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FINITE GROWTH IN TERMOELASTIC MATERIALS

Joan O'Connor Blanco
Lavinia Maria Alves Borges
Fernando Pereira Duda
Mechanical Engineering-COPPE, Universidade Federal do Rio de Janeiro, Av. Horacio Macedo, 2030, Centro de Tecnologia, Sala G-204, P.O. Box 68503, CEP 21941-914, Rio de Janeiro, RJ, Brazil
joan@ufrj.br
lavinia@mecanica.ufrj.br
duda@mecanica.coppe.ufrj.br

Melchor Rodriguez Madrigal
Mechanical Engineering, Polytechnic Institute "Jose Antonio Echeverria", Calle 127, s/n, CUJAE, Marianao, Habana, Cuba
melchor@mecanica.cujae.edu.cu

Abstract. *In this work is presented and reviewed the finite growth theory for thermo-elastic materials by meaning of stress induced growth using continuum mechanics theory. The material growth will be regulated by the activation of homeostatic, growth and reabsorption surfaces. An application in biomechanics through biological tissue growth theory is presented in an example modeling a ventricular hypertrophy.*

Keywords: *continuum mechanics, growth, reabsorption*

1. INTRODUCTION

In the last years, bioengineering has become in one of the most active research areas. A large amount of research work related with bio-materials, tissue engineering regeneration and implants and prostheses and its adaptation inside the human body, is done and is increasing and in continuos expansion.

Also, the development of finite element softwares and CAD computational tools to generate complicated geometrical bio-forms with an optimal approximation to the physical real form of human organs and tissues, allows, for example; to estimate the hip prostheses behavior inside the femoral bone medular canal or predict the stresses behavior at bone level and its effect in the periodontal ligament (which serve as a connective tissue between a teeth and the bone matrix) when teeth are submitted to loads when using orthodontic appliances, and much more examples of the use of the finite element method (FEM) in applications for hard (as cortical and trabecular bone) or soft tissues (skin, muscle, arteries) on different medical disciplines. It is possible to know the stress and strain scenario and also displacements for biological systems or systems formed by an organ and an implant or prostese coupled to the organ, when submitted to loads with a very good approximation (Prendergrast (1997), Huiskes (2000)) (assuming high level of accuracy of the geometrical models and material properties and models, without entering in a detailed description), but, up to this point, it is not possible to predict if bone or soft tissue grows or reabsorb when loaded. So, a powerful tool for medicine specialists will be a computational tool which allow them to predict, with reliable approximation, if tissue growth or reabsorption will occur, and its growth or reabsorption rates for different load levels, because the success of the implant or prostese depends on growth.

As known, mathematical formulation of growth using continuum mechanics involves kinematic, balance laws and constitutive relations. The objective of this work is a study and a revision of the growth theory for thermo-elastic materials presented by other authors and its treatment using continuum mechanics, starting from the model for living bone in an early work presented by Cowin and Hegedus (1975), passing through the kinematic treatment to describe surface growth based on growth velocities described by Skalak *et al.* (1982), the extension of this work including the effect of incompatible growth presented by Rodriguez *et al.* (1994) using the multiplicative decomposition of the deformation gradient, up to the theory for the activation of growth and reabsorption process induced by stress levels proposed by Vignes and Papadopoulos (2010) based on classical growth approach mentioned above, and other works related with thermo-mechanics of plasticity developed by Casey (1998) or with thermo-mechanics of volumetric growth (Epstein and Maugin (2000), Klisch and Hoger (2003)) which can be considered as procedures well established.

This revision includes: the presentation of balance laws; mass, linear and angular momentum and energy balances for a thermo-elastic material with growth, the multiplicative decomposition of the deformation gradient, the constitutive equations and the thermodynamic analysis and entropy balance. Also is presented an example of finite growth adaptive heart growth due to abnormal loads (cardiac hypertrophy)

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2. BALANCE LAWS

2.1 Continuum growth configurations

Let X a material point of a body B occupying a region in Euclidean space, where **X** and **x** are the position vectors of X relatives to the fixed origin 0 in reference and current configurations $\mathcal{B}$ and $\mathcal{B}_t$. The mapping $\mathbf{x} = \chi(\mathbf{X}, t)$ is considered one-to-one in **X** for fixed *t*, so invertible, then: $\mathbf{X} = \chi^{-1}(\mathbf{x}, t)$ as shown in Fig. 1. Deformation gradient in the material point of $\mathcal{B}_t$ relative to $\mathcal{B}$ is defined as $\mathbf{F} = \nabla\chi$. The volumetric Jacobian of the mapping χ_t is $J(\mathbf{X}, t) = \det\nabla\chi_t(\mathbf{X}) > 0$.

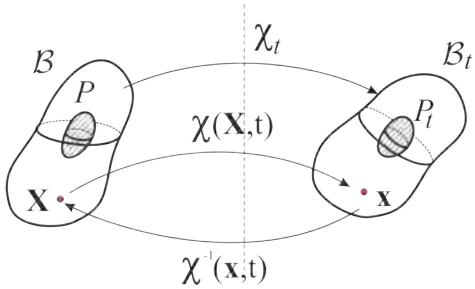

Figure 1. Reference and current configurations for continuum growth

2.2 Mass Balance

Mass in the current and reference configurations are defined as

$$M(P) = \int_{P_t} \rho dv, \quad M(P) = \int_{P} \rho_K dV \tag{1}$$

where, $\rho = \rho(\mathbf{x}, t)$ and $\rho_K = \rho_K(\mathbf{X}, t)$ are the mass densities in the current and in the reference configurations. Using the volume relation $dv = JdV$ and the localization theorem yields, $\rho = \rho_K J$

