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INFILTRATION IN POROUS MEDIA: A HEURISTIC SOLUTION OF THE RICHARDS EQUATION

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Abstract. *In this paper, we consider a transient one-dimensional flow problem of water in unsaturated porous media, modelled by the non-linear Richards equation. Constitutive relations of Van Genuchten will be employed and a hybrid method of Padé approximants and Adomian decomposition is developed. We show in this study, that the solution obtained by Adomain's procedure did not converge. The divergence was circumvented proposing a heuristic solution in parametrised form for the non-linear Richards equation with the objective to calculate the one-dimensional vertical transient flow. This solution is optimized via least squares and the non-linear Newton method and evaluated using the non-linear Richards equation. The results are compared to the profiles of the numerical matrix potential in the literature. The matrix potential profile generated by the heuristic solution is close to the numerical solution. This solution may define an initialization of a recursion scheme for a modified Adomian method. This scheme will be applied to the non-linear Richards equation.*

Keywords: *transient flow problem, divergence of Adomian's scheme, heuristic non-linear Richards solution*

1. INTRODUCTION

In engineering, knowledge about infiltration and water movement in soil emerges as a preventive measure, both to control the destructive action of water on foundations, dams and pavements, and to predict the behaviour of flow and transport of pollutants. It has also been designed to protect underground aquifers, which are the main funding sources of water for irrigation and production areas. The need for mathematical modelling and simulation are justified by the interest to evaluate the effects of water management, such as soil drainage, irrigation and sub-irrigation plans to water and balance agricultural production. In civil engineering, the estimated seepage through dams and infiltration losses can also be an application for these simulations. The saturation of the soil under decks, the drainage foundations or flooded areas, evaluation of loss in storage reservoirs for soil infiltration, as well as contamination of groundwater by the use of pesticides and toxic material can also be cited as examples of such simulations.

Mathematical modelling of the infiltration process in porous media is substantiated by the equations of Richards, or Fokker-Planck. Both equations are highly non-linear. Thus, analytical solutions to the equations are extremely difficult to find, so that linearisation or numeric schemes are frequently used to get approximate descriptions of the problem. In order it turn prognostics more efficient, it is important to consider field observations, because these are necessary for identification of constitutive relations governing the phenomenon and are used in theoretical formulations for computer simulations. Basically, the best-known models that relate soil parameters (conductivity, matrix potential and soil moisture), are the models found in refs. Brooks and Corey (1964), Genuchten (1980) and Gardner (1958). The Van Genuchten model provides more satisfactory results than others when compared with experimental data, but due to its functional form proposed solutions have limited applicability for this model. On the other hand, the other two models result in simplified equations, leading to a case of linearised governing equation and deriving a series of solutions as in analytical works by Lomen and Warrick (1978), Sander *et al.* (1988), Srivastava and Yeh (1991), Basha (1999), Chen *et al.* (2001) and Basha (2002). However, most of these solutions are limited to cases with uniform initial conditions and infinite domain. Furthermore, they are bound to considerations restricted to information on the diffusivity of water and dependency relationships

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between hydraulic conductivity, soil moisture and matrix potential. Thus, for a more general approach quite often it was necessary to resort to numerical models.

A model discretisation equation, which describes the flow in the unsaturated zone, can be performed numerically by finite difference techniques or finite elements. In these techniques, a scheme of discretisation is applied to a system of nodal points that map the region-time soil depth considered. Implementing the initial and boundary conditions appropriately, leads to a set of algebraic equations that can be solved by various techniques (Feddes *et al.*, 1988). Although numerical methods are powerful tools for resolutions of complex and non-linear equations, they generally do not provide sufficient information about the solution, also suffer problems of numerical errors and conservation of mass (Mannich, 2008).

In this paper, we analyse a problem of one-dimensional and transient flow of water in unsaturated medium, modelled by the Richards equation in its non-linear fashion. To this end, the constitutive relations of Van Genuchten are employed and a hybrid method of Padé approximants and Adomian decomposition is applied. The Adomian method allows in principal analytical representations to obtain solutions of non-linear problems without linearisation or model discretisation. Although the method of Adomian was applied successfully in non-linear problems in the field of infiltration of porous media, some authors define it to solve the Richards equation, but only in its linear form, or restricted by various simplifying assumptions and linearisation resulting in model parameter relations for soils (conductivity, matrix potential and soil moisture), (Serrano, 2004), (Pamuk, 2005), (Nasseri *et al.*, 2008).

In this work we show, that Adomian's method is limited for the generalised situation. Because of this limitation, we present a heuristic solution for a parametrised non-linear equation of Richards. This solution is optimized via least squares and Newton's method and evaluated by the non-linear Richards equation. The profiles of the potential matrix are then compared to numerical findings by Wendland and Pizarro (2010).

