



COMPARATIVE STUDY OF WENO AND SPECTRAL VOLUME METHODS FOR COMPRESSIBLE FLOWS

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Abstract. *A comparative study of two numerical formulations for high-order reconstruction on unstructured grids is performed. The Weighted Essentially Non-Oscillatory (WENO) and the Spectral Volume (SV) methods are considered for the spatial discretization of the 2-D Euler equations. Solution discontinuities pose a major challenge for research on high-order methods since reconstruction requires special treatment to preserve high order of accuracy and high resolution. The test cases include problems with strong shock waves and other discontinuities which provide a comparative assessment of the resolution capability of the tested schemes for typical aerospace flows. A novel formulation is presented that uses both the WENO and SV discretization methods for reconstruction, depending on the flow characteristics, to provide a continuous high-order solution field, regardless of discontinuities. The results obtained show that it is possible to couple these methods and drastically reduce the computational time and improve the overall quality of the solution. Furthermore, the results provide qualitative data on the proposed method and the results of the present effort could provide valuable guidelines for future developments regarding high-order methods for unstructured meshes.*

Keywords: CFD, high-order, Spectral Volume, WENO

1. INTRODUCTION

High-order reconstruction methods capable of handling unstructured grids are highly sought in computational fluid dynamics since solving several real world problems requires mesh discretizations over complex geometries. In recent years, several efforts have been made by the CFD group of Instituto de Aeronáutica e Espaço (DCTA/IAE) for the development of high-order unstructured grid schemes, such as the Weighted Essentially Non-Oscillatory (WENO) and Spectral Volume (SV) methods, for the solution of aerodynamic flows (Wolf and Azevedo, 2007).

The computational solver uses the 2-D Euler formulation in a cell centered finite volume context for unstructured meshes, with an implicit scheme for time integration. The CFD tools developed in the group and used in the present work have been verified and validated previously (Breviglieri *et al.*, 2010). Furthermore, detailed analyses of the actual order of accuracy for the various schemes can also be found in the cited references.

In the current paper, literature test cases are addressed to provide data regarding computational efficiency, accuracy and flow feature resolution of the SV and WENO methods. Among these are the unsteady supersonic internal flow, known as the forward-facing step problem (Woodward and Colella, 1984) and the RAE 2822 transonic airfoil simulation. The comparative simulations of the SV and WENO methods are third-order accurate. The current research is focused on the shock-capturing and high-order accuracy capabilities of the discretization schemes.

Furthermore, a hybrid method, comprising the SV reconstruction on smooth regions and WENO reconstruction on discontinuous regions is derived in this research as a proposition to improve the quality of the solution, compared with regular limiter formulations, and reduce the computational time required by the WENO method. In other words, the WENO reconstruction is used in regions in which the SV method would require a limiter. Hence, no limiter is actually implemented in the SV formulation, but the hybrid methodology switches to a WENO scheme in the regions in which the limiter was supposed to be active. A comparative analysis of the results for the flow cases addressed in this paper should be helpful in devising guidelines for future developments regarding high-order methods for unstructured meshes.

2. Spatial Discretization

The SV and WENO method formulation used in this research follows the work in [Wolf and Azevedo \(2007\)](#) and [Breviglieri *et al.* \(2010\)](#) for a two-dimensional unstructured framework. Both schemes use discontinuous polynomials as basis function to achieve high-order reconstruction, and, similar to second or higher order finite volume methods, it reduces to first-order in the worst case. A Riemann solver is used to compute numerical fluxes at cells interfaces, which provides the numerical dissipation and physical propagation characteristics to the methods.

This section presents the hybrid approach to perform limited reconstruction for the high-order method considered in this research. In the context of the Euler formulation, it is necessary to limit some reconstructed properties at flux integration points in order to maintain stability and convergence of the simulation, if the flow solution contains discontinuities. The present limiting technique involves two stages. First, the solver must find out and mark “troubled cells” which are, in the second stage, limited. The limiter technique here proposed employs the 3rd-order SV method globally on the mesh and the 3rd-order WENO reconstruction procedure to the troubled cells. It is expected that, across discontinuities, the WENO method would be able to find a smooth stencil for polynomial reconstruction, which would retain the high-order aspect of the reconstruction.

Several markers, or sensors, were developed and employed for unstructured meshes over the past decades. For an in-depth review, the interested reader is referred to [Qiu and Shu \(2006\)](#). The limiter marker used in the present work is termed Accuracy-Preserving TVD (AP-TVD) marker. Once the mesh element is confirmed as a troubled cell, its polynomial, can no longer be used in any flux integration point, because the property function is no longer smooth within such element. Hence, it is of utmost importance to limit as few elements as possible. To that end, the marker is designed to check for the flux integration points in each cell and mark those that do not satisfy the monotonicity criterion. The polynomial of a troubled cell is, then, replaced by a new polynomial, obtained with the WENO method. It should be pointed out that, for the quadratic reconstruction, the WENO method must cope with different cell topology, resulting from the SV method cells partitioning procedure. Moreover, the underlying program logic and data structure have to be carefully designed to account for the hybrid nature of the approach. The hybrid method must reduce the computational time while maintaining high-order accuracy and lower resource usage. Therefore, a large amount of time has been spent on fine-tuning the solver structure to account for both schemes operating at once.

For the quadratic reconstruction, the limiting process can be summarized in the following stages:

1. For a given spectral volume, SV_i , compute the minimum and maximum cell averages using a local stencil which includes its node neighbors, *i.e.*,

$$\begin{aligned} Q_{min,i} &= \min \left(Q_i, \min_{1 \leq r \leq nb} Q_r \right) \\ Q_{max,i} &= \max \left(Q_i, \max_{1 \leq r \leq nb} Q_r \right) \end{aligned} \quad (1)$$

2. The i -th cell is considered as a possible troubled cell if

$$p_i(x_{rq}, y_{rq}) > 1.001 Q_{max,i} \quad \text{or} \quad p_i(x_{rq}, y_{rq}) < 0.999 Q_{min,i}. \quad (2)$$

The 1.001 and 0.999 constants are not problem dependent. They are simply used to overcome machine error, when comparing two real numbers, and to avoid the trivial case of when the solution is constant in the neighborhood of the spectral volume considered.