The rate change of mass, which can be non-zero due to volumetric mass sources and mass fluxes can be defined as

$$\frac{d}{dt}M(P) = \frac{d}{dt}\int_{P_t} \rho(\mathbf{x}, t)dv = \int_{P_t} \rho\Gamma dv + \int_{\partial P_t} m da \tag{2}$$

where; Γ is the rate change of mass per unit mass and m is the mass flux into P_t per unit surface area through the boundary ∂P_t. Using Reynolds transport theorem

$$\int_{P_t} (\dot{\rho} + \rho div\mathbf{v} - \rho\Gamma)dv = \int_{\partial P_t} m da \tag{3}$$

where **v** is the velocity of the body, and *m* is the mass flux in $(\mathbf{x}, t)$ which depends only on the outward normal **n** to ∂P_t: $m = m(\mathbf{x}, t; \mathbf{n}) = -\mathbf{m}(\mathbf{x}, t) \cdot \mathbf{n}$. Using the Cauchy tetrahedron argument and the divergence theorem:

$$\int_{\partial P_t} m da = -\int_{P_t} div\mathbf{m} dv$$

Substituting and using the Localization theorem, the local form of the spatial mass balance is obtained

$$\dot{\rho} + \rho div\mathbf{v} = \rho\Gamma - div\mathbf{m} \tag{4}$$

and the equivalent local form in the reference configuration

$$\dot{\rho_K} = \rho_K\Gamma - Div\mathbf{M} \tag{5}$$

2.3 Linear momentum balance

The linear momentum balance law for growth can be written as

$$\frac{d}{dt}\int_{P_t} \rho\mathbf{v} dv = \int_{P_t} \rho\mathbf{b} dv + \int_{\partial P_t} \mathbf{t} da + \int_{P_t} (\rho\Gamma)\tilde{\mathbf{v}} dv - \int_{\partial P_t} (\mathbf{m} \cdot \mathbf{n})\tilde{\mathbf{v}} da \tag{6}$$

where; $\mathbf{b}$ is the body force per unit mass, $\mathbf{t}$ is the traction force per unit of area acting on the boundary ∂P_t, $\tilde{\mathbf{v}}$ is the velocity of the new mass entering the body. The first and the second terms represents linear momentum changes due to body forces and traction and the third and fourth terms represents changes in linear momentum accompanying the additional mass through volumetric sources and mass fluxes.

An equivalent form of the linear momentum balance is obtained using $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}} - \mathbf{v}) + \mathbf{v}$, and substituting in Eq. (6) as proposed by Epstein and Maugin (2000)

$$\frac{d}{dt}\int_{P_t} \rho \mathbf{v} dv = \int_{P_t} \rho \mathbf{b} dv + \int_{\partial P_t} \mathbf{t} da + \int_{P_t} (\rho\Gamma)\mathbf{v} dv - \int_{\partial P_t} (\mathbf{m}\cdot\mathbf{n})\mathbf{v} da + \int_{P_t} \rho\tilde{\mathbf{b}} dv + \int_{\partial P_t} \tilde{\mathbf{t}} da \tag{7}$$

where:

$$\tilde{\mathbf{b}} = \Gamma(\tilde{\mathbf{v}} - \mathbf{v}), \quad \tilde{\mathbf{t}} = (\mathbf{m}\cdot\mathbf{n})(\tilde{\mathbf{v}} - \mathbf{v})$$

being $\tilde{\mathbf{b}}$ body force per mass unit associated to the irreversible momentum changes due to volumetric sources, and $\tilde{\mathbf{t}}$ is the traction per unit surface area corresponding to the irreversible momentum changes from surface fluxes. Where the third and fourth terms represents the momentum changes due to the new mass entering the body with the same velocity of the body and the last two terms are associated to irreversible momentum changes due to volumetric and surface sources.

With Eq. (7), Reynolds transport theorem and mass balance, and using the divergence and localization theorems, the local form of the spatial linear momentum balance equation is obtained

$$\rho\dot{\mathbf{v}} = \rho\overline{\mathbf{b}} + div\overline{\mathbf{T}} \tag{8}$$

where:

$$\overline{\mathbf{b}} = \mathbf{b} + \tilde{\mathbf{b}} + \frac{1}{\rho}(\mathrm{div}\,\mathbf{m})\mathbf{v}, \quad \overline{\mathbf{T}} = \mathbf{T} - \mathbf{v}\otimes\mathbf{m} + \tilde{\mathbf{T}}$$

$\overline{\mathbf{b}}, \overline{\mathbf{T}}$: Effective body forces and Cauchy stress. $\mathbf{T}$ is the Cauchy tensor that is related to the traction vector $\mathbf{t} = \mathbf{Tn}$

The referencial form is obtained using $Jdiv\overline{\mathbf{T}} = Div\overline{\mathbf{P}}$, where $\overline{\mathbf{P}} = \overline{\mathbf{T}}\mathbf{F}$ is the first Piola-Kirchhoff stress tensor, hence

$$\rho_K\dot{\mathbf{v}} = \rho_K\overline{\mathbf{b}} + Div\overline{\mathbf{P}} \tag{9}$$

where:

$$\overline{\mathbf{P}} = \mathbf{P} - \mathbf{v}\otimes\mathbf{M} + \overline{\mathbf{T}}\mathbf{F} \tag{10}$$

however, computationally it is convenient to use the second Piola-Kirchhoff stress tensor: $\overline{\mathbf{S}} = \mathbf{F}^{-1}\overline{\mathbf{P}}$

2.4 Angular momentum balance

Angular momentum balance law for growth can be written as

$$\begin{aligned}\frac{d}{dt}\int_{P_t} x\times\rho\mathbf{v} dv \;=\; & \int_{P_t} x\times\rho\mathbf{b} dv + \int_{\partial P_t} x\times\mathbf{t} da + \int_{P_t} x\times(\rho\Gamma)\mathbf{v} dv - \int_{\partial P_t} x\times(\mathbf{m}\cdot\mathbf{n})\mathbf{v} da + \int_{P_t} x\times\rho\tilde{\mathbf{b}} dv \\ & + \int_{\partial P_t} x\times\tilde{\mathbf{t}} da\end{aligned} \tag{11}$$

