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FURTHER APPLICATIONS OF A VARIATIONAL METHOD TO THE VERTICAL HYDRODYNAMIC IMPACT OF AXISYMMETRIC BODIES

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Abstract. *This paper presents the application of a variational method to the classical hydrodynamic impact problem within the so-called Generalized von Kármán Model (GvKM). The extended Lagrange equation, valid for variable mass systems enables one to consistently treat the impact problem by considering the added mass tensor defined in the bulk of the liquid and excluding the jets. The solution of the nonlinear dynamic equation of the impacting motion depends on the determination of the added mass tensor and its derivative with respect to time at each integration time step. This is done through a variational method technique that leads to a second-order error approximation for the added mass if a first-order error approximation is sought for the velocity potential. This technique was detailed in a previous work by the authors. An application of the variational method for a prolate ellipsoid is here addressed.*

Keywords: *variational numerical method, hydrodynamic impact, axisymmetric bodies*

1. INTRODUCTION

First studies of the hydrodynamic impact problem began in the 30s with the works of von Kármán (1929) and Wagner (1931), motivated by studying the impact of seaplane floats during landing onto the water. They studied the impact problem by approximating the impact body by a two-dimensional flat disc, considered to be collapsed onto the plane $z = 0$. In this approach, the contact surface solid-liquid, which is three-dimensional in its original form, is considered as a contact plane. Furthermore, Wagner took into account the free surface elevation, i.e., the piled-up water effects; which are not considered in the von Kármán approach.

Besides von Kármán and Wagner approaches, different models of impact have been studied, such as the Generalized von Kármán model (GvKM), the Generalized Wagner Model (GWM), and the Modified Logvinovich Model (MLM). In the GvKM, the exact body boundary conditions are fulfilled but the wet correction, i.e the free surface elevation, is not taken into account. This model has been recently used by Malenica and Korobkin (2007) for ship hulls during water impact. The GWM is considered as a generalization of the Wagner model, where the body boundary condition is exactly satisfied and the condition of the free surface is imposed on the horizontal lines at the splash-up height; see Zhao, *et al.*, 1996, Faltinsen and Chezhian (2005) and Korobkin (2004). In the MLM, the body boundary condition is satisfied at $z = 0$, but the exact body shape is considered a posteriori in the calculation of the hydrodynamic loads; see Malenica and Korobkin (2007), and Korobkin and Malenica (2005).

In the context of a GvKM, the purpose of the present paper is to discuss the application of a variational numerical method, published in Santos, *et al.*, 2013[1], based on the previous work by Pesce and Simos (2008), to address the hydrodynamic impact problem of a prolate ellipsoid (spheroid) during its vertical entrance into the water. According to Santos, *et al.*, 2013, the potential problem of hydrodynamic impact, characterized by the dominance of inertial forces, is formulated by assuming the liquid surface as equipotential, what allows the analogy with the infinity frequency limit in the usual free surface oscillating floating body problem. The vertical impact force is then calculated from the added mass variation with the penetration depth, obtained through the variational method. In this approach, the body boundary conditions are exactly satisfied, instead of approximating the impact body by an equivalent flat plate, what implies that the original three-dimensional shape of the body is taken into account. However, the effects of the local free surface elevation are not considered in the present work.

An account of the relative importance of buoyancy forces is given and illustrated through a simple free fall example.

2. MATHEMATICAL FORMULATION

Consider that the fluid is initially at rest. The initial instant $t = 0$ is defined as the instant when the body touches the free surface at a single point, taken as the origin of a Cartesian coordinate system $Oxyz$. The fluid is assumed inviscid and the flow irrotational, such that a potential scalar function, $\phi(x, y, z, t)$, defines the velocity field. Body forces are

[1] In that paper, the technique was detailed and applied to other axisymmetric bodies (sphere and oblate ellipsoid).

assumed to be negligible. The usual formulation of the problem follows from the consideration of a) the incompressibility condition of the liquid, b) the impermeability condition of the body, c) the kinematic and dynamic boundary conditions of the free surface, d) the evanescence condition for the velocity potential $\phi = \phi(x, y, z, t)$ and for the free surface elevation $z = \eta(x, y, t)$, and the initial condition for both. Further details about the mathematical formulation can be found in Korobkin (1988), Faltinsen (1990) and Casetta (2004). Within the Generalized von Kármán Model, the formulation of the problem can be simplified as follows; see Fig. 1 for definitions of V, S_B and S_F.

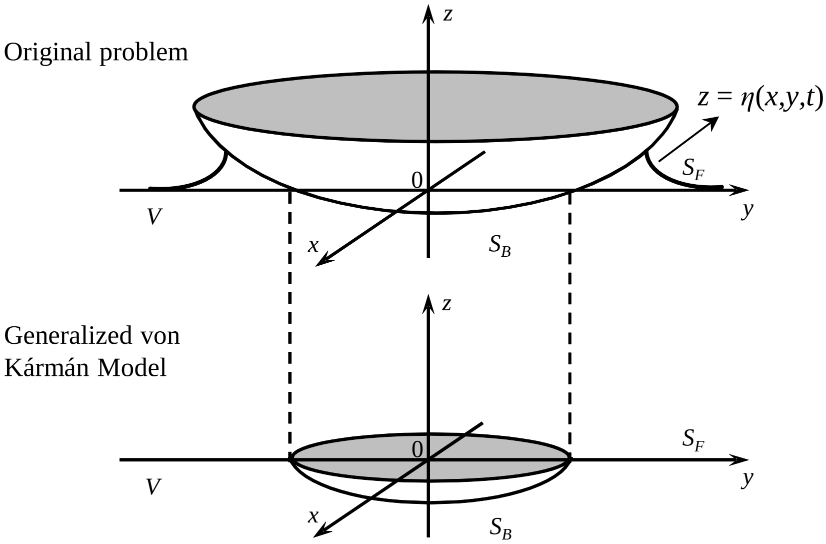

Figure 1. Sketch of the three-dimensional contact area in the Generalized von Kármán Model (GvKM).
Adapted from Santos, *et al*., 2013.