3. Recompute the properties at the quadrature points of the i -th cell with the WENO polynomial.

The limited reconstruction is based on primitive variables $\{p, u, v, e_i\}^T$. Here, p is the pressure, u and v are the Cartesian velocity components, and e_i is the internal energy. Once these properties are available from the limited reconstruction, the vector of conserved variables is easily obtained to resume the numerical flux integration. Such formulation does not require any input from the user and is not derived for a particular problem.

3. RESULTS

The results presented here attempt to verify the stability and resolution of the high-order methods for aeronautical applications. For the presented results, density is made dimensionless with respect to the free stream condition, $\rho = \frac{\rho}{\rho_\infty}$, and pressure is made dimensionless with respect to the density times the speed of sound squared, $p = \frac{p}{\rho_\infty c_\infty^2}$. Moreover, the L_{max} norm of the residue for the mass equation is used for the convergence history plots. All test cases are executed on an Intel Xeon 2.6 GHz 64-bit Linux workstation. The solver is serial, Fortran 90 code with dynamic memory allocation compiled with optimized settings for the Intel compiler. The results are computed with the implicit matrix-free method with a CFL value of 1×10^6 , unless otherwise stated.

3.1 RAE 2822 Airfoil

The transonic flow over a RAE 2822 airfoil with 2.31 deg. angle-of-attack and free stream Mach number, $M_\infty = 0.729$, is simulated with the 2nd-order MUSCL scheme and 3rd-order SV with different limiter formulations. The minmod, hierarchical moment limiter (Breviglieri *et al.*, 2010) and WENO formulations are considered in this test case. The computational mesh has 5378 cells, of which 132 elements lie on the airfoil surface, as necessary to properly describe the airfoil geometry, especially on the leading edge. This mesh is shown in Fig. 1.

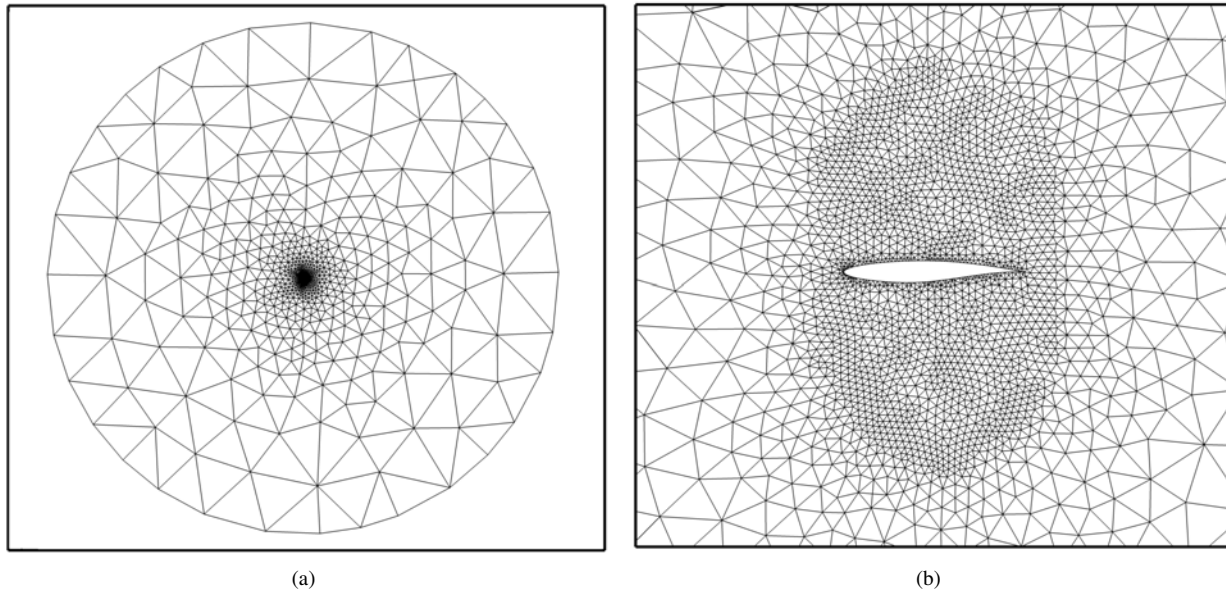


Figure 1. Computational domain and RAE 2822 airfoil mesh detail.

Figure 2 shows Mach contours around the airfoil for the various methods considered. The figures share the same color map. The hybrid scheme presents a sharp shock resolution. However, when one considers the residual history in Fig. 3, the oscillatory behavior of the WENO method is observed for the hybrid scheme. The figure presents the residue of the mass equation normalized by the residue of the first iteration. The 3rd-order SV minmod method also halts convergence but provides a C_ℓ value closer to the expected one, around 0.8, than the hybrid scheme. This difference might be due to the fact that, for the minmod simulation, the residual dropped 3 orders of magnitude while the hybrid scheme halts convergence at around half of that value. A similar behavior is observed for the hierarchical limiter.

The C_p distribution is presented in Fig. 4 for the various numerical methods, along with experimental data. Clearly, the experimental data shows a smooth transonic shock, due to shock wave–boundary layer interaction, which is an important phenomenon for such supercritical airfoils. The present computations cannot reproduce such physics, since the Euler equations are used. The hybrid method presents similar results compared to the hierarchical limiter formulation, regarding shock position and strength, as well as the undershoot, although it is closer to the experimental data in the suction region. With regard to the limiter formulation of the hybrid method, it is active only in the discontinuity region, as shown in Fig. 5. The minmod limiter is active all around the region of interest, while, in the hybrid scheme, the active limiter is restricted to the shock region. Hence, the minmod limiter renders the solution 1st-order accurate at such cells, whereas the hybrid scheme, by using the WENO reconstruction, should be able to retain the high-order accuracy of the original method.