Using the Reynolds transport theorem and considering the mass balance and the linear momentum balance equations in local forms, the angular momentum balance equation is obtained

$$\overline{\mathbf{T}} = \overline{\mathbf{T}}^{\mathrm{T}} \tag{12}$$

and its equivalent in the reference configuration

$$\overline{\mathbf{P}}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\overline{\mathbf{P}}^{\mathrm{T}} \tag{13}$$

2.5 Energy Balance

The equation of energy balance can be deduced from the conventional power balance postulate (Gurtin *et al.* (2010)) in which the external expended power in P_t is balanced by the summation of the internal power expended into P_t and the kinetics energy of P_t, taking the form

$$\frac{d}{dt}\int_{P_t} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} dv + \int_{P_t} \overline{\mathbf{T}}\cdot\mathbf{L} dv = \int_{P_t} \rho\overline{\mathbf{b}}\cdot\mathbf{v} dv + \int_{\partial P_t} \bar{\mathbf{t}}\cdot\mathbf{v} da + \int_{P_t} \frac{1}{2}(\rho\Gamma - div\mathbf{m})\mathbf{v}\cdot\mathbf{v} dv \tag{14}$$

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The last term of the Eq. (14) is the supply of kinetic energy in P_t due to the added mass.

The total energy balance is determined from the thermodynamic first law (Cowin and Hegedus (1975)) which represents the balance that describes the interaction between the internal energy and the kinetic energy of P_t, the rate of the expended power in P_t and the heat transferred to P_t

$$\begin{aligned}\frac{d}{dt}\int_{P_t}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dv+\int_{P_t}\rho u dv &= \int_{P_t}\rho\bar{\mathbf{b}}\cdot\mathbf{v}dv+\int_{\partial P_t}\bar{\mathbf{t}}\cdot\mathbf{v}da+\int_{P_t}\rho(r+r_i)dv-\int_{\partial P_t}(\mathbf{q}+\mathbf{q}_i)\cdot\mathbf{n}da \\ &+ \int_{P_t}\frac{1}{2}(\rho\Gamma-div\mathbf{m})\mathbf{v}\cdot\mathbf{v}dv+\int_{P_t}(\rho\Gamma-div\mathbf{m})\tilde{u}dv\end{aligned} \quad (15)$$

where; $u,\tilde{u}$: are; the internal energy per unit of mass of the existent mass and the internal energy per unit of mass of the added mass; r: heat supply per mass unit; $\mathbf{q}$: heat flux into P_t per surface area unit; and $r_i,\mathbf{q}_i$: are the irreversible heat terms; r_i per unit mass and, $\mathbf{q}_i$ per unit surface area. $r_i,\mathbf{q}_i$ are take into account for the expended energy in the growth process. The last term of Eq. (15) is the rate at which the internal energy is added in the region P_t with the new mass.

The equivalent form of the energy balance using $\tilde{u}=(\tilde{u}-u)+u$, as proposed by Epstein and Maugin (2000), is

$$\begin{aligned}\frac{d}{dt}\int_{P_t}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dv+\int_{P_t}\rho u dv &= \int_{P_t}\rho\bar{\mathbf{b}}\cdot\mathbf{v}dv+\int_{\partial P_t}\bar{\mathbf{t}}\cdot\mathbf{v}da+\int_{P_t}\rho(r+\tilde{r})dv-\int_{\partial P_t}(\mathbf{q}+\tilde{\mathbf{q}})\cdot\mathbf{n}da \\ &+ \int_{P_t}\frac{1}{2}(\rho\Gamma-div\mathbf{m})\mathbf{v}\cdot\mathbf{v}dv+\int_{P_t}(\rho\Gamma-div\mathbf{m})u dv\end{aligned} \quad (16)$$

where: $\tilde{r}=(\rho\Gamma-div\mathbf{m})(\tilde{u}-u)+r_i,\quad \tilde{\mathbf{q}}=\mathbf{q}_i$

being $\tilde{r}$ the irreversible total heat supply (in the special case that the added mass has identical internal energy of that of the existence mass, then not other dissipative effects are associated to the growth process, so $\tilde{r}\mapsto 0$ and $\tilde{\mathbf{q}}\mapsto 0$) Using again the conventional power balance postulate and substituting the external power terms by the sum of internal power and the kinetic energy rate in Eq. (16), and applying the Reynolds transport and the localization theorems, the local form of the spacial energy balance is then given by

$$\rho\dot{u}=\overline{\mathbf{T}}\cdot\mathbf{L}+\rho(r+\tilde{r})-div(\mathbf{q}+\tilde{\mathbf{q}}) \quad (17)$$

the equivalent referencial form is

$$\rho_K\dot{u}=\overline{\mathbf{P}}\cdot\dot{\mathbf{F}}+\rho_K(r+\tilde{r})-Div(\mathbf{q}_K+\tilde{\mathbf{q}}_K) \quad (18)$$

where $\mathbf{q}_K$ e $\tilde{\mathbf{q}}_K$ are reference heat flux vectors given by: $\mathbf{q}_K=J\mathbf{F}^{-1}\mathbf{q},\quad \tilde{\mathbf{q}}_K=J\mathbf{F}^{-1}\tilde{\mathbf{q}}$

3. CONSTITUTIVE EQUATIONS

The mass, linear and angular momentum, and energy balances provides the eight equations and the twenty-six unknowns $\{\chi,\rho,\overline{\mathbf{T}},u,\mathbf{q},\theta,\Gamma,\mathbf{m},\tilde{r},\tilde{\mathbf{q}}\}$ where the first eighteen unknowns $\{\chi,\rho,\overline{\mathbf{T}},u,\mathbf{q},\theta\}$ correspond to the set of variables for a conventional thermo-elastic material. Of the remaining eight unknowns; four describe mass sources and mass flow $\{\Gamma,\mathbf{m}\}$ and four describe the irreversible process $\{\tilde{r},\tilde{\mathbf{q}}\}$, and extend the model to thermo-elastic materials with growth. The components of the body force $\mathbf{b}$, heat supply r and absolute temperature θ are assumed known. Constitutive laws which describes the behavior of the idealized material and relates the kinematic, mechanic, thermal and growth fields provides the remaining relations needed to mathematically close the system of equations of the problem.