$$\Delta\phi = 0 \text{ in the liquid domain } V, \tag{1}$$

$$(\nabla\phi - \boldsymbol{U}) \cdot \boldsymbol{n} = 0 \text{ on the wetted surface } S_B, \tag{2}$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \text{ on the free surface } S_F, \tag{3}$$

$$\frac{\partial \phi}{\partial t}, \phi = 0 \text{ on the free surface } S_F, \tag{4}$$

$$\phi, \ \eta \to 0 \text{ as } \|\boldsymbol{x}\| \to \infty, \tag{5}$$

$$\phi(x, y, t = 0) = 0, \ \eta(x, y, t = 0) = 0, \ S_B(x, y, t = 0) = 0. \tag{6}$$

where convective terms were neglected in Eqs. (3) and (4). Equation (4) is valid except at the intersection line of the body with the free surface. Therefore, the vertical hydrodynamic impact force is given by

$$F = \int_{S_B} p n_z \mathrm{d}S = -\rho \int_{S_B} \frac{\partial \phi}{\partial t} n_z \mathrm{d}S = -\rho \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_B} \phi n_z \mathrm{d}S, \tag{7}$$

being convective terms accordingly neglected.

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3. NUMERICAL PROCEDURE

The solution of the nonlinear dynamic equation of the impacting motion depends on the determination of the added mass tensor and its derivative with respect to time at each integration time step. The potential problem of hydrodynamic impact, characterized by the dominance of inertial forces, is here formulated by assuming the liquid surface as equipotential, what allows the analogy with the infinity frequency limit in the usual free surface oscillating floating body problem (see Newman, 1978). Thus, within the GvKM, recalling the classical result that the added mass of the impacting body can be written in terms of that corresponding to a double body, its value can be calculated at each instant of time, see Newman (1978) and Fig. 2. The added mass of the double body is associated to the penetration depth (and wetted portion) of the impacting body, as shown in Fig. 2(b),(c).

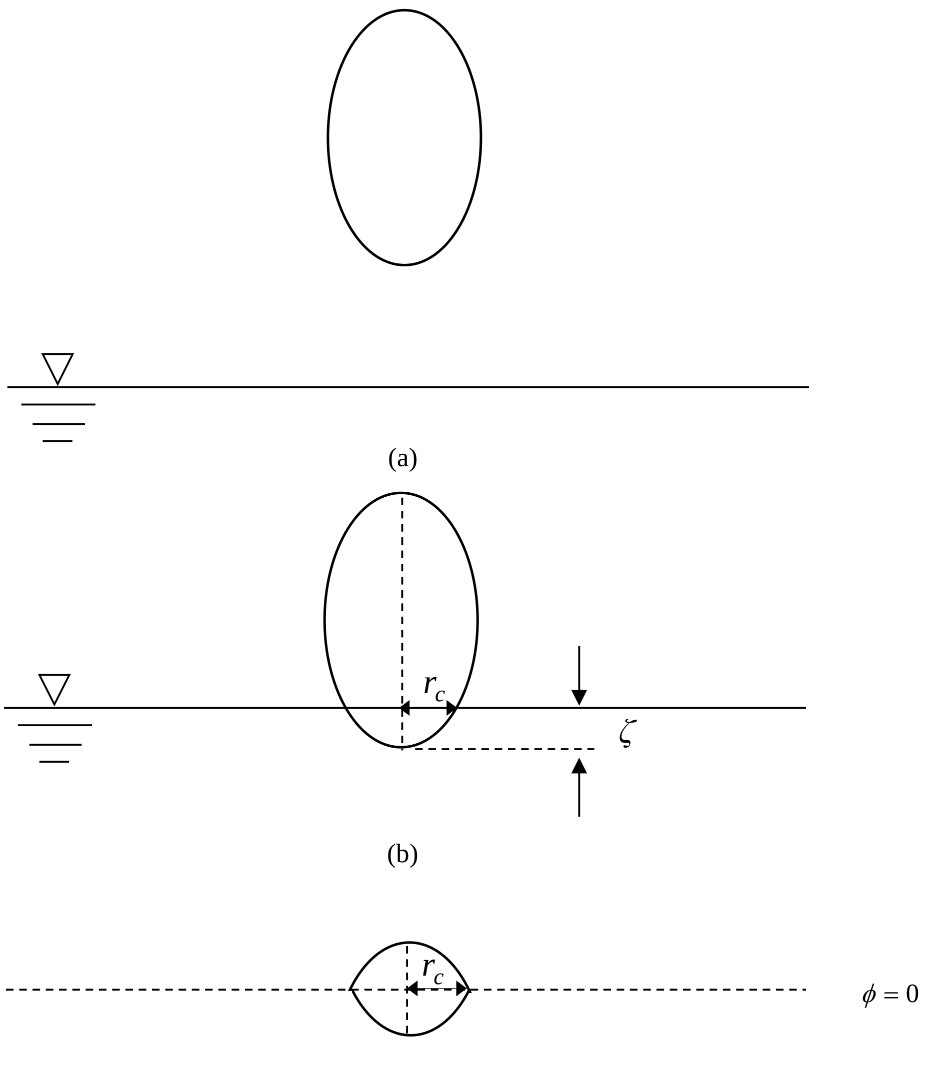

Figure 2. Sketch of the computational strategy: (a) before impact; (b) body impacting the water surface; (c) double body, which is symmetric to the plane $z = 0$.