The benchmark data is presented in Table 1. The time column displays the required wall-clock time to C_ℓ convergence, normalized by the hybrid method wall-clock time, at around 6000 iterations. The memory column shows the required memory for each method normalized by the most expensive scheme, namely the 3rd-order hybrid SV method. The authors emphasize that these 6000 iterations actually took around 50 wall-clock minutes in the reference computer described earlier. One can observe, in Fig. 3, that the C_ℓ curve for the hybrid test case shows a convergent behavior. The MUSCL scheme is extremely cheap compared to the hybrid method. However, one must consider that it is, at most, second-order accurate. Given the fact that the MUSCL scheme is commonly employed in industry, such benchmark provides insights into the high-order schemes overall accuracy and computational cost. The SV with minmod limiter is also cheaper than the hybrid approach with nearly the same memory usage, as well as the hierarchical limiter.

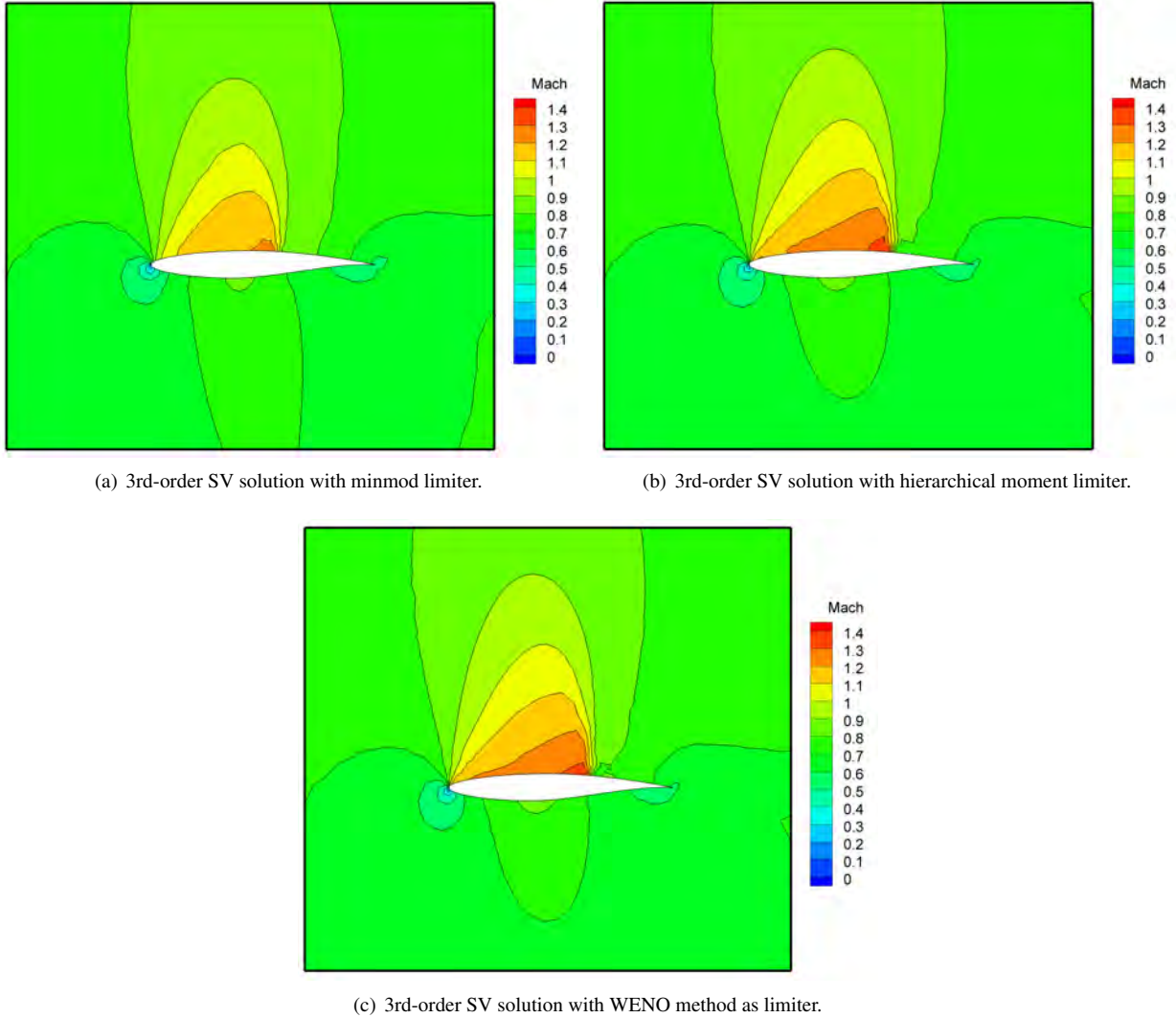


Figure 2. Mach contours for RAE 2822 airfoil at $M_\infty = 0.729$ and $\alpha = 2.31$ deg.