3.1 Multiplicative decomposition of the deformation gradient

In the modeling of thermo-elastic materials with growth for finite deformations, the multiplicative decomposition of the deformation gradient proposed by Rodriguez *et al.* (1994), can be expressed as

$$\mathbf{F}=\mathbf{F}_e\mathbf{F}_g \quad (19)$$

where; $\mathbf{F}_g$: is the local mapping of the material in the reference configuration $\mathcal{B}$ to a local maximally unloaded intermediate configuration (generally incompatible); $\mathcal{B}_g$: intermediate configuration, is the collection of the local intermediate configurations and captures growth deformation; $\mathbf{F}_e$: is the elastic mapping from intermediate configuration into de the global compatible current configuration $\mathcal{B}_t$ of the body (as shown in Fig. (2))

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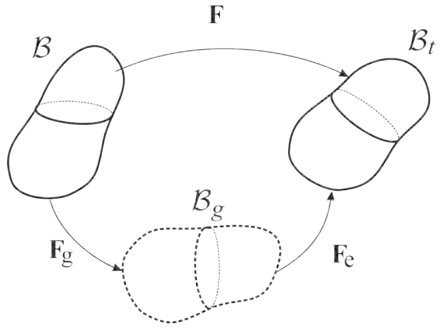

Figure 2. Multiplicative decomposition of the deformation gradient

Mass is preserved from $\mathcal{B}_g$ to $\mathcal{B}_t$ (from Eq. (19)), so $M(\mathcal{B}_g) = M(\mathcal{B}_t)$, then using $\rho = \rho_K J$ and localization theorem

$$\rho_K = \rho_g J_g \tag{20}$$

where ρ_g is the growth density (density in the intermediate configuration) and $J_g = det(\mathbf{F}_g)$ is the Jacobian of the growth deformation gradient. Taking the material time derivative of Eq. (20)

$$\dot{\rho}_K = \dot{\rho}_g J_g + \rho_g \dot{J}_g = \dot{\rho}_g + \dot{\rho}_g (J_g - 1) + (\rho_g - \rho_0)\dot{J}_g + \rho_0 \dot{J}_g \tag{21}$$

Where exist two cases to consider; the so called "density preserving" growth and "volume preserving" growth or densification. In the density preserving case, the mass growth is assumed that occurs through volume changes at constant density $(\rho_g = \rho_0)$ in this case, Eq. (21) reduces to

$$\dot{\rho_K} = \rho_0 \dot{J}_g \tag{22}$$

In preserving volume growth case, mass growth occurs through density changes at constant volume $J_g = 1$ then

$$\dot{\rho_K} = \dot{\rho_g} \tag{23}$$

4. Constitutive equations for thermo-elastic materials with growth. Finite growth theory

It is well known, from medical disciplines such as orthopaedics, orthodontics that growth and reabsorption occurs as a results of different tension stimulus, or more specifically with the increase or decrease of tension levels (Cowin and Hegedus (1975), Huiskes (2000)). Assuming this fact as an starting point, and given an arbitrary material point X in a tensional homeostatic state $\overline{\mathbf{S}}_0$ at absolute temperature θ_0 of a body B of a thermo-elastic material with growth in the tension-temperature space $S = \{\overline{\mathbf{S}}_0, \theta_0\}$, the theory of finite growth proposed by Vignes and Papadopoulos (2010) that is based on an standard procedure established in plasticity by Casey (1998) is based on the following hypothesis (schematically shown in Fig. (3))

1) The existence of an open set $S_0 \in \mathrm{R}^7$ in stress-temperature space containing $\{\overline{\mathbf{S}}_0, \theta_0\}$ such that the material behaves as a conventional thermo-elastic material within this region. S_0 is assumed simply connected and bounded by a smooth and oriented hyper-surface δS_0 called Homeostatic surface.

2) The existence of an open set $S_r \in \mathrm{R}^7$ in stress-temperature space containing $S_0 : S_0 \subseteq S_r$, as in 1, S_r is assumed simply connected and bounded by a smooth and oriented hyper-surface δS_r called reabsorption activation surface.

3) The existence of an open set $S_g \in \mathrm{R}^7$ in the stress-temperature space containing $S_r : S_r \subseteq S_g$, also S_g is assumed simply connected and bounded by a smooth and oriented hyper-surface δS_g called growth activation surface.

Where the homeostatic state is considered as a particular stable state in which a body is capable to return naturally to this state when submitted to small perturbations by intrinsic biological regulation.

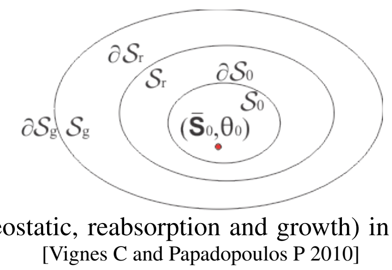

Figure 3. Sets and hyper-surfaces (homeostatic, reabsorption and growth) in the stress-temperature space *Adapted from [Vignes C and Papadopoulos P 2010]

The region $\overline{\mathbf{S}}_0$ models a range of normal stress-temperature loading or activity about the homeostatic state in which produce no growth response and no growth or reabsorption occurs. In this region, material behaves as a conventional

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thermo-elastic material. In view of this, it is expected that internal energy u, the stress $\overline{\mathbf{S}}$ and the heat flux $\mathbf{q}$ will depend only on the Green deformation tensor $\mathbf{E}$ and the temperature θ as describe by the constitutive equations:

$$u = \hat{u}(\mathbf{E}, \theta), \quad \overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta), \quad \mathbf{q}_K = \hat{\mathbf{q}}_K(\mathbf{E}, \theta, \nabla\theta) \tag{24}$$

where the response function of heat flux is subjected to the condition: $\hat{\mathbf{q}}_K(\mathbf{E}, \theta, 0) = 0$

Green deformation tensor is: $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$. For a thermo-elastic material with growth, changes in strain and temperature may initiate growth process (or reabsorption) at the activation surface $\delta \mathrm{S}_g$ (or $\delta \mathrm{S}_r$), this process modified the stress state and the body internal energy, so constitutive equations must take into account this coupling.