2. BASICS AND MATHEMATICAL MODEL

A porous medium may be defined as a volume configured by discontinuities in its interior, which can be distributed in various ways according to their frequency, shape and dimension. These discontinuities are generically denoted by voids or pores, whereas the latter is most commonly used to refer to spherical voids. A porous medium may be occupied by the liquid phase and gas phase, thus determining a system three-phase, with the coexistence of these three phases. Details and illustrations may be found in ref. (Franciss, 1980). A porous medium is said permeable when it allows water transportation forming channels that cross the geological environment. This flow is generally associated to transmissivity.

2.1 Mathematical formulation

The governing equation is established from the Darcy-Buckingham equation and the continuity equation. The Darcy-Buckingham is described mathematically by

$$\overrightarrow{q} = -K(\theta)\nabla\phi, \tag{1}$$

where $\overrightarrow{q}$ - specific flow (m/s); $K(\theta)$ - hydraulic conductivity (m/s) depending on soil moisture (m^3/m^3) e ϕ - hydraulic potential (m). Libardi (2005) state that the Darcy equation is a particular case of the Darcy-Buckingham for the condition of a saturated soil. Moreover, Eq. (1) implies isotropy and no saturation with respect to $K(\theta)$ and although the hysteresis is disregarded in determining the physical relations, Eq. (1) has still a large mathematical complexity due to the non-linearity present in the hydraulic conductivity, i.e. the main difference in the flow in saturated soils, modelled by the traditional Darcy equation. Another important point is that this equation is established to condition steady state or dynamic equilibrium. However, most situations in nature are transient and to describe this situation we use the continuity equation.

$$\frac{\partial\theta}{\partial t} = -\nabla(\overrightarrow{q}) \tag{2}$$

Combining the Darcy-Buckingham with the continuity equation and using $\phi = \psi + z$, where ψ is the matrix potential and z the depth, one gets

$$C(\psi)\frac{\partial\psi}{\partial t} = \nabla[K(\psi)\nabla\psi] + \frac{dK(\psi)}{d\psi}\frac{\partial\psi}{\partial z}, \tag{3}$$

where $C(\psi) = \frac{d\theta}{d\psi}$ is called hydraulic capacity. Equation (3) is known as the Richards equation. This equation governs the movement of water in unsaturated soil and can be applied in the whole domain even for distinct saturated and unsaturated areas in accordance with Wendland (1991).

2.2 Simulated model

This work makes use of the model of Genuchten (1980), which according to Quispe (2008) is capable of characterizing the zone of capillary rise and still has the ability to retain zero to saturation condition. The relationship between volumetric

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water content (θ) and the matrix potential (ψ) is established as $\psi = \frac{1}{\alpha}\left(\frac{1}{S(\theta)^{1/m}} - 1\right)^{1/q}$, where $m = 1 - 1/q$ is the effective saturation; α, q are parameters dependent on soil and $S(\theta) = (\theta - \theta_r)/(\theta_s - \theta_r)$, and: θ_s and θ_r are respectively, soil moisture the saturated and residual soil water content. The relationship between K and θ is given by $K(\theta) = K_s S(\theta)^{1/2}\left[1 - \left(1 - S(\theta)^{1/m}\right)^m\right]^2$, where K_s is the saturated hydraulic conductivity. One may write K as a direct function of ψ

$$K(\psi) = K_s((\alpha\psi)^q + 1)^{-m/2}[1 - (1 - [(\alpha\psi)^q + 1]^{-1})^m]^2. \tag{4}$$

The specific water capacity of the soil is given by:

$$C(\psi) = \frac{mq\alpha^q(\theta_r - \theta_s)\psi^{q-1}}{[1 + (\alpha\psi)^q]^{m+1}}. \tag{5}$$

The soil parameters that will be used for the simulations in this work are the same as presented by Celia *et al.* (1990): $\alpha = 3.35m^{-1}; q = 2; K_s = 9.92 \times 10^{-5} ms^{-1}; \theta_s = 0.368m^3m^{-3}$ and $\theta_r = 0.102m^3m^{-3}$ [1]. The sample soil is $L = 1m$ in length and initial and boundary conditions are $\psi(z, 0) = -10m, -L \leq z \leq 0$; $\psi(0, t) = -0.75m$ and $\psi(-L, t) = -10m$ for $t > 0$.

2.3 Padé approximants

The terms $C(\psi)$ and $K(\psi)$ are the terms that define the non-linearity of the Richards equation and from the expressions (4) and (5) one notes that the non-linearity is not trivial. Although the model of Van Genuchten is widely used in the numerical simulations, due to its complexity in functional form, algebraic procedures are complicated, thus making analytical solutions rare. In this work, the terms C and K are approximated via Padé approximants that should turn the model more tractable from the algebraic point of view, and obtain convergence by the Adomian decomposition method. Note, the parameters C and K are phenomenological relations, so that the representation by Padé approximants can be used without loss of generality in the problem. The Padé approximant $[m/n]_{f(x)}$ of a given function $f(x)$ is defined by $[m/n]_{f(x)} = p_m(x)/q_n(x)$, where $p_m(x)$ is a polynomial of degree m and $q_n(x)$ of degree n. Then the hydraulic conductivity and water capacity, shown in Eq. (4) and Eq. (5), are approximated by Padé's rational functions.

