laminar flow is shown. As one can observe, the CLSA solution and the $H_{0,0}/H_{0,0}/H_{0,0}$ do not provide a good approximation. Moreover, the $H_{1,1}/H_{0,0}/H_{0,0}$ and $H_{1,1}/H_{0,0}/H_{1,1}$ overestimate the mean temperature. On the other hand, the $H_{1,1}/H_{1,1}/H_{1,1}$ approximation scheme shows a very good agreement with the exact solution.

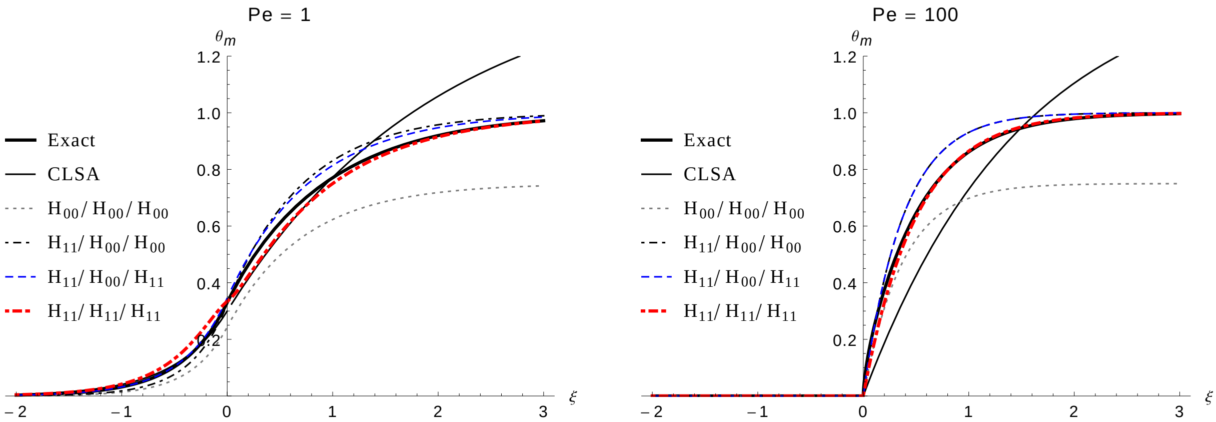

Figure 2. Comparison between different lumped approximation schemes and exact solution for different Péclet numbers for a laminar flow.

Subsequently, figure 3 shows a comparison of the maximum absolute error resulting from the different approximate solutions for a range of values of the Péclet number. The left plot corresponds to the plug-flow case while the other corresponds to the laminar flow case.

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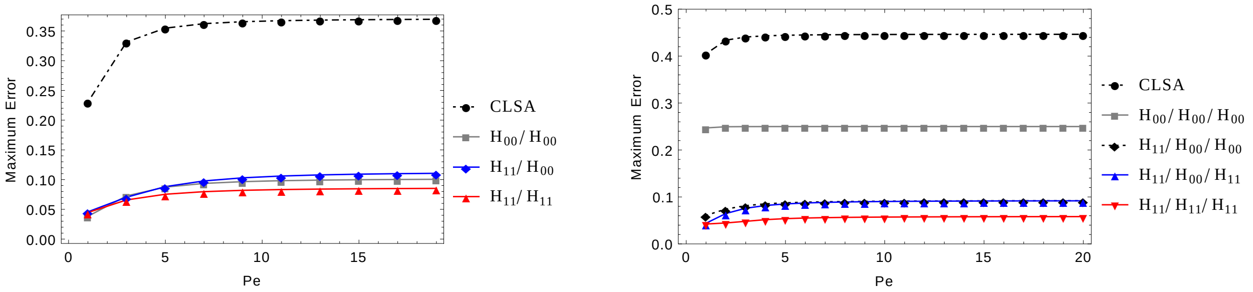

Figure 3. Comparison between maximum error of different lumped approximation schemes and exact solution for different Péclet numbers.

Finally, figure 4 shows a comparison between different schemes and the exact solution at the discontinuity point. For the plug-flow case (left plot) all schemes underestimate the mean temperature. For the laminar flow case, the $H_{1,1}/H_{1,1}/H_{1,1}$ scheme shows a good agreement with the exact solution for the entire Péclet number range. One should also observe that, as $\mathrm{Pe} \to \infty$, the mean stream temperature at $\xi = 0$ tend to zero for all approximations.

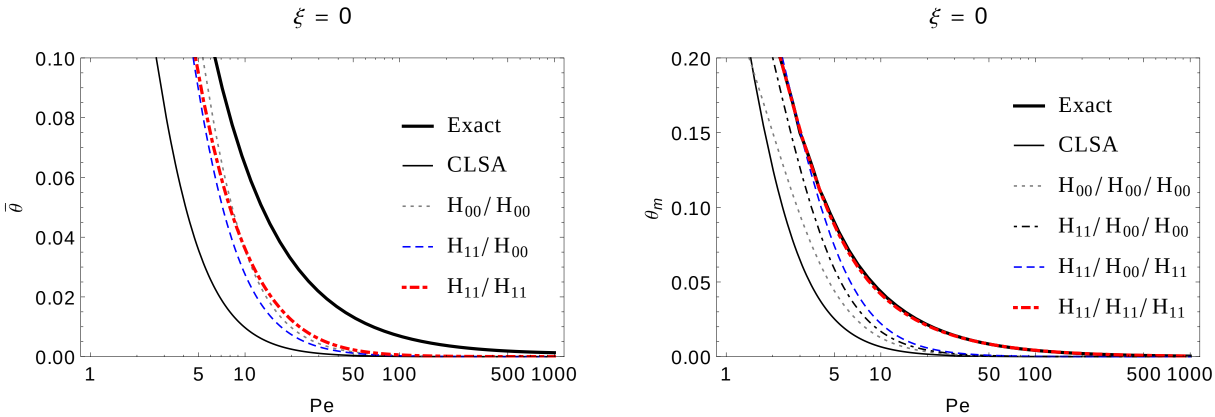

Figure 4. Comparison between different lumped approximation schemes and exact solution for different Péclet numbers.

6. CONCLUSIONS

This paper presented an alternative approach for calculating the mean stream temperature for dynamically-developed and thermally-developing flow with a discontinuous wall boundary condition, comprising an extended version of the Graetz problem. An approximate analytical methodology, based on the CIEA, was employed, based on the Hermite approximation formulas. The simplified case of plug-flow and a laminar flow case were presented for illustrating the methodology and comparing the results with an exact solution with no approximations. The results showed that some approximation schemes lead to very good agreement with the two-dimensional Graetz (exact) solution. It was also seen that, for the approximation schemes that do not provide a good agreement, there is a tendency where the approximate solutions underestimate the exact solution at the discontinuity point.

7. ACKNOWLEDGEMENTS

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