To characterize growth, constitutive equations depends on; growth deformation gradient, a scalar measure of isotropic growth-induced hardening κ, the symmetric second order tensor α which measure kinematic growth-induced hardening and growth density and also the total deformation gradient and temperature. The growth variables are defined in the intermediate configuration, so for thermo-elastic materials with growth, the constitutive responses are

$$u = \hat{u}(\mathbf{F}, \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g), \quad \overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{F}, \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g), \quad \mathbf{q}_K = \hat{\mathbf{q}_K}(\mathbf{F}, \theta, \nabla\theta, \mathbf{F}_g, \kappa, \alpha, \rho_g) \tag{25}$$

After manipulate the equations (using the invariance observer principle Gurtin *et al.* (2010)) the response function of $\mathbf{S}$ is given by

$$\overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta, \mathrm{G}) \tag{26}$$

where: $\mathrm{G} = (\mathbf{E}_g, \kappa, \alpha, \rho_g)$ is the set of growth variables and $\mathbf{E}_g = \frac{1}{2}(\mathbf{F}_g^T\mathbf{F}_g - \mathbf{I})$ is the growth deformation tensor

For the rest of response functions using the same arguments used for Eq.(26)

$$u = \hat{u}(\mathbf{E}, \theta, \mathrm{G}), \quad \mathbf{q}_K = \hat{\mathbf{q}}_K(\mathbf{E}, \theta, \nabla\theta, \mathrm{G}) \tag{27}$$

Same dependencies are assumed for heat supply and heat flux irreversible terms

$$r_i = \hat{r}_i(\mathbf{E}, \theta, \mathrm{G}), \quad \mathbf{q}_{Ki} = \hat{\mathbf{q}}_{Ki}(\mathbf{E}, \theta, \nabla\theta, \mathrm{G}) \tag{28}$$

with the restrictions

$$\hat{r}_i(\mathbf{E}, \theta, \mathrm{G}) \mid_{\dot{\mathrm{G}}=0} = 0, \quad \mathbf{q}_{Ki} = \hat{\mathbf{q}}_{Ki}(\mathbf{E}, \theta, \nabla\theta, \mathrm{G}) \mid_{\dot{\mathrm{G}}=0} = 0 \tag{29}$$

The established restrictions means that not occur growth or reabsorption ($\dot{\mathrm{G}} = 0$), so, no energy is expended in growth or reabsorption process and there no mass entering or leaving the body.

Assuming $\overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta, \mathrm{G})$ is invertible for fix values of θ e G, the deformation tensor can be expressed as

$$\mathbf{E} = \hat{\mathbf{E}}(\overline{\mathbf{S}}, \theta, \mathrm{G}) \tag{30}$$

This expression allows for any constitutive expression described in the strain-temperature space be transformed into the stress-temperature space and viceversa.

5. Thermodynamics analysis and entropy balance

Thermodynamics assumptions add restrictions to the constitutive equations. For elastic-thermo-plastic materials with finite deformations, an entropy function can be constructed and use the second law of thermodynamics to obtain those restrictions (Casey (1998), Vignes and Papadopoulos (2010)). Considering a thermo-elastic material with growth submitted to an arbitrary homothermal process for fix G. The energy balance equation Eq. (18) reduces to

$$\rho_K \dot{u} = \overline{\mathbf{P}} \cdot \dot{\mathbf{F}} + \rho_K r = \overline{\mathbf{S}} \cdot \dot{\mathbf{E}} + \rho_K r \tag{31}$$

Clausius-Duhem integral as a consequence of the second law is

$$\int_{t_0}^{t} \frac{r}{\theta} dt = \int_{t_0}^{t} \frac{1}{\theta}(\dot{u} - \frac{\overline{\mathbf{S}} \cdot \dot{\mathbf{E}}}{\rho_K}) dt \tag{32}$$

Defining a potencial $\eta = \hat{\eta}(\mathbf{E}, \theta, \mathrm{G})$, from path independence Clausius-Duhem integral, the entropy function is

$$\dot{\eta} = \frac{r}{\theta} = \frac{1}{\theta}(\dot{u} - \frac{\overline{\mathbf{S}} \cdot \dot{\mathbf{E}}}{\rho_K}) \tag{33}$$

Helmholtz free energy is introduced for all homothermal process with fixed G

$$\psi = \hat{\psi}(\mathbf{E}, \theta, \mathrm{G}) = u - \eta\theta \tag{34}$$

Gibbs equation for an homothermal process for fixed G is obtained from Eq. (31)

$$\rho_K \dot{\psi} = \overline{\mathbf{S}} \cdot \dot{\mathbf{E}} - \rho_K \eta \dot{\theta} \tag{35}$$

expanding the material derivatives

$$\rho_K (\frac{\partial \hat{\psi}}{\partial \theta} + \eta)\dot{\theta} + (\rho_K \frac{\partial \hat{\psi}}{\partial \mathbf{E}} - \overline{\mathbf{S}}) \cdot \dot{\mathbf{E}} = 0 \tag{36}$$

valid for all values of $\dot{\mathbf{E}}$ and $\dot{\theta}$, which are rate-independent, so the Gibbs relations are

$$\eta = \hat{\eta}(\mathbf{E}, \theta, \mathrm{G}) = -\frac{\partial \hat{\psi}}{\partial \theta}, \qquad \overline{\mathbf{S}} = \hat{\overline{\mathbf{S}}}(\mathbf{E}, \theta, \mathrm{G}) = \rho_K \frac{\partial \hat{\psi}}{\partial \mathbf{E}} \tag{37}$$