The approaches are established to accurately represent the condition of the profiles defined by the original expressions for compact functions. Due to the asymptotic behaviour of the terms K, and C (Figure 1) it is assumed that the Padé approximant must contain the higher polynomial degree in the denominator. Further to minimize the degree of the polynomials the expansion is used at a point different than zero. Thus, a good approximation for the functions presented in the region of interest is

$$\begin{array}{lll} C(\psi): & [4/6], & \text{the expansion point } \psi = -0.2 \\ K(\psi): & [3/7], & \text{the expansion point } \psi = -1.0 \end{array}$$

Figure 1 highlights the consistency between the approaches via Padé and original expressions (Eq. (4) and Eq. (5)).

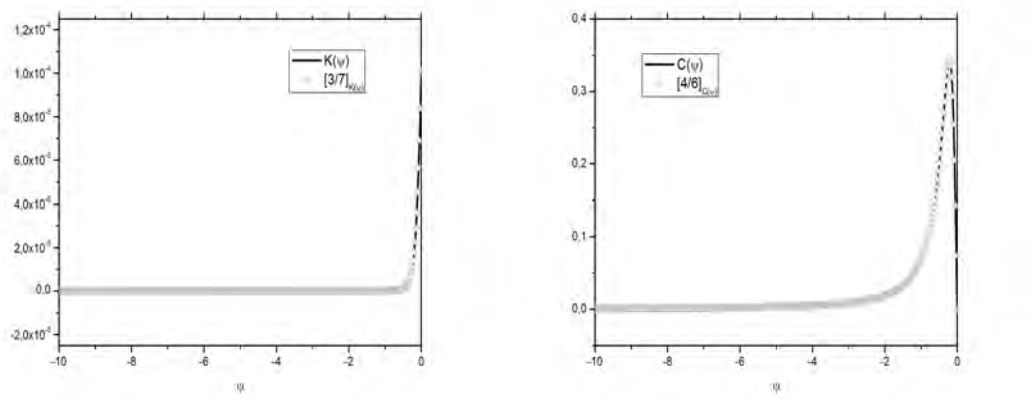

Figure 1. *Comparison of approximants* $[3/7]_{K(\psi)}$ *and* $[4/6]_{C(\psi)}$ *with the original expressions.*

[1] In the case of θ_s e θ_r: m^3 e m^{-3} correspond respectively to the volume of water and air voids.

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2.4 Adomian decomposition

This subsection shows the failed attempt to determine the one-dimensional transient flow of water in a non-saturated or without linearisation by Adomians method. In general, the technique is based on the decomposition of a solution in a number of functions. The non-linear term is approximated by a recursive scheme defined according to the form of the non-linearity. It is noteworthy that, as an approximate solution for an infinite series, there is need for a truncation of the series as a practical solution. Because the solution is a truncated series, the validity of the method is restricted to the radius of convergence of the series. It is noted also that, once established convergence, the solution is exact (Adomian, 1994).

Although the method of Adomian has successful applications in physics and mathematics, it is worth mentioning the recent work of Mello (2010), Petersen (2011) and Silva (2011) which, for non-linear phenomena, the method established a good approximation for the required solution with only a few terms by truncating the series. However, in this study, the procedure does not reach satisfactory results by not obtaining convergence for the solution in the region of interest. Several attempts to implement the method of Adomian in different ways, do not result in satisfactory solutions. The following is only one version of the method is shown.

The equation to be solved for the matrix potential is the Richards equation

$$C(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial \psi}{\partial z} + 1\right)\right] \tag{6}$$

Let $C(\psi) = \overline{C} + \mathcal{C}(\psi)$ and $K(\psi) = \overline{K} + \mathcal{K}(\psi)$, where $\overline{C}$ and $\overline{K}$ are constants and medium $\mathcal{C}(\psi)$ and $\mathcal{K}(\psi)$ functions that carry information of the matrix potential (ψ). Now, grouping the linear and non-linear terms

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{\overline{C}}{\overline{K}}\frac{\partial \psi}{\partial t} = \frac{1}{\overline{K}}\left[\mathcal{C}(\psi)\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial z}\left(\mathcal{K}(\psi)\frac{\partial \psi}{\partial z}\right) - \frac{\partial \mathcal{K}(\psi)}{\partial z}\right]. \tag{7}$$

Thus, the left side of Eq. (7) represents the linear terms, which are used to find the initialization of the recursion from the decomposition. Now, the right side is the non-linearity and is used as correction scheme in the recursion. This equation highlights the strong non-linearity present in the phenomenon (the right side of the equation) which is visible when it is separated from the linear (left side of the equation). Note that, in front of the correction term, there is an amplification factor $1/\bar{K}$, i.e., if $\bar{K}$ is a small term (as seen in Fig. 1), then $1/\bar{K}$ becomes a problematic factor because it reaches high values and spoils the determination of Adomian polynomials.