Pre-calculation and a former interpolation approach was adopted to calculate the added mass, instead of solving the problem at each instant of integration. Thus, for each penetration depth, the added mass coefficient of the double body is calculated by the variational method to potential flows around three-dimensional bodies in unbounded fluid, see Pesce and Simos (2008). This technique leads to a second-order error approximation for the added mass if a first-order error approximation is sought for the velocity potential. This method is an example of desingularized numerical techniques, through which the velocity potential is approximated in a sub-space of finite dimension, formed by trial functions derived from elementary potential solutions, such as poles, dipoles, and vortex rings, which are placed inside the body. Dipoles and rings of dipoles are employed hereafter, as trial functions; see Appendix A. A summary of the variational method presented by Pesce and Simos (2008) is presented in Appendix B, where the complete formulation

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of the variational method can be found. Further details concerning its application to the impact problem are given in Santos, *et al*., 2013.

With the added mass function, interpolated from pre-calculated values for a range of predefined penetration positions ζ^*, the equation of motion for the vertical impact problem can be solved. Through the viewpoint of the Lagrangian formalism, the vertical impact force acting upon a rigid body is given by

$$F_I = -\frac{d}{dt}\left(M_b \dot{\zeta}\right), \tag{8}$$

where M_b is the added mass defined within the bulk of the liquid; i.e., excluding the jets; see Casetta, *et al*., 2011. On the other hand, the force applied to the body is given by

$$F_I = m\frac{d\dot{\zeta}}{dt}, \tag{9}$$

where m is the mass of the body. Equations (9) and (8) lead to the following equation of motion

$$\left(m + M_b\right)\ddot{\zeta} + \frac{dM_b}{d\zeta}\dot{\zeta}^2 = 0. \tag{10}$$

Let the dimensionless time, position, and added mass be defined as

$$t^* = U_0 t / b;\ \ \zeta^* = \zeta / b;\ M_b^* = M_b / m_D, \tag{11}$$

where b is the maximum radius of the impacting body, $\beta = m / m_D$ is the specific mass and m_D is the mass of liquid displaced by the totally immersed body; U_0 is the vertical velocity at the very first instant of impact, $t^* = 0^-$. Equation (10) takes, then, the dimensionless form,

$$\ddot{\zeta}^* + \frac{\frac{dM_b^*}{d\zeta^*}}{\beta + M_b^*}\dot{\zeta}^{*2} = 0. \tag{12}$$

Equation (12), integrated under initial conditions $\zeta^*(0) = 0$; $\dot{\zeta}^*(0) = 1$, leads to the determination of the impact force. Notice, from (11), that velocity is normalized by U_0, acceleration by bU_0^{-2} and force by $m_D^{-1}U_0^{-2}b$. The functions $(M_b^*(\zeta^*); \frac{dM_b^*}{d\zeta^*}(\zeta^*))$ are previously determined or, alternately, might be calculated at each time step.

If the buoyancy force is considered, the equation of motion is readily deducible as

$$\ddot{\zeta}^* + \frac{\frac{dM_b^*}{d\zeta}}{\beta + M_b^*}\dot{\zeta}^{*2} + \frac{gb}{U_0^{\ 2}}\frac{\mu_D^*}{\beta + M_b^*} = 0, \tag{13}$$

with

$$\mu_D^*(\zeta^*) = \frac{\mu_D(\zeta)}{m_D}, \tag{14}$$

where g is the acceleration of gravity and $\mu_D(\zeta)$ the mass of liquid displaced as function of the penetration depth of the body. Let, as in Pesce, *et al*., 2006, an impact Froude number be defined as

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$$F_R = \frac{U_0}{\sqrt{gb}}. \tag{15}$$

Thus, Eq. (13) can be written in the following form,

$$\ddot{\zeta}^* + \frac{\frac{dM_b^*}{d\zeta}}{\beta + M_b^*}\dot{\zeta}^{*2} + \frac{1}{F_R^{\ 2}}\frac{\mu_D^*}{\beta + M_b^*} = 0. \tag{16}$$

The buoyancy effect is then related to the inverse of the impact Froude number squared (usually a very high value), what makes it relatively small if compared to the inertial effects at the very beginning of the water entry problem. This will be illustrated. In fact, the ratio between inertia and buoyancy forces may be shown to be of order $F_I/F_B \approx O(F_R^{\ 2}) >> 1$; Pesce, *et al.*, 2006. If, e.g., the impacting body is dropped (in vacuum) from the rest to the free surface, from a height H, such that $U_0 = \sqrt{2gH}$, the impact Froude number can be simply calculated as

$$F_R = \sqrt{\frac{2H}{b}}, \tag{17}$$

the square of which may take high values.