The energy balance equation written in terms of Helmholtz free energy and entropy using the Gibbs relations is

$$\rho_K \dot{\eta}\theta = \rho_K (r + \tilde{r}) - Div(\mathbf{q}_K + \tilde{\mathbf{q}}_K) - \rho_K (\frac{\partial \hat{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \hat{\psi}}{\partial \kappa}\dot{\kappa} + \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \hat{\psi}}{\partial \rho_g}\dot{\rho}_g) \tag{38}$$

Integrating and using the Reynolds transport theorem, the global balance of entropy is obtained

$$\begin{aligned} \frac{d}{dt}\int_P \rho_K \eta dV \quad &= \quad \int_P \rho_K \frac{r + \tilde{r}}{\theta} dV - \int_{\partial P} \frac{\mathbf{q} + \tilde{\mathbf{q}}}{\theta} dA - \int_P \frac{(\mathbf{q} + \tilde{\mathbf{q}}) \cdot \nabla\theta}{\theta^2} dA + \int_P (\rho\Gamma - Div\mathbf{M})\eta dV \\ &- \quad \int_P \rho_K (\frac{\partial \hat{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \hat{\psi}}{\partial \kappa}\dot{\kappa} + \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \hat{\psi}}{\partial \rho_g}\dot{\rho}_g) dV \end{aligned} \tag{39}$$

assuming that Clausius-Duhem inequality is a valid form of the second law for thermo-elastic materials with growth, which includes an additional entropy entering the body with the new mass

$$\frac{d}{dt}\int_P \rho_K \eta dV \geq \int_P \rho_K \frac{r}{\theta} dV - \int_{\partial P} \frac{\mathbf{q}}{\theta} dA + \int_P (\rho\Gamma - Div\mathbf{M})\eta dV \tag{40}$$

using again the Localization theorem yields

$$\rho_K \dot{\eta}\theta \geq \rho_K r - Div\mathbf{q}_K + \frac{\mathbf{q}_K \cdot \nabla\theta}{\theta} \tag{41}$$

and substituting

$$\rho_K \tilde{r} - Div\tilde{\mathbf{q}}_K - \rho_K (\frac{\partial \hat{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \hat{\psi}}{\partial \kappa}\dot{\kappa} + \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \hat{\psi}}{\partial \rho_g}\dot{\rho}_g) - \frac{\mathbf{q}_K \cdot \nabla\theta}{\theta} \geq 0 \tag{42}$$

Considering an arbitrary homothermal process without irreversible heat terms

$$\frac{\partial \hat{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \hat{\psi}}{\partial \kappa}\dot{\kappa} + \frac{\partial \hat{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \hat{\psi}}{\partial \rho_g}\dot{\rho}_g \leq 0 \tag{43}$$

The fact that all terms were independent of temperature gradient leads to conclusion that the inequality is valid for all process without irreversible heat, so, a process for a thermo-elastic material with heat not irreversible, leads to the standard heat conduction Fourier equation.

$$-\mathbf{q}_K \cdot \nabla\theta \geq 0 \tag{44}$$

6. Stress induced finite growth in human tissues. Example of growth in cardiovascular system

In this section is developed an example that is associated with adaptive heart growth due to abnormal loads, know as cardiac hypertrophy, where there is a ventricular enlargement as a result of volume overload or high filling pressure (Guccione *et al.* (1991), Rodriguez *et al.* (1994), Taber (1995)).

In general terms, blood deoxygenated comes from the body through cava veins and fills the right atrium, which contracts sending blood to the right ventricle which contracts and sends the blood to the lungs through pulmonary artery. Blood oxygenated back to the heart through pulmonary veins filling the left atrium which contracts and sends blood to the left ventricle that when contracts, sends blood to the circulatory system through the aorta as schematically represented in Fig. (4.1), more details can be founded in an excellent review paper presented by Taber (1995).

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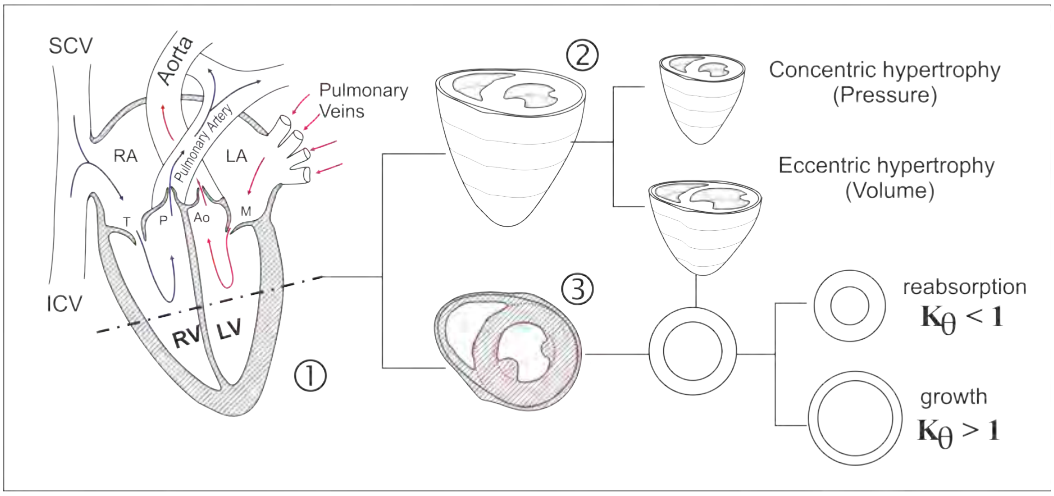