Decomposing Eq. (7) creates a recursive scheme, where the initial and boundary conditions will be considered in the first equation of the form

$$\begin{aligned}
\frac{\partial^2 \psi_0}{\partial z^2} &- \frac{\overline{C}}{\overline{K}}\frac{\partial \psi_0}{\partial t} = 0 \\
\frac{\partial^2 \psi_1}{\partial z^2} &- \frac{\overline{C}}{\overline{K}}\frac{\partial \psi_1}{\partial t} = f_0(\psi_0) \\
&\vdots \\
\frac{\partial^2 \psi_i}{\partial z^2} &- \frac{\overline{C}}{\overline{K}}\frac{\partial \psi_i}{\partial t} = f_{i-1}(\psi_{i-1}) \\
&\vdots
\end{aligned} \tag{8}$$

To determine the first contribution to the recursive scheme, ψ_0, consider the first equation scheme (8). Thus, solving through separation of variables, the graphical representation of ψ_0 is shown in Fig. 2.

Once established the first contribution ψ_0 of the solution, the function representing the non-linearity $f_0(z,t)$ which is used as a correction term can now be determined, which is shown in Fig. 3. The figure shows, on a logarithmic scale, the non-linearity expressed by the function $f_0(z,t)$ for the times $t = 10^{-4}$, $t = 10^{-3}$ and $t = 10^{-1}$.

As can be seen in Fig. 3, the non-linearity expressed by the function $f_0(z,t)$ varies "violently" by different orders of magnitude and the results that differ from the real problem. Thus, the correction term is an aggravating factor for the convergence of the series generated by the decomposition method. To determine the second contribution of the recursive scheme, ψ_1, consider the second equation scheme (8). Thus, solving by Laplace transform, the graphical representation of ψ_1 is shown in Fig. 4.

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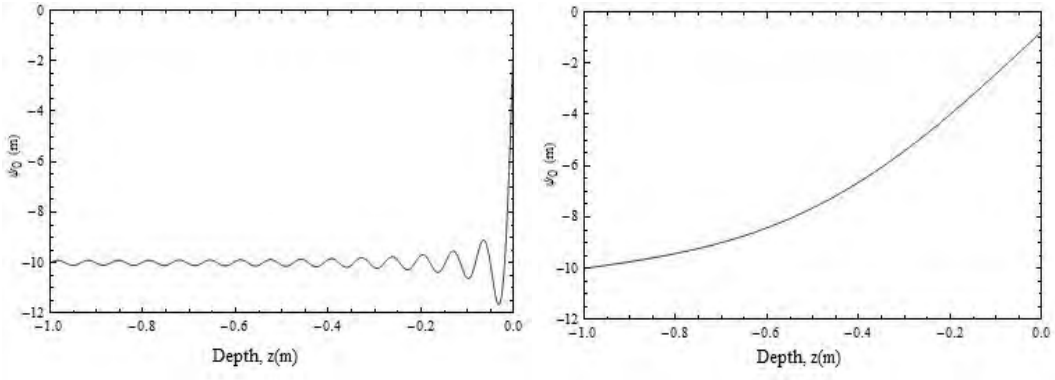

Figure 2. *Graphic* ψ_0 *for the times* $t = 0$ *and* $t = 10^{-3}$.

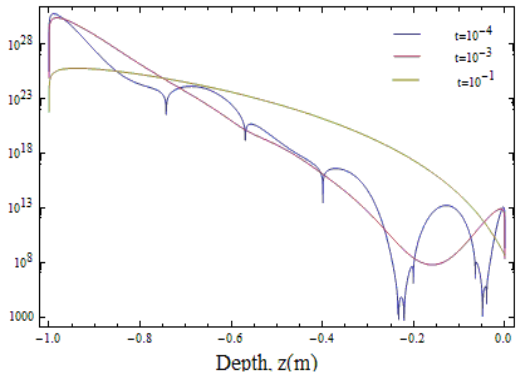

Figure 3. $f_0(z,t)$ *for the times* $t = 10^{-4}$, $t = 10^{-3}$ *and* $t = 10^{-1}$.