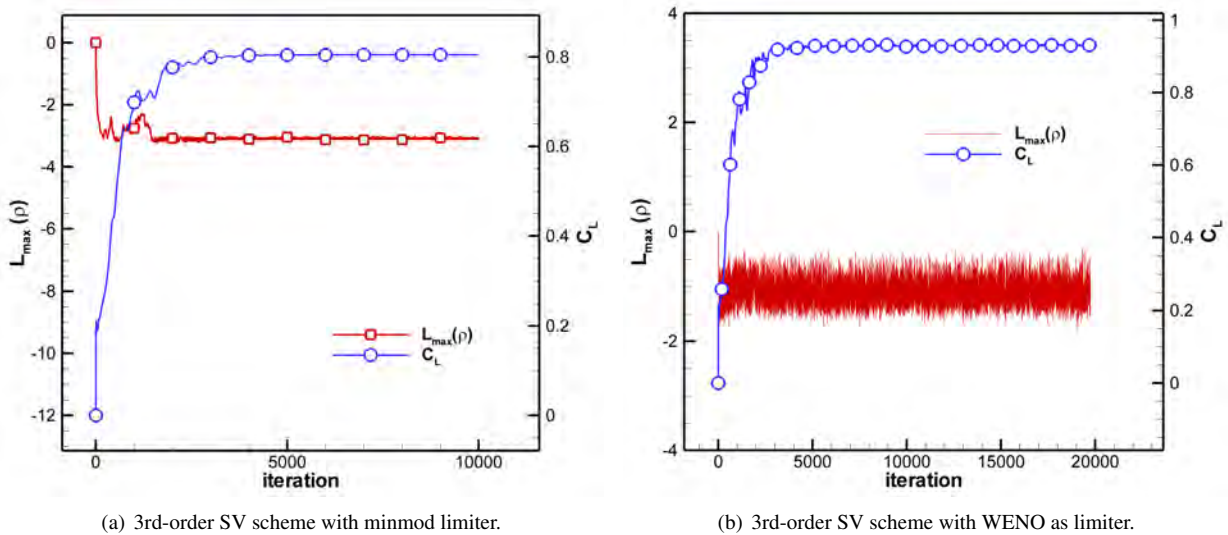


Figure 3. Residue history for the RAE 2822 airfoil simulations at $M_\infty = 0.729$ and $\alpha = 2.31$ deg.

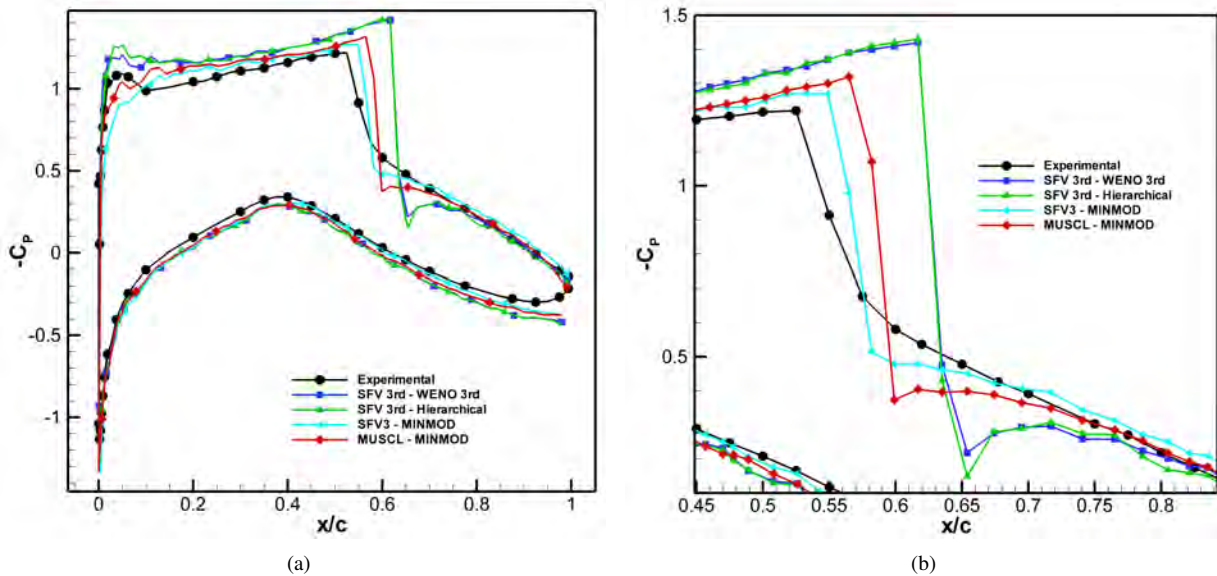
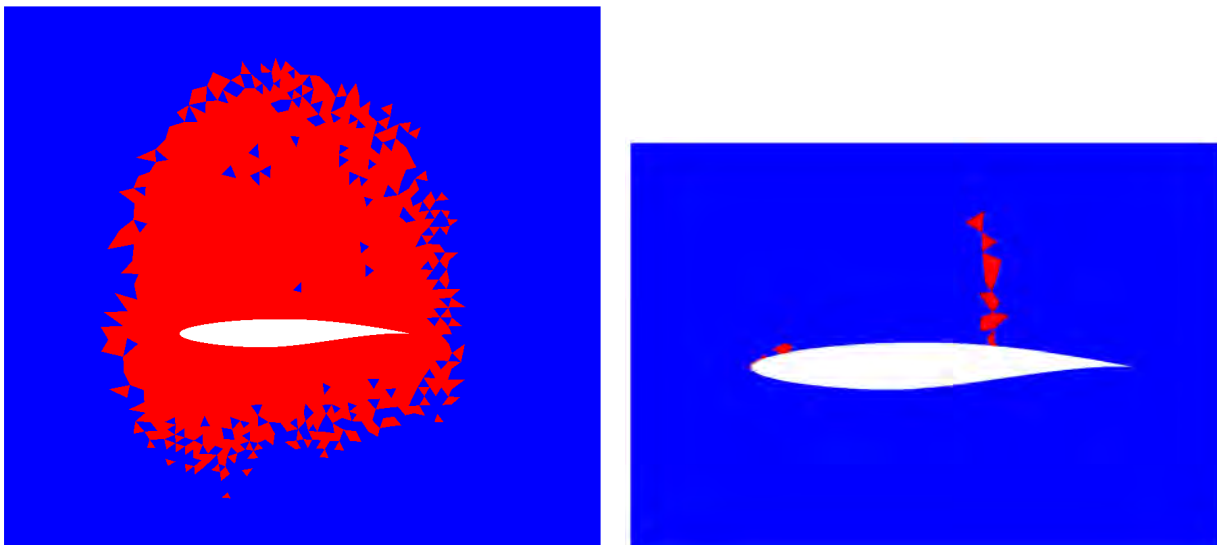


Figure 4. Experimental and numerical C_p with detailed shock region for the RAE 2822 airfoil at $M_\infty = 0.729$ and $\alpha = 2.31$ deg.