Figure 4. Schematic of mature heart; components, blood circulation and ventricular section cut. 1) Frontal section cut of mature heart. 2) Isometric cut, ventricular enlargement as a result of volume overload or high filling pressure. 3) Cross section cut, left ventricle cross section [dotted line] and circumferential stretch ratio [K_θ] describing growth and reabsorption) * The heart scheme (1) is adapted from [Taber L A 1995], being; ISC, SVC: inferior and superior cava veins; RA, LA: Right and Left Atriums; RV, LV: Right and Left Ventricles; T, P, Ao and M are tricuspid, pulmonary, aortic and mitral valves

The model presented, is used to know how growth leads to residual stress and how stress may determine a growth pattern in a tissue. A growth displacement field is specified in an unloaded cylindrical tube and residual stress fields resulting from different growth fields are studied. This model consists of a cylindrical tube (of the left ventricle [LV] see Fig. 4) considered incompressible, elastic and isotropic used to illustrate how the circumferential growth give rise to a transmural distribution of residual stresses that would cause the cylinder to change shape when cut (Choung and Fung (1986), Rodriguez *et al.* (1994)).

The residual stress present in the cylinder after growth can be calculated assuming that growth strains generates stresses similar to those of loading, then a constitutive equations for isotropic material can be used. For growth deformation gradient $\mathbf{F}_g$ the following displacement field is prescribed:

$$\rho = \mathrm{R}, \quad \varphi = \mathrm{K}_\theta(\mathrm{R})\Theta, \quad \xi = \mathrm{Z} \tag{45}$$

The point P in the reference configuration $\mathcal{B}$ (stress-free) has coordinates $(\mathrm{R}, \Theta, \mathrm{Z})$, the growth deformation gradient $\mathbf{F}_g$ maps the original state $\mathcal{B}$ in a new locally stress-free state in an intermediate configuration $\mathcal{B}_g$ where P has coordinates (ρ, φ, ξ) as shown in Fig. (5.a)

The term $\mathrm{K}_\theta(\mathrm{R})$ is the circumferential growth stretch ratio which depends on the radius and is assumed constant, so, when $\mathrm{K}_\theta > 1$ growth occurs and when $\mathrm{K}_\theta < 1$ reabsorption occurs. Displacements field is incompatible as shown in Fig. (5. b, c) for values of $\mathrm{K}_\theta < 1$ and $\mathrm{K}_\theta > 1$, in reabsorption case; a discontinuity appears which is not permissible if the total growth deformation must be compatible. In growth case, a material superposition appears which cannot be occur in a compatible deformation.

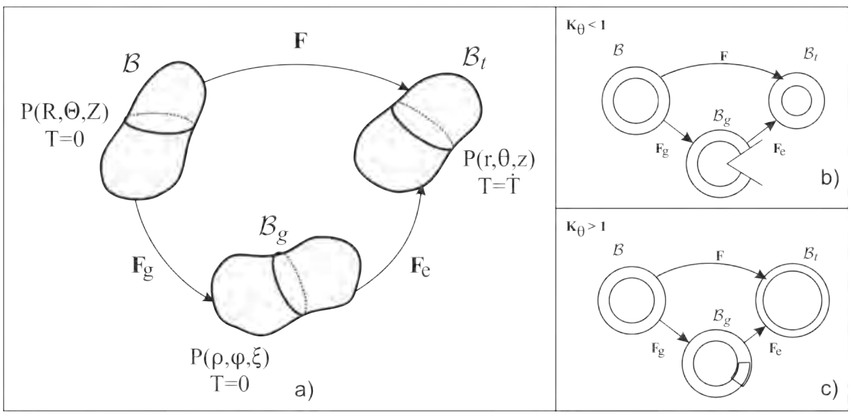

Figure 5. a) Multiplicative decomposition of the growth deformation gradient and b), c) Cylindrical models of the left ventricle after uniform circumferential growth. *b), c) Adapted from [Rodriguez *et. al* 1994]

Using the displacements field established in Eq. (45) the growth deformation gradient is defined as:

$$\mathbf{F}_g = \begin{bmatrix} \dfrac{\partial\rho}{\partial \mathrm{R}} & \dfrac{1}{\mathrm{R}}\dfrac{\partial\rho}{\partial\Theta} & \dfrac{\partial\rho}{\partial \mathrm{Z}} \\ \rho\dfrac{\partial\varphi}{\partial \mathrm{R}} & \dfrac{\rho}{\mathrm{R}}\dfrac{\partial\varphi}{\partial\Theta} & \rho\dfrac{\partial\varphi}{\partial \mathrm{Z}} \\ \dfrac{\partial\xi}{\partial \mathrm{R}} & \dfrac{1}{\mathrm{R}}\dfrac{\partial\xi}{\partial\Theta} & \dfrac{\partial\xi}{\partial \mathrm{Z}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\rho}{\mathrm{R}}\mathrm{K}_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

In order to establish the overall growth deformation field compatibility, an adicional elastic component is required that will be used in the constitutive equations of the material to determine the residual stress that must satisfy equilibrium and zero stress boundary condition