4. RESULTS

This section aims at illustrating further applications of the variational method to the vertical impact problem. The three-dimensional impact of a prolate ellipsoid is here taken. The application of the variational method to the vertical impact of an oblate ellipsoid was presented in a previous work in Santos, *et al.*, 2013. The results were separated into two sections: the calculation of the added mass coefficient through the variational method and the integration of the equation of motion during the vertical impact of the body. Numerical results were obtained by routines programmed in Matlab®.

4.1 Added mass results for a prolate ellipsoid

The equation of the impacting prolate ellipsoid of revolution, centered at the origin of a Cartesian coordinate system is given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \tag{18}$$

where a is the horizontal and b the vertical semi-diameters, respectively, and $b > a$.

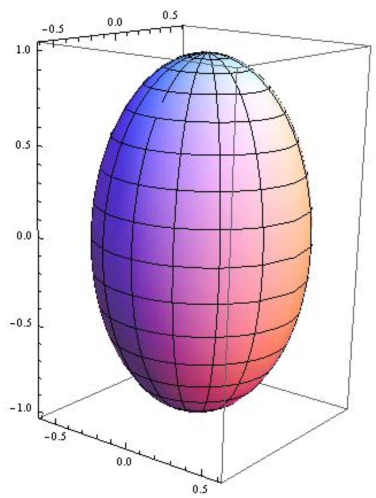

Figure 3. Prolate ellipsoid $\left(b/a = 1/0.6 \cong 1.67\right)$, being a the semi-diameter and b on the axis of revolution z.

Figure 3 shows geometric details of the body. For this geometry, the mass of liquid displaced as function of the penetration depth of the body (see Eqs. (13) and (14)) is given by

$$\mu_D(\zeta) = \frac{\rho \pi a^2}{3b^2}\zeta^2\left(3b - \zeta\right). \tag{19}$$

Predefined penetration depths are considered, with $0 < \zeta^* \leq 0.5$ and $\zeta^* = \zeta / b$. Although the impact force reaches its maximum value at the initial stage, i.e., at small penetration depths, moderate values for ζ^* were also considered, for both, completeness, and to illustrate the numerical convergence process as function of the body "roundness".

The choice of the trial functions is based on the physics of the flow pattern around the double body. Since the elementary solution of a dipole leads to the flow due to a sphere, the systematic procedure used in this work is to include a single vertical dipole at the origin in order to emulate the flow around an 'inscribed sphere' with radius equal to the penetration depth, ζ, and dipoles placed along circular rings – or, simply, rings of dipoles - to represent the whole double body surface. All trial functions are placed at the symmetry plane $z = 0$. The set of N_{TF} trial functions is composed by a single vertical dipole and a number of $\left(N_{TF} - 1\right)$ circular concentric discrete rings of dipoles; see Fig. 4 for details. Note that continuous rings of dipoles could be also employed or even higher order elementary singularity lines. The vertical dipole is positioned at the origin and the rings of dipoles placed around the symmetry axis. The radii of the rings of dipoles, R_j, $1 \leq j \leq \left(N_{TF} - 1\right)$, is given by $R_j = \left[0.045 + \left(j-1\right)\Delta R\right] r_c$, where $\Delta R\left(N_{TF} - 2\right) = 0.93$ and r_c is the double body radius, which is associated with the considered penetration depth, see Fig. 2(b),(c). Therefore, the radius of the rings ranges from $4.5\%\, r_c$ to $97.5\%\, r_c$.

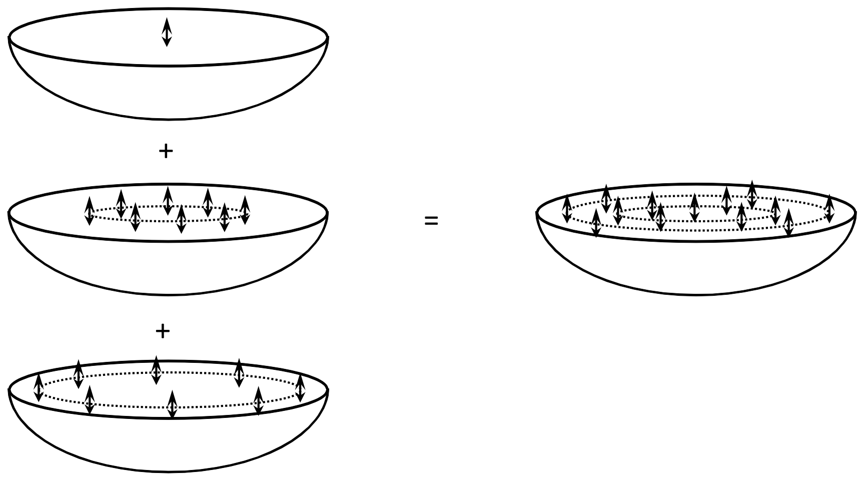

Figure 4. Sketch of the placement of the trial functions at the plane $z = 0$. The symbol $\updownarrow$ represents a single vertical dipole and the circular ring of discrete dipoles is symbolized by the dotted line.