(a) 3rd-order SV troubled cells in red for the minmod limiter.

(b) 3rd-order SV troubled cells in red with WENO as limiter.

Figure 5. Limited cells for pressure reconstruction for the RAE 2822 airfoil at $M_\infty = 0.729$ and $\alpha = 2.31$ deg. Blue contour represent unlimited reconstruction.

Table 1. RAE 2822 Airfoil Benchmark.

Method	Order	Limiter	time	memory
MUSCL	2	minmod	0.10	0.58
SV	3	minmod	0.22	0.98
SV	3	Hierarchical	0.25	0.98
SV	3	WENO	1.0	1.0

3.2 Forward-Facing Step Problem

The forward-facing step test case is designed to resolve unsteady, complex oblique shock reflections, pertinent to supersonic variable-geometry jet engine intakes. Due to computational constraints faced by [Emery \(1968\)](#), the test case was designed to be numerically simple to set up. The initial conditions are uniform throughout the domain, and the inlet boundary condition is supersonic. Such setup is actually difficult to perform experimentally, due to the unrealistic combination of initial and boundary conditions. However, quantitative results are available for a range of numerical methods in [Woodward and Colella \(1984\)](#), which makes this a good test case for validating the capabilities of any flow solver, independently of the numerical method used, and, hence, for the present study.

The artificial nature of the test case setup helps to evaluate the robustness of the spatial discretization algorithm combined with the limiter technique. Due to the strong shock reflection at the lower face of the step during the first few iterations, it is difficult to maintain positivity of pressure and density using various numerical schemes. Furthermore, the edge of the forward step is a singular point of the Prandtl-Meyer expansion fan generated by the flow over the step. The continuity and momentum equations can disobey the second law of thermodynamics through an expansion fan. This introduces numerical difficulty in the form of a non-physical expansion shock, which at high Mach numbers and small cell sizes can yield negative pressures and densities in the solver.

The 2-D configuration is 3 dimensionless length unit long and 1 unit wide, with a step of 0.2 unit high located at 0.6 length units from the configuration inlet. The inflow and outflow boundary conditions are both supersonic, so the solver does not have to account for waves leaving the domain at the entrance boundary, or entering the domain at the exit boundary. An uniform Mach number of 3.0 is set as inflow at $t = 0$ dimensionless time units. The simulation is unsteady and a constant Δt value of 1×10^{-4} was used. The mesh has 4536 triangular cells and is presented in Fig. 6. This simulation considered the 3rd-order SV and WENO methods. Moreover, the SV scheme employed different limiter formulations, namely the minmod and the hybrid SV/WENO approach here proposed.

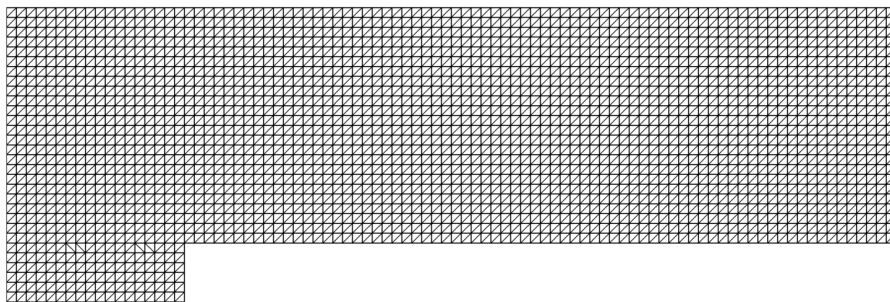
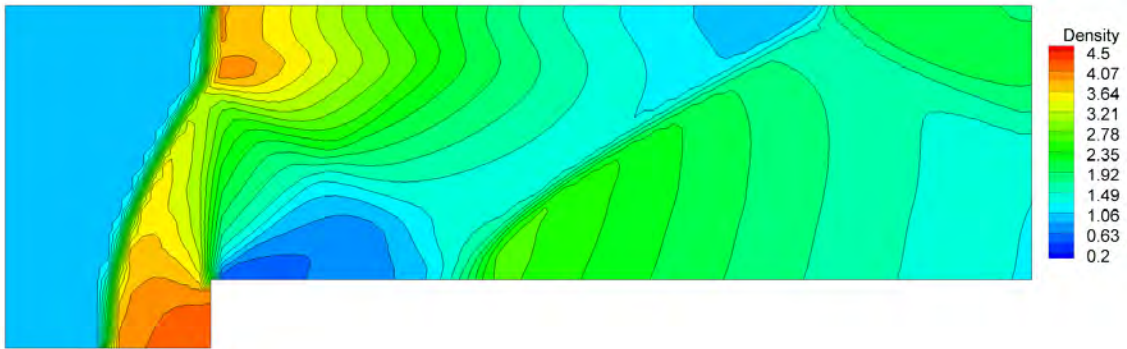


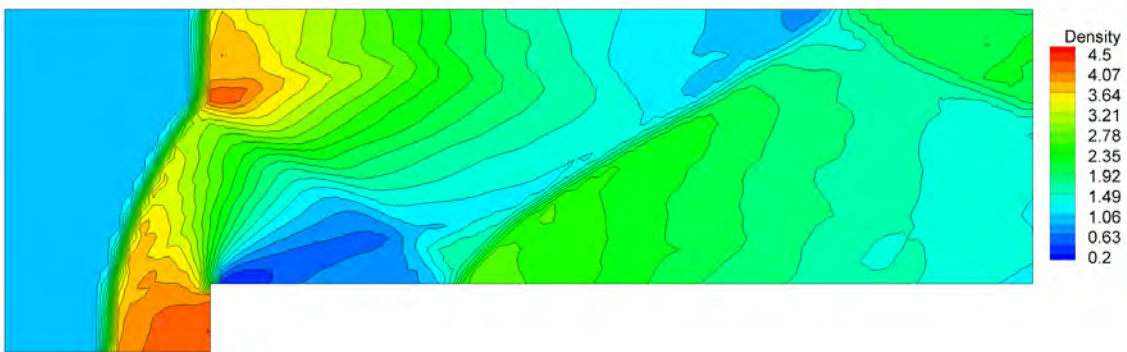
Figure 6. Computational mesh for the forward facing step simulation.