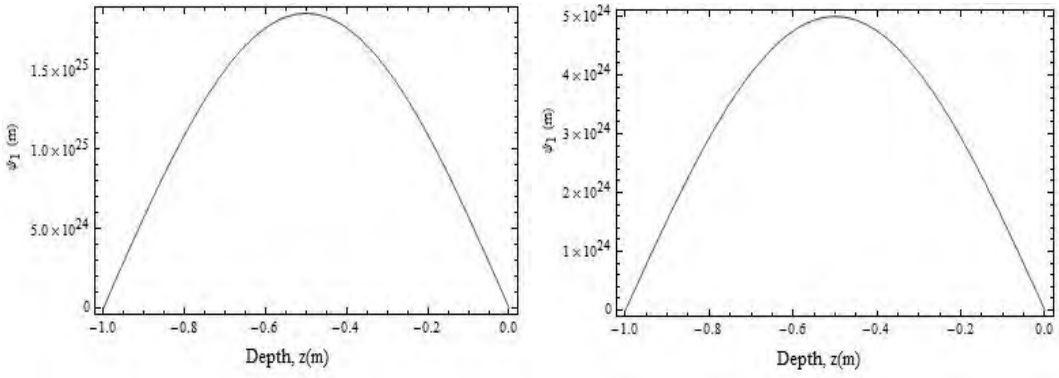

Figure 4. *Graphic* ψ_1 *for the times* $t = 10^{-4}$ *and* $t = 10^{-1}$.

As noted, in Fig. 4, the second contribution directly affected the convergence of the solution, returning a profile of high magnitude. This problem is associated with the already mentioned fact that a big factor appears in front of each recursive correction. Therefore, in this study, the method of Adomian is not effective to correct an initial solution of a linear differential equation, since this phenomenon has a non-linearity with a dominant feature. Consequently, from these findings, the authors suggest a new formulation for obtaining the analytical solution or analytical representation to the non-linear Richards equation. This solution must be built following two steps: First, one must determine an approximate initial solution. Then establish a new recursive Adomian scheme, which is initialized by the previously determined approximate solution. The remainder of this work is to determine a heuristic solution, that defines the recursion initialisation for the non-linear Richards equation.

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2.5 Heuristic solution

Next we will provide a heuristic solution parametrised for non-linear equation. This solution is obtained by reverse engineering, i.e. take as its starting point an existing product, in this case, numerical data available in the literature. Thus, we adopted a new methodology in order to obtain a compact analytic representation for a non-linear phenomenon. This solution can be corrected in a recursive manner to reproduce the solution in the limit of infinite recursion depth.

The numerical data which will be used as a starting point in this methodology were obtained by Wendland and Pizarro (2010) (Fig. 5). In the current study, the authors determine the matrix potential, ψ, from the non-linear partial differential equation of Richards, for one dimension, with temporal variation by applying a scheme of Finite Elements. The constitutive relations between the parameters θ, ψ and K were given by Van Genuchten relations. The parameters of the soil together with the boundary conditions ($\psi(0,t) = -0.75m$ and $\psi(-L,t) = -10m$ for $t > 0$), the initial conditions ($\psi(z,0) = -10m, -L \leq z \leq 0$) and soil sample were the same as employed by (Celia *et al.*, 1990).

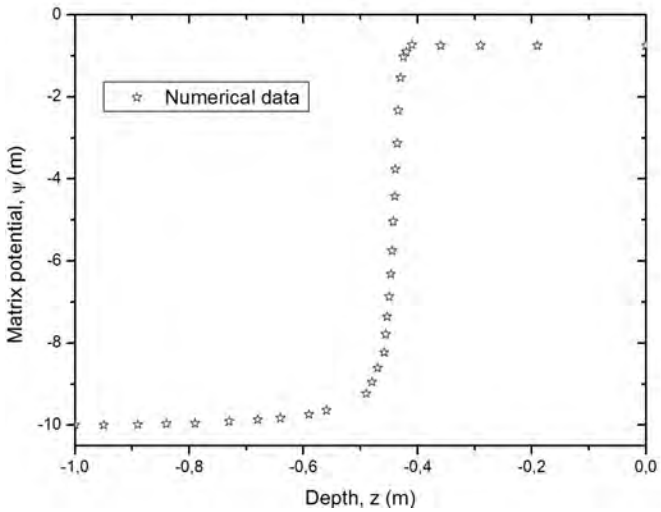

Figure 5. *Approximate numerical solution* (Wendland and Pizarro, 2010).

One observes that the matrix potential, ψ, determined by the approximate numerical solution has a behaviour similar to a hyperbolic tangent. With this solution heuristic for the Richards equation, developed from the behaviour of the hyperbolic tangent like function and to approximate the general solution, the parameters are optimized in order to satisfy the Richards equation. At first, the parameters are assigned the hyperbolic tangent function in order to reproduce the Fig. 5 considering aspects as non symmetry, asymptotes, translational (vertical and horizontal) and smoothness. Thus, a proposal acceptable solution can be given by Eq. (9):

$$\psi_h(z,t) = -a_1 \tanh\left(a_2\left(10^{a_3(-z)}+1\right)(-a_4/(1+t)+a_5-z)\right) - a_6, \tag{9}$$