Obviously, a systematic inclusion of rings of discrete dipoles can improve the results because the emulated flow approximates that one around the exact double body surface. This can be seen in Fig. 5, which illustrates a convergence study for the dimensionless added mass, $M_b^* = \dfrac{M_b}{\frac{4}{3}\rho\pi a^2 b}$, and the weak solution boundary condition error, ε_{bc}, as a function of N_{TF}. The ε_{bc} parameter is given by

$$\varepsilon_{bc} = \left| \frac{\int_{S_B} \nabla\tilde{\phi} \cdot \mathbf{n}\,\mathrm{d}S - \int_{S_B} \mathbf{U} \cdot \mathbf{n}\,\mathrm{d}S}{\int_{S_B} \mathbf{U} \cdot \mathbf{n}\,\mathrm{d}S} \right|, \tag{20}$$

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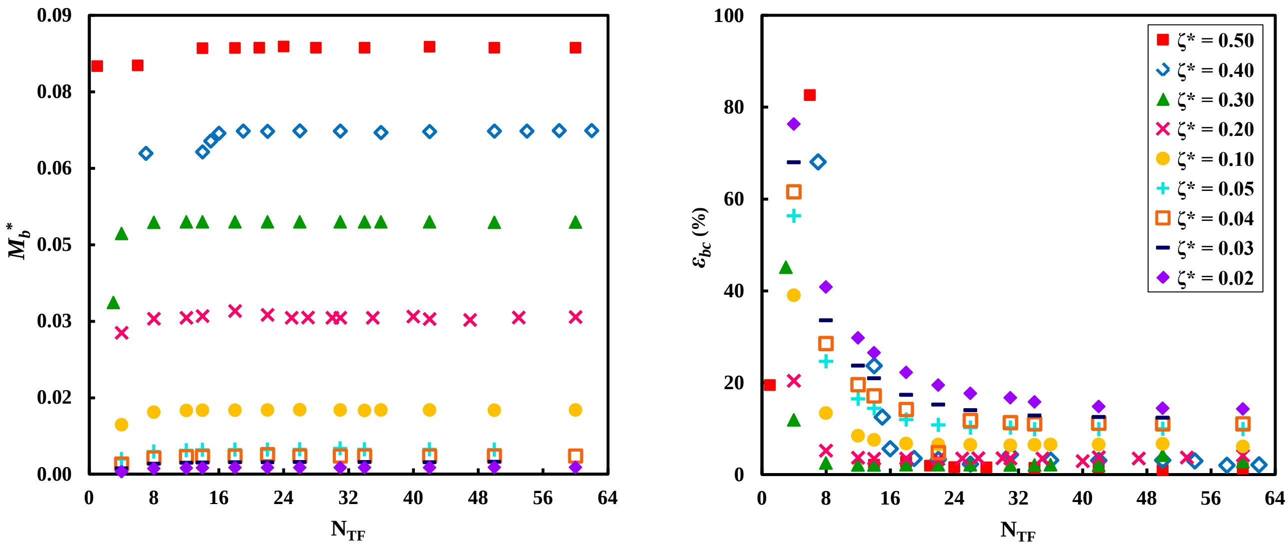
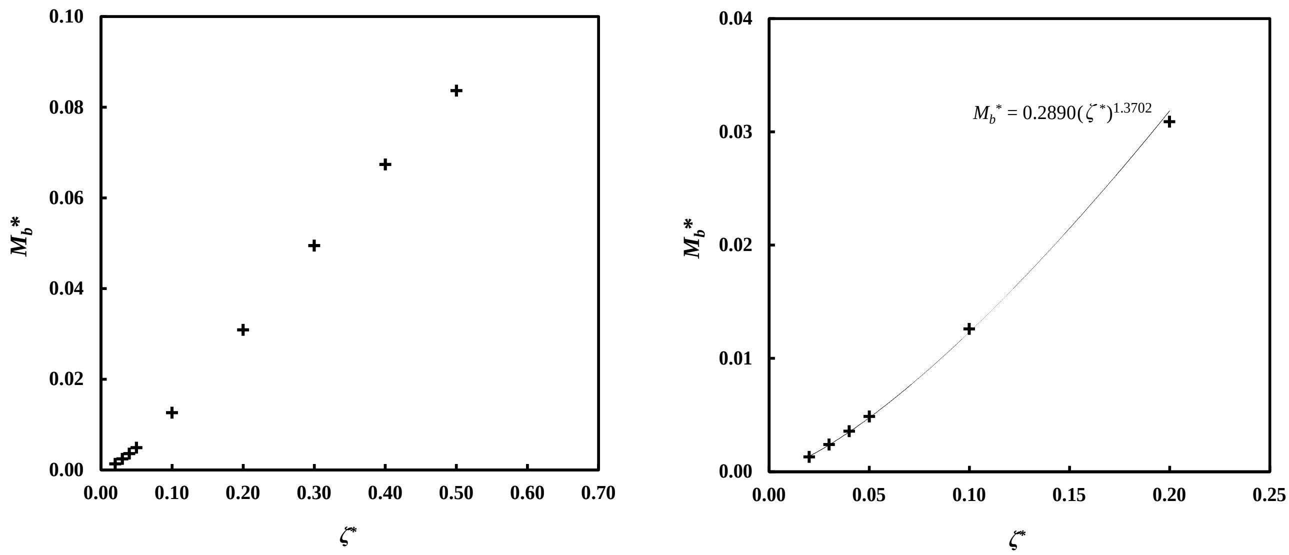
where $\tilde{\phi}$ is the numerical solution obtained with the variational method. This can be viewed as the error of the weak solution (obtained through the variational method) regarding the satisfaction of the boundary condition on the body surface; see Santos, *et al*., 2013.