The more interesting structures of the flow develop at time equal to $t = 4.0$ dimensionless time units. At such time instant, a detached shock evolves to a lambda shock that reflects in the upper surface of the channel. A contact discontinuity is created past the lambda shock and both the reflected shock and the contact discontinuity move downstream along the channel. The reflected shock is again reflected at the lower wall as well as near the end of the channel. The contact discontinuity interacts with the shock that reflected in the lower wall. This interaction occurs in the region near the end of the channel, just upstream of the last reflection of the shock. At the corner region, there is also a weak oblique shock wave that ends the expansion region due to a Prandtl-Meyer expansion fan. This weak shock interacts with the first reflected shock near the lower wall of the channel. This leads typically to a flow structure that resembles a non-physical shock-boundary layer interaction in the region where the second shock reflection occurs. The solution in this region is very dependent on the treatment applied to the corner of the step. In the present work, no special treatment is applied to the corner of the step, contrary to the original reference of [Woodward and Colella \(1984\)](#). One can see that the results here presented are similar to those presented in [Abgrall \(1994\)](#) and [Sonar \(1997\)](#), where, as in this work, no special treatment is applied to the corner of the step. The use of a solution limiter formulation is required for the SV method to preserve the positivity of the reconstruction.

Results at time $t = 4.0$ are shown in Figs. 7 and 8, in terms of density and Mach number contours, for the SV scheme. As the reader can observe, the SV method solution with minmod limiter tends to smooth out the shock waves and presents an unrealistic density gradient at the step face region. Moreover, the normal shock profile at the upper wall is disturbed. The hybrid method, on the other hand, does not produce such disturbances in the flow field and it has sharper shock resolutions overall compared to the minmod solution. One important aspect of the hybrid solution is that the first shock reflection after the step face does not build up to a lambda-shock structure, as is common with highly-dissipative schemes, as observed in [Breviglieri and Azevedo \(2012\)](#).

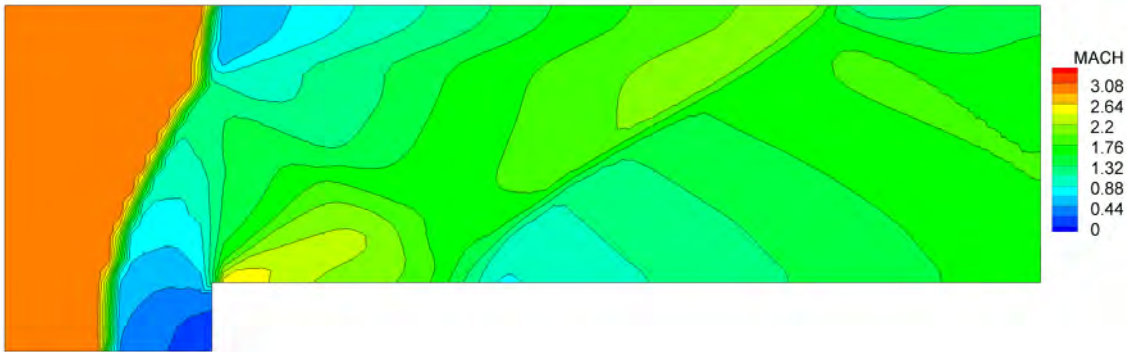


(a) 3rd-order SV solution with minmod limiter.

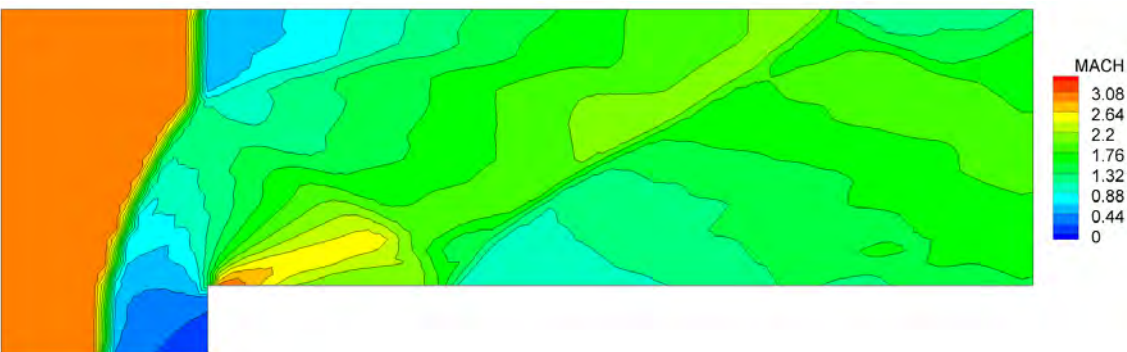


(b) 3rd-order SV solution with WENO method as limiter.

Figure 7. Density contours for the forward facing step problem.



(a) 3rd-order SV solution with minmod limiter.



(b) 3rd-order SV solution with WENO method as limiter.