where ψ_h is the heuristic matrix potential. The parameter a_1 is responsible for the lower and upper limit of the matrix potential; a_2 establishes the maximum slope of the profile; $\left(10^{a_3(-z)}+1\right)$, whose free parameter a_3 corrects the concave and convex regions near the inflection point, due to the need for the time representation in the stationary condition and heuristic solution is added the term $a_4/(1+t)$ where the parameter a_4 determines the rate at which the solution reaches a stationary mode and parameters a_5 and a_6 are respectively the factors responsible for translations in horizontal and vertical axis. Thus, the matrix potential ψ_h is given as a function of $\psi_h = \psi_h(z,t;\{a_i\})$, where $i = 1,2,3,4,5,6$. To adjust the curve it is necessary to minimize the differences between the results generated by the proposed function and the data obtained in numerical simulation. However, to avoid cancellation, the difference for each given value must be positive. One way is to minimize the squared difference. Thus, heuristic solution presented in Eq. (9) is optimized using the method of non-linear least squares.

The least squares problem is to find a that minimizes the expression

$$G(a) = \frac{1}{2}\sum_{k=1}^{m}(r_k(a))^2 = \frac{1}{2}\|r(a)\|^2 \tag{10}$$

where $a = [a_1, a_2, \ldots, a_6]^T$, $r(x)$ represents the difference between the data set and the function proposed ψ_h. m is assumed to be adjusted points for the curve. In order to optimise the parameters of the least squares method Newton's

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method was employed. Since $r(a)$ is continuously differentiable twice

$$A(a)\,\epsilon\,\mathcal{R}^{m\times 6} \quad \text{sendo} \quad A(a)_{kj} = \frac{\partial r_k(a)}{\partial a_j}. \tag{11}$$

The first derivative of $G(a)$ is of the form

$$\nabla G(a) = \sum_{k=1}^{m} r_k(a)\nabla r_k(a) = A(a)^T r(a); \tag{12}$$

where $\nabla r_k(a)$ represents the vector with the first derivatives with respect to the parameters. Regarding the second derivative

$$\begin{aligned} \nabla^2 G(a) &= \sum_{k=1}^{m} \nabla r_k(a)\nabla r_k(a)^T + r_k(a)\nabla^2 r_k(a) \\ &- A(a)^T A(a) + \sum_{k=1}^{m} r_k(a)\nabla^2 r_k(a), \end{aligned} \tag{13}$$

where $\nabla^2 r_k(a)$ is the Hessian matrix of second derivatives $r_k(a)$.

Consider a_i an approach to the solution of Eq. (10). According to Eq (12) and Eq. (13) having a quadratic approximation of $G(a)$ centred on a_i

$$\begin{aligned} Q_i(a) &= G(a_i) + \nabla G(a_i)^T(a-a_i) + \frac{1}{2}(a-a_i)^T\nabla^2 G(a_i)(a-a_i) \\ &= \frac{1}{2}r(a_i)^T r(a_i) + r(a_i)^T A(a_i)(a-a_i) \\ &+ \frac{1}{2}(a-a_i)^T\left(A(a_i)^T A(a_i) + \sum_{k=1}^{m} r_k(a_i)\nabla^2 r_k(a_i)\right)(a-a_i). \end{aligned} \tag{14}$$

Substituting $G(a)$ by $Q(a)$ and taking $\nabla Q_i(a) = 0$

$$a_{i+1} = a_i - \left(A(a_i)^T A(a_i) + \sum_{k=1}^{m} r_k(a_i)\nabla^2 r_k(a_i)\right)^{-1} A(a_i)^T r(a_i). \tag{15}$$

The point a_i is not necessarily a solution to Eq. (10), because $G(a_i)$ is approximated by a quadratic model, but it is given as a good approximation. The quadratic model should be repeated now centered in a_{i+1}. This iterative process is Newton's method. The iterative equations are

$$a_{i+1} = a_i + \Delta_i, \quad i = 0, 1, \ldots \tag{16}$$

with Δ_i must satisfy the equation

$$\left(A_i^T A_i + \sum_{k=1}^{m} r_k(a_i)\nabla^2 r_k(a_i)\right)\Delta_i = -A_i^T r_k. \tag{17}$$

In Newton's method, convergence is local, thus the initial approximation for the iterative process should be close enough to the solution to ensure convergence to a stationary point. Furthermore, the vector Δ_i can not point in the direction in which the G decreases, and there is no guarantee for convergence to the minimum G. However, Newton's method is considered efficient and robust.

After optimization of the parameters evaluating the heuristic solution is measured by a test for self-consistency. Suppose the following condition if ψ is the solution of the original problem $\Omega_R[\psi] = 0$, where Ω_R is the differential operator of the Richards equation. However, because the solution is a heuristic approximation, then $\Omega_R[\psi_h] = R$, where R the remainder is to establish equality. Thus, to characterize ψ_h as solution R should be minimized. If self-consistency does not offer a good response, then the least squares method shall be performed again, now using the optimised parameter as initial approximation for Newton's iterative scheme. The initial approximation for the iterative Newton's method was established graphically through observations to numerical data. The parameters are initially approximated: $a_1 = 4, a_2 = 40, a_3 = -1, a_4 = 0.9, a_5 = 0.5$ and $a_6 = 5$.