(a) (b)

Figure 5. Numerical results and convergence study for a prolate ellipsoid $(b/a = 1/0.6 \cong 1.67)$, as function of N_{TF} (the number of trial functions) (a) dimensionless added mass; (b) weak solution boundary condition error. The legend in (b) is also used in figure (a).



(a) (b)

Figure 6. Dimensionless added mass for a prolate ellipsoid $(b/a = 1/0.6 \cong 1.67)$ as function of the dimensionless penetration depth, ζ^*: (a) full penetration range; (b) small penetration range and power fitting.

Figure 5(a) shows that the convergence is really fast. For all penetration depths, few trial functions are necessary for a satisfactory convergence to the added mass values. Moreover, a systematic inclusion of inner equi-spaced rings of dipoles improves the result by reducing the error ε_{bc}; see Fig. 5(b). Recall that the error in the added mass is of order $O(\varepsilon_{bc}^2)$; Pesce and Simos (2008). The errors in the converged values of added mass are then less than 2,5%. Optimal values for ε_{bc} are obtained (Fig. 5(b)) and the corresponding added mass results are presented as a function of ζ^* in Fig. 6(a). A power fitting for $M_b^*(\zeta^*)$ is then determined, for small penetration depths, as shown in Fig. 6(b), from which the derivative $\frac{dM_b^*}{d\zeta^*}(\zeta^*)$ may be promptly obtained. The fitted power functions $(M_b^*(\zeta^*); \frac{dM_b^*}{d\zeta^*}(\zeta^*))$ are then used during the integration of the equation of motion, Eq.(12), at each time step.

4.2 Vertical impact force

With the functions $M_b^*(\zeta^*)$ and $\frac{dM_b^*}{d\zeta^*}(\zeta^*)$ determined, Eq. (12) is integrated under initial conditions $\zeta^*(0) = 0$ and $\dot{\zeta}^*(0) = 1$. Figure 7 shows position, velocity and acceleration for different values of specific mass, β. Notice that, *if no buoyancy effect is considered*, nondimensional acceleration and impact force are equal to each other, i.e., $F_I^* = m_D^{-1}U_0^{-2}bF_I = \ddot{\zeta}^*$.