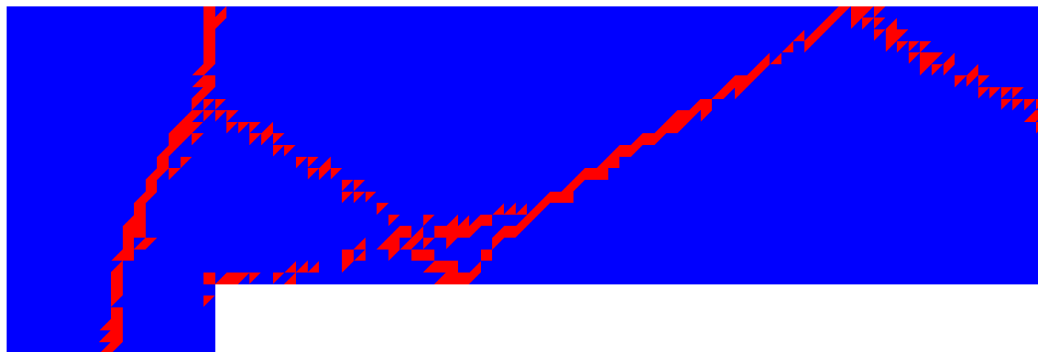
Figure 8. Mach number contours for the forward facing step problem.

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The poor resolution of the minmod limiter can be explained as in the previous airfoil cases that is, most of the cells are reconstructed with 1st-order accuracy, as Fig. 9 indicate. Note that most of the domain is limited for the minmod case, whereas the hybrid scheme restricts the limited reconstruction to those cells in the discontinuity regions. Moreover, even such cells are reconstructed with the WENO scheme, which seeks for the smoothest polynomial away from discontinuities to preserve the high-order aspect of the method.



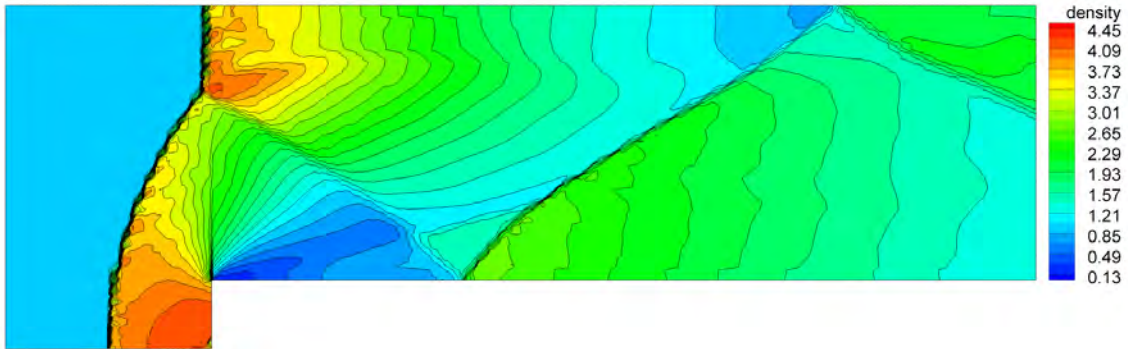
(a) 3rd-order SV troubled cells in red for the minmod limiter.



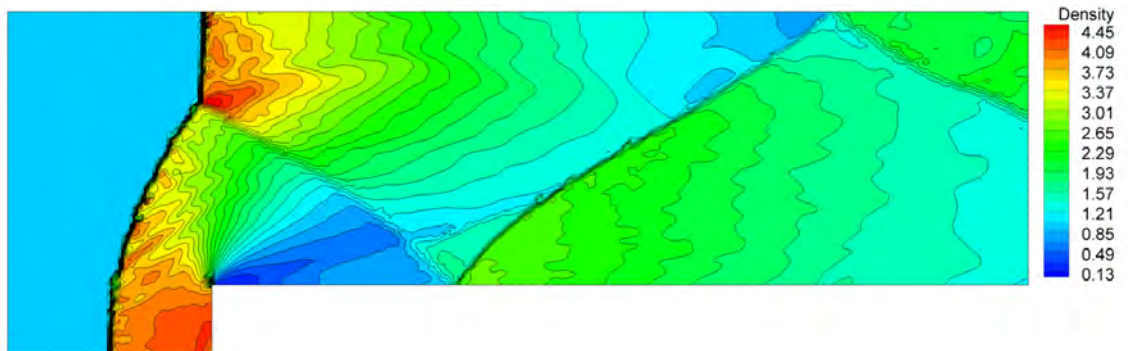
(b) 3rd-order SV troubled cells in red with WENO as limiter.

Figure 9. Limited cells for pressure reconstruction, at the last iteration, for the forward facing step simulation. Blue contour represent unlimited reconstruction.

One often overlooked aspect of high-order methods regards the flow field visualization. The previous figures present the results on the initial spectral volume (SV) mesh, as in Fig. 6. However, to fully exploit the high-order resolution of these schemes, it is important to investigate the inner-cell property distribution. To that end, the authors developed a visualization tool which uses the SV and WENO polynomials to map the high-order solution to a finer-grained mesh built specifically for visualization purposes. Such tool allows one to compare quantitatively the flow property distributions obtained with different methods. The visualization mesh is composed of refined cells obtained with successive triangulation of the original mesh elements. Since the solution polynomials are continuous across its element boundaries, it can be used to obtain the properties values within such limits. For those cells that lie on a limited and discontinuous region, the WENO polynomial is used instead. Figures 10 and 11 present such visualization for density and Mach number contours, now for the 3rd-order WENO and hybrid methods. The figures have the same color map range. One can observe in much more detail the shock resolution and reflection along the domain. The WENO density solution shows a similar behavior to the minmod solution on the step face region. One can also note that the WENO reconstruction produces oscillations right after the shock waves, whereas they tend to propagate further into the domain for the hybrid scheme. Although the solutions are not effectively the same, both schemes are able to capture the relevant physics expected for this problem. The difference lies on the computational effort to obtain such solutions using the WENO and hybrid method. The WENO scheme is over ten times more expensive than the hybrid method. Details are given in Table 2.