In Fig. 6 we note that the initially approximate function still diverges from the profile of the matrix potential, however, the results are not as distant to numerical data, especially for regions far from the surface. Therefore, this approach can be used as a starting point for a Newton iterative process. In the Fig. 7 shown the conditions of self-consistency test, the initial approximation for three different times. In such a test, it is interesting to observe the performance of the solution before

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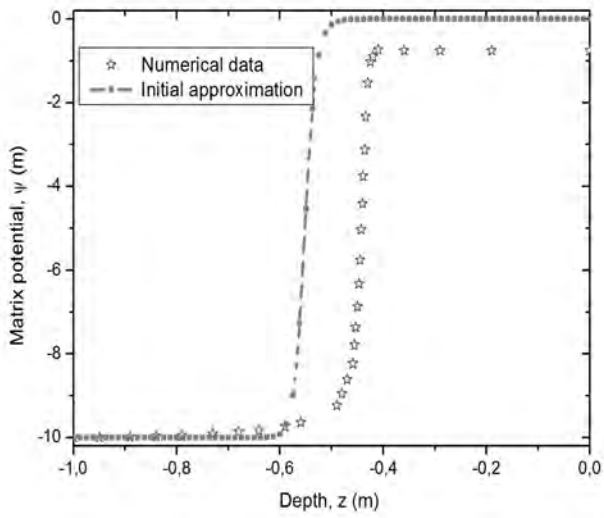

Figure 6. *Graphic of initial approximation compared to numerical data (Figure 5).*

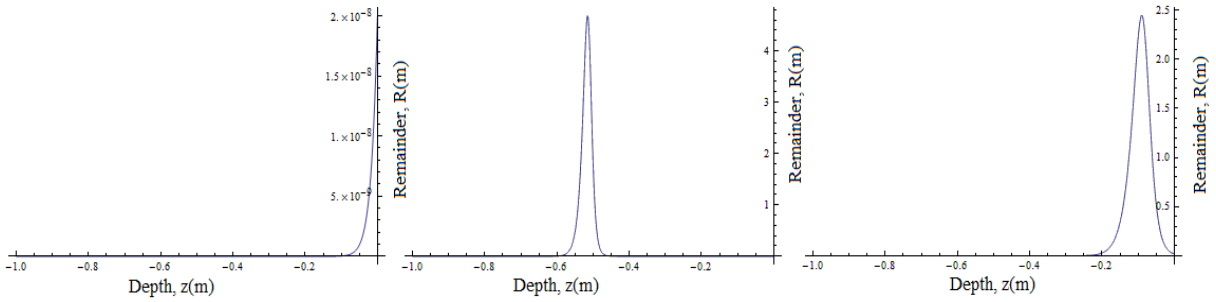

Figure 7. *self-Consistency test for the times* $t = 10^{-4}$, $t = 1$ *e* $t = 1.7$.

optimization and the evolution of the parametrised solution that achieves a satisfactory result. Figure 7 it is observed that the tests show acceptable results for small times.

Established the initial approach, the method of least squares is then applied and its solution obtained by non-linear Newton method. Following the methodology shown in the subsection, the following results are obtained: $a_1 = 4.5386, a_2 = 47.4330, a_3 = -1.2318, a_4 = 0.9443, a_5 = 0.7941$ and $a_6 = 5.2459$.

The parametrised heuristic solution, optimized and evaluated by the test of self-consistency, is shown in Fig 8. Note that the optimum solution shows acceptable results when compared to numerical data of Wendland and Pizarro (2010), thus providing an initial representation for the matrix potential already close to the true solution.

The temporal evolution of the potential profile matrix is shown in Fig. 9, where it is possible to observe the consistency with respect to the initial condition.

The conditions for the self-consistency test are shown in Fig. 10. Note that for small times, the remainder R accuses small numerical values. For longer times, the remainder R reaches higher results, but acceptable for the proposed heuristic solution.

Thus, it was possible to reproduce the profile of the matrix potential ψ accusing little difference in the tests of self-consistency and we obtained a useful model for the unsaturated zone of soils.