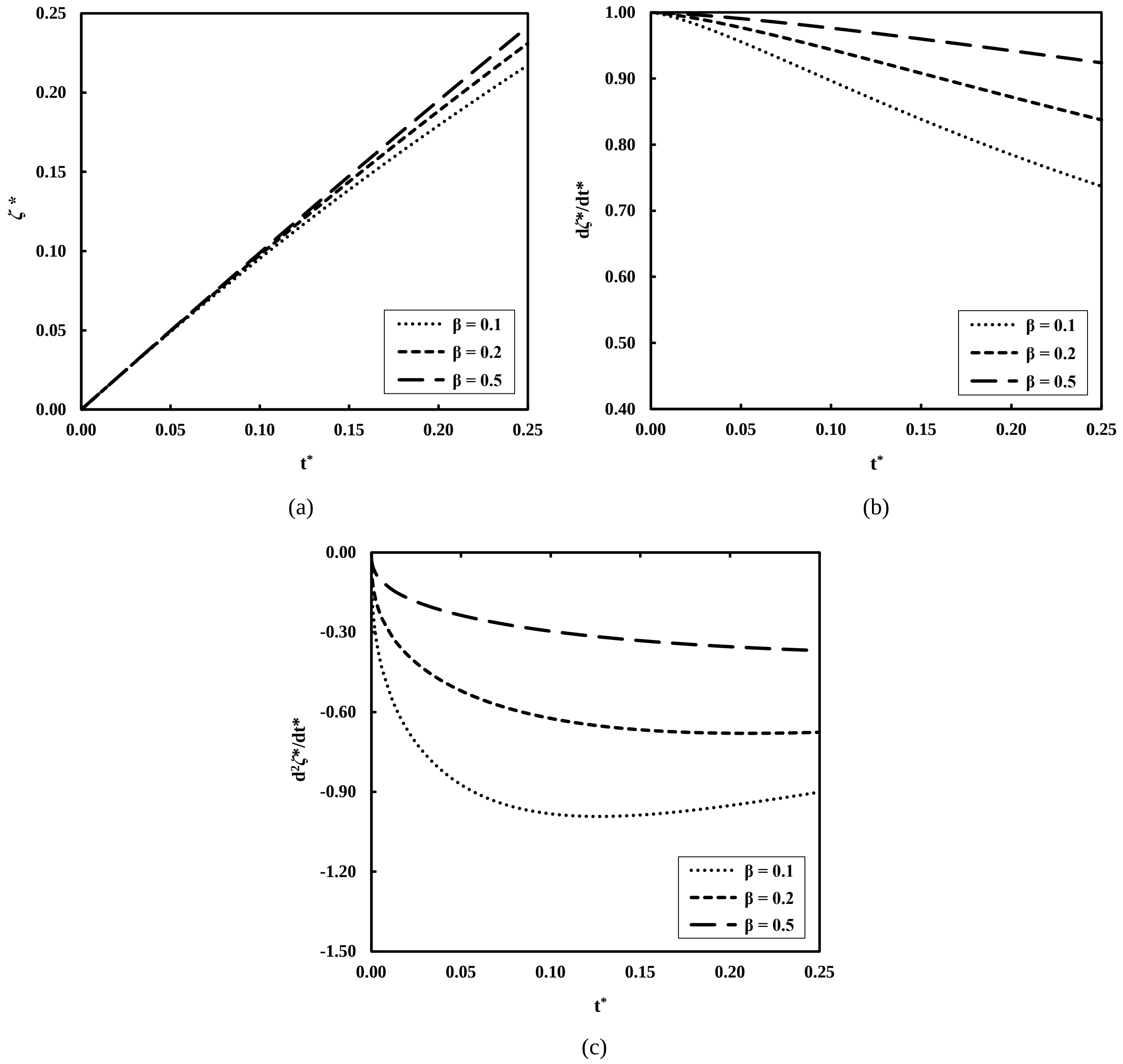


Figure 7. Dimensionless penetration, velocity and acceleration for an impacting prolate ellipsoid $(b/a = 1/0.6 \cong 1.67)$ vs. dimensionless time, for three values of specific mass. Equation of motion without buoyancy force, Eq. (12).

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The buoyancy force effect is taken into account in Fig. 9. The prolate ellipsoid is supposed to be dropped from rest, from a height equal to the vertical (major) semi-axis (H/b=1), as depictured in Fig. 8. As anticipated, the buoyancy effect is indeed small, even for a relatively small dropping height.

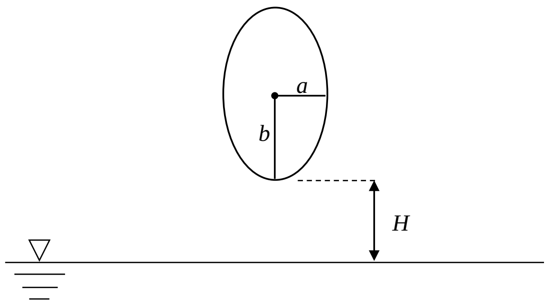

Figure 8. The prolate ellipsoid dropped from the rest.

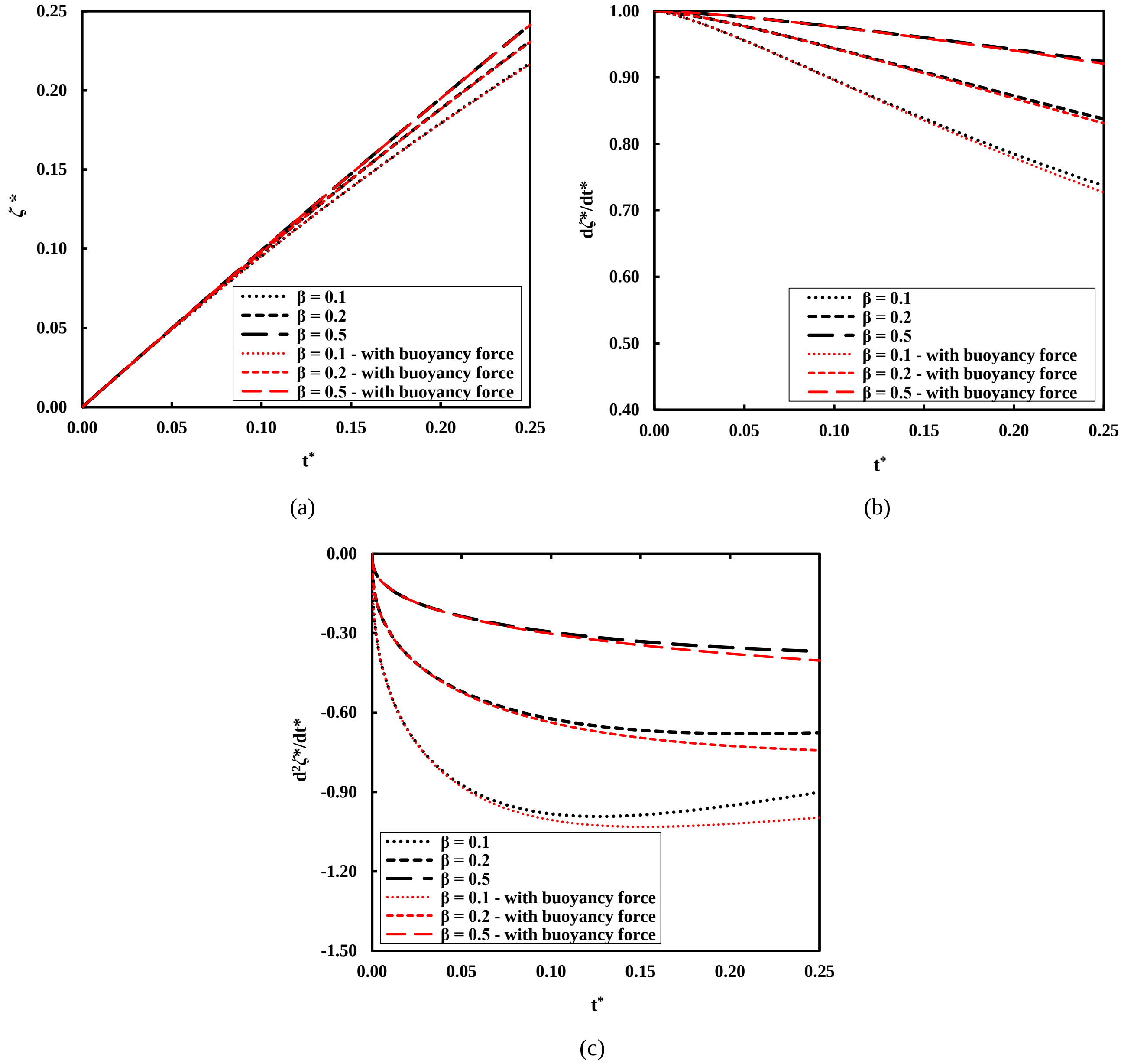

Figure 9. Dimensionless penetration, velocity and acceleration of an impacting prolate ellipsoid $(b/a = 1/0.6 \cong 1.67)$ vs. dimensionless time, for three values of specific mass. Equation of motion with and without buoyancy force; Eqs. (**13**) and (12), respectively.