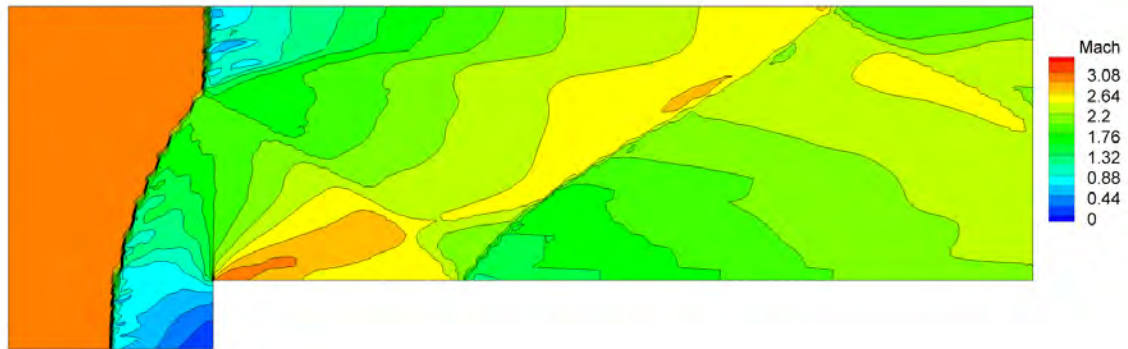


(a) 3rd-order WENO solution.

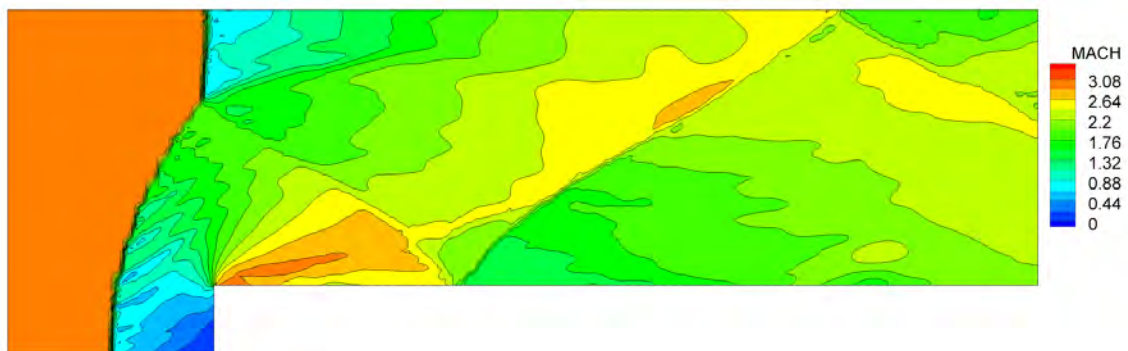


(b) 3rd-order SV solution with WENO as limiter.

Figure 10. Density contours on fine mesh for visualization.



(a) 3rd-order SV WENO solution.



(b) 3rd-order SV solution with WENO as limiter.

Figure 11. Mach number contours on fine mesh for visualization.

The benchmark data is presented in Table 2. The time column displays the required wall-clock time to reach $t = 4.0$ time units of the simulation, again normalized by the hybrid method computational time. The memory column shows the required memory for each method normalized by the most expensive one, which, in this case is also the 3rd-order hybrid SV method. Such computation actually took around 6.7 wall-clock hours in the reference computer described earlier. The MUSCL scheme is extremely cheap compared to the hybrid method. However, as also previously discussed, one must consider that it is at most second-order accurate and, once more, provides a reference to compare with low-order CFD applications typically used in industry. The SV with minmod limiter is also cheaper than the hybrid approach, with nearly the same memory usage, but it severely degrades the quality of the solution. Furthermore, as Fig. 9 indicates, most of the domain is 1st-order accurate due to the order reduction associated with the active limiter. The WENO simulation results, however, have similar quality as that provided by the hybrid method, but at a much higher computational costs. As indicated in Table 2, the computational costs of the WENO calculation is about 10 times larger than that of the hybrid scheme simulation, due to the search procedure of WENO scheme. At each time-step, a reconstruction stencil is built for every cell in the domain. The hybrid method uses the fixed stencil for polynomial reconstruction of the SV for most of the domain, and only on the limited cells, the WENO search algorithm is employed. This also results in a reduced memory requirement, as indicated in the Table below.

Table 2. Forward Facing Step Benchmark.

Method	Order	Limiter	time	memory
MUSCL	2	minmod	0.03	0.64
SV	3	minmod	0.56	1.02
WENO	3	-	10.5	4.57
SV	3	WENO	1.0	1.0

4. CONCLUSIONS

The present research has presented results with the comparison of high-order WENO and SV methods. Considering the calculations performed so far, each method has characteristics that excel over the other scheme. The WENO search algorithms, for instance, represent a real challenge for computational efficiency. On the other hand, WENO methods require no limiter technique for dealing with the strong shocks that appear in supersonic and hypersonic flows. The SV method uses a simplified reconstruction process, but it requires a limiter formulation in order to be stable for the same applications. The limiting process typically yields a reduction in the order of accuracy near discontinuities and considerable cost increases.

A hybrid method is proposed to combine the advantages of both methods into a unified framework that can employ an implicit time march scheme on unstructured meshes. The possibility of high-order uniformity over the domain typical of WENO solutions along with the efficiency and robustness of the SV method is explored in this paper. The results have shown that it is possible to couple these methods and drastically reduce the computational time and improve the overall quality of the solution. Furthermore, the results presented in this paper provide qualitative data on the proposed method and draw directions to further investigate some of its shortcomings. For instance, it is clear from the present results that some work is needed in order to improve the solution smoothness with the hybrid scheme. Moreover, convergence rate improvements, extension to curved boundary elements and formal order measurement studies are tasks that still remain to be performed. In any event, the current results are encouraging, and they indicate that such hybrid approach might be applied to 4th- and possibly higher-order methods.

5. ACKNOWLEDGEMENTS

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