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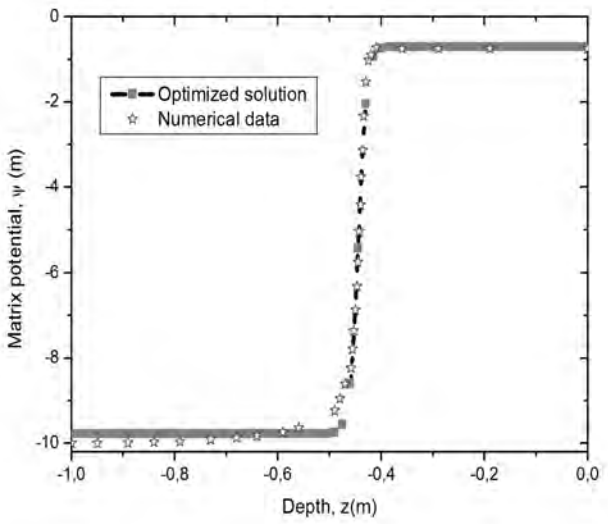

Figure 8. *Graph heuristic solution optimized compared to numerical data of Wendland and Pizarro (2010).*

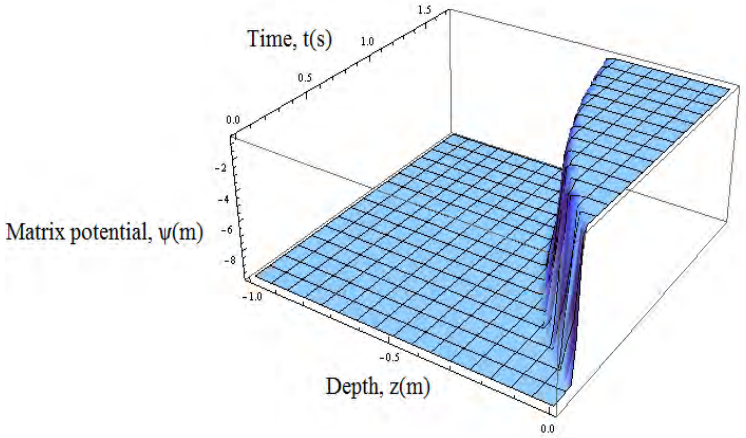

Figure 9. *Graphic matrix potential dependent of space and time.*

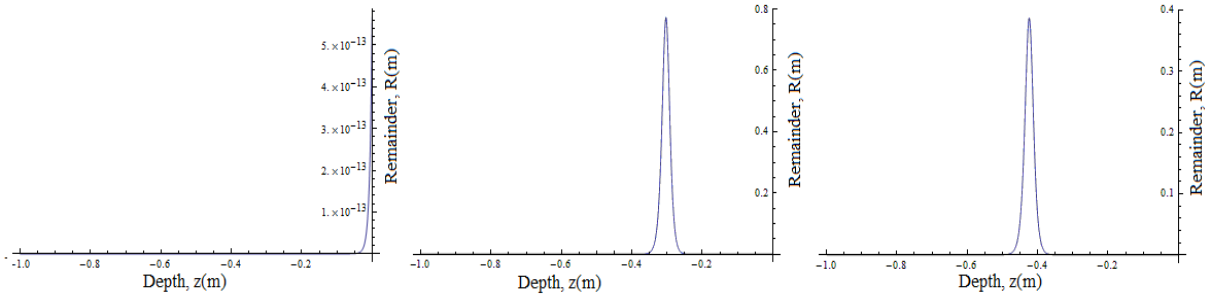

Figure 10. *self-consistency test for the times* $t = 10^{-4}$, $t = 1$ *e* $t = 1.7$.

2.6 Conclusions and future prospects

Based on our discussion, we conclude that the hybrid model, the Padé approximants and Adomian decomposition, does not converge to the solution of the non-linear Richards equation. Although the Padé approximants have provided good approximation to the hydraulic conductivity and water capacity, the decomposition method of Adomian offers no efficiency in determining the profile of the matrix potential, because of a lack of convergence of the decomposition series.

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The heuristic solution, which was presented in Eq 9, when optimized by the method of least squares and non-linear Newton's method, give good results in obtaining the matrix potential profile, accusing small differences compared with the numerical solution and evaluated by the original equation.

For the continuation of this work, the heuristic parametrised solution, is interpreted as a first step to obtain an analytical solution or analytical representation of the non-linear Richards equation. To this end, we determine a differential equation for ψ_h, $\Omega_h[\psi_h] = 0$, where Ω_h is the differential equation operator whose solution is the heuristic matrix potential ψ_h. Then, the difference to the Richards equation is defined as a source term $(\Omega_R - \Omega_h)[\psi] = \mathcal{R}[\psi]$, where the source $\mathcal{R}[\psi]$ can be a differential operator. This defines the following recursive scheme

$$\begin{aligned} \Omega_h[\psi_0] &= 0 \\ \Omega_h[\psi_1] &= \mathcal{R}[\psi_0] \\ &\vdots \\ \Omega_h[\psi_i] &= \mathcal{R}[\psi_{i-1}], \end{aligned} \tag{18}$$

where, the solution is given by

$$\psi = \lim_{n\to\infty} \sum_{i=0}^{n} \psi_i. \tag{19}$$

The heuristic solution $\psi_0 = \psi_h$ is used as a recursion initialisation, Eq. 18.

Note, the fact that the profile of the potential generated by the heuristic solution matrix is close to the numerical solution, it is well plausible that the source term, originated by the difference of differential equations, is only a small correction and the Adomian scheme is likely to converge.

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