5. CONCLUSIONS

This paper presented new results of the application of a variational method to a vertically impacting prolate ellipsoid, formulated under the so-called Generalized von Kármán Model (GvKM). Following a previous work by the authors, Santos, *et al*., 2013, and using rings of discrete dipoles as trial functions, the added mass of the body was determined, after a numerical convergence study, and the equations of motions integrated. Illustrative examples were given and the relative effect of buoyancy forces addressed. Further studies might include a comprehensive calculation of a whole family of ellipsoids and improvements regarding the proper consideration of the so-called wet-surface correction.

6. ACKNOWLEDGEMENTS

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8. RESPONSIBILITY NOTICE

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APPENDIX A. TRIAL FUNCTIONS

In the present study, dipole and rings of dipoles are employed as trial functions. In polar cylindrical coordinates, they can be respectively written as

Dipole (see Lamb, 1932)

$$\phi(r, z; a) = \frac{1}{2}\left(\frac{a^2}{r^2 + z^2}\right)^{3/2} z \,. \tag{A.1}$$

Rings of dipoles

The velocity potential of the i-th dipole displaced (in r') at the plane $z = 0$ with respect to the origin of the coordinate system, see Fig. 10 , is given by

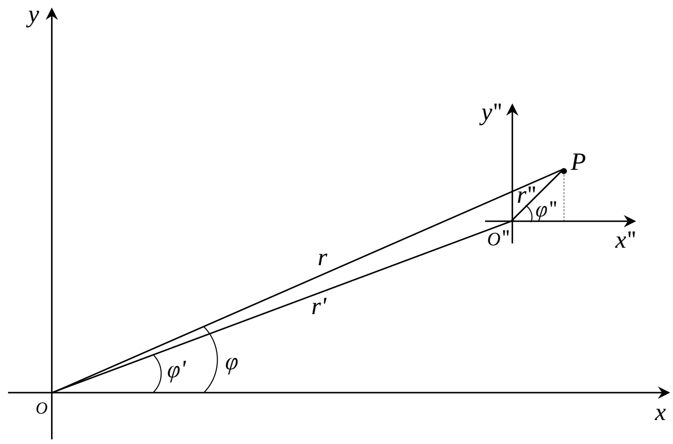

Figure 10. Sketch of the displaced dipole at the plane $z = 0$. Adapted from Santos, *et al*., 2013.

$$\phi_i\left(r_i'', z''; a\right) = \frac{1}{2}\left(\frac{a^2}{r_i''^2 + z''^2}\right)^{3/2} z'' \,, \tag{A.2}$$

and the potential of the ring of dipoles (with radius r', see Fig. 10) can be written as a sum of dipoles displaced with respect to the origin (see Eq. (A.2)). This leads to

$$\phi\left(r'', z''; a\right) = \sum_{i=1}^{n_d} \phi_i\left(r_i'', z''; a\right), \tag{A.3}$$

where n_d is the number of dipoles that represents the discrete ring. In Eqs. (A.2) and (A.3), $r_i''^2 = r^2 + r'^2 - 2rr'\cos\left(\varphi - \varphi_i'\right)$ and $z'' \equiv z$. The rings of discrete dipoles which are used in this work are positioned at the plane $z = 0$, with radius r' and $\varphi_i' = \dfrac{2\pi\left(i-1\right)}{n_d}$.

APPENDIX B. FORMULATION OF THE VARIATIONAL METHOD

See Pesce and Simos (2008), for details. Let $\tilde{\phi}(\mathbf{r})$ be a numerical approximation of $\phi(\mathbf{r})$ and $\left\{T_j(\mathbf{r});\, j = 1,..., N\right\}$ a linearly independent set of trial functions which satisfy the Laplace equation and the proper evanescence condition. Writing

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$$\tilde{\phi}(\mathbf{r}) = \sum_{j=1}^{N} q_j T_j(\mathbf{r}) \,, \tag{B.1}$$

and from Eqs. (1)-(6), the weak equation is presented as

$$G(\phi,\psi) = V(\psi) \;;\;\; \forall\, \psi \in W_2^{(1)}(\mathrm{V}) \,, \tag{B.2}$$

where

$$\begin{aligned} V(\psi) &= \int_{S_B} \psi U_n \mathrm{d}S \\ G(\phi,\psi) &= \int_{S_B} \nabla\phi \cdot \mathbf{n}\psi \, \mathrm{d}S \end{aligned} \,. \tag{B.3}$$

The weak equation corresponds to a linear algebraic system in the unknown coefficients $\{q_j;\, j = 1,...,N\}$, i.e.

$$\begin{aligned} &\mathbf{Gq} = \mathbf{V} \\ &\mathbf{G} = \left[G(T_i, T_j)\right] \\ &\mathbf{q} = \left\{q_j\right\} \\ &\mathbf{V} = \left\{V(T_i)\right\} \end{aligned} \,. \tag{B.4}$$