The field which maps the state $\mathcal{B}_g$ in the final state $\mathcal{B}_t$ where P has coordinate $(\mathrm{r}, \theta, \mathrm{z})$ is:

$$\mathrm{r} = \mathrm{r}(\rho), \quad \theta = \eta_\theta(\rho)\varphi, \quad \mathrm{z} = \epsilon \mathrm{z} \tag{47}$$

then the elastic component of the growth deformation gradient is given by:

$$\mathbf{F}_e = \begin{bmatrix} \dfrac{\partial \mathrm{r}}{\partial\rho} & \dfrac{1}{\rho}\dfrac{\partial \mathrm{r}}{\partial\varphi} & \dfrac{\partial \mathrm{r}}{\partial\xi} \\ \mathrm{r}\dfrac{\partial\theta}{\partial\rho} & \dfrac{\mathrm{r}}{\rho}\dfrac{\partial\theta}{\partial\varphi} & \mathrm{r}\dfrac{\partial\theta}{\partial\xi} \\ \dfrac{\partial \mathrm{z}}{\partial\rho} & \dfrac{1}{\rho}\dfrac{\partial \mathrm{z}}{\partial\varphi} & \dfrac{\partial \mathrm{z}}{\partial\xi} \end{bmatrix} = \begin{bmatrix} \dfrac{d\mathrm{r}(\rho)}{d\rho} & 0 & 0 \\ 0 & \dfrac{\mathrm{r}}{\rho}\eta_\theta & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \tag{48}$$

A suitable choice of the term $\eta_\theta(\rho)$ allows the deformation gradient total growth is compatible restoring the compatibility of displacements Θ. In the case where K_θ is a constant, the simplest choice is $\eta_\theta(\rho) = 1/\mathrm{K}_\theta$.

The incompressibility constraint is only applied to the elastic component of the growth deformation gradient, so the third invariant of the right Cauchy-Green stress tensor $\mathbf{C}$ is the unit:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} \left(\dfrac{d\mathrm{r}(\rho)}{d\rho}\right)^2 & 0 & 0 \\ 0 & \left(\dfrac{\mathrm{r}}{\rho}\eta_\theta\right)^2 & 0 \\ 0 & 0 & (\epsilon)^2 \end{bmatrix} \tag{49}$$

then;

$$\mathbf{I}_3 = \left[\frac{d\mathrm{r}(\rho)}{d\rho}\frac{\mathrm{r}}{\rho}\eta_\theta\epsilon\right]^2 = 1 \tag{50}$$

which can be integrated to obtain an expression for the growth radius r

$$\int \mathrm{r}d\mathrm{r} = \int \frac{\mathrm{K}_\theta}{\epsilon}\rho d\rho \tag{51}$$

being $\rho = \mathrm{R}$ and $\eta_\theta = \dfrac{1}{\mathrm{K}_\theta}$; hence

$$\frac{\mathrm{r}^2}{2} + \mathrm{C}_1 = \frac{\mathrm{K}_\theta}{\epsilon}\frac{\mathrm{R}^2}{2} \qquad \Leftrightarrow \qquad \mathrm{r} = \sqrt{\frac{\mathrm{R}^2\mathrm{K}_\theta}{\epsilon} + \mathrm{C}_2} \tag{52}$$

The components of the Green-St. Venant strain tensor referred to the coordinates of the growth configuration $\mathcal{B}_g$ are calculated from the expression:

$$\mathbf{E} = \frac{1}{2}\left[\mathbf{F}_e{}^{\mathrm{T}}\mathbf{F}_e - \mathbf{I}\right] = \begin{bmatrix} \left(\dfrac{d\mathrm{r}(\rho)}{d\rho}\right)^2 - 1 & 0 & 0 \\ 0 & \left(\dfrac{\mathrm{r}}{\rho}\eta_\theta\right)^2 - 1 & 0 \\ 0 & 0 & (\epsilon)^2 - 1 \end{bmatrix} \tag{53}$$

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then;

$$\mathbf{E}_{\rho\rho} = \frac{1}{2}\left[\left(\frac{\mathrm{RK}_\theta}{\mathrm{r}\epsilon}\right)^2 - 1\right]; \qquad \mathbf{E}_{\varphi\varphi} = \frac{1}{2}\left[\left(\frac{\mathrm{r}}{\mathrm{RK}_\theta}\right)^2 - 1\right]; \qquad \mathbf{E}_{\xi\xi} = \frac{1}{2}\left[(\epsilon)^2 - 1\right] \tag{54}$$

For this example it is assumed the stress-strain relationship developed by Rodriguez *et al.* (1994) and Guccione *et al.* (1991) where it is used the strain energy function proposed by Choung and Fung (1986):

$$\mathrm{W} = \frac{\mathrm{C}}{2}\left(\mathrm{e}^{\mathrm{Q}} - 1\right) \tag{55}$$

where C is a material constant and Q is a function of the principal deformation components which define the material symmetry of the tissue considered, for the isotropic case de function Q is given by:

$$\mathrm{Q} = 2\mathrm{b}_1\left(\mathbf{E}_{\rho\rho} + \mathbf{E}_{\varphi\varphi} + \mathbf{E}_{\xi\xi}\right) \tag{56}$$

where b_1 is a constant which depends on material (material constant values from Guccione *et al.* (1991); being b_1=4.24 and C=0.765 kPa). Stress is obtained from:

$$\mathbf{T}_{ij} = \frac{1}{2}\mathbf{F}_{iS}\mathbf{F}_{jT}\left(\frac{\partial \mathrm{W}}{\partial \mathbf{E}_{ST}} + \frac{\partial \mathrm{W}}{\partial \mathbf{E}_{TS}}\right) - \mathrm{p}\delta_{ij} \tag{57}$$

where p is the hydrostatic pressure which enters in the constitutive equations as a Lagrange multiplier. The components of $\mathbf{T}_{ij}$ are determined from expressions 48, 54, 55, 56:

$$\mathbf{T}_{rr} = \frac{1}{2}\left(\frac{\mathrm{RK}_\theta}{\mathrm{r}}\right)^2\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\left(\frac{\mathrm{RK}_\theta}{\mathrm{r}}\right)^2 + \left(\frac{\mathrm{r}}{\mathrm{RK}_\theta}\right)^2 - 2\right]}\right] - \mathrm{p}(r) \tag{58}$$

$$\mathbf{T}_{\theta\theta} = \frac{1}{2}\left(\frac{\mathrm{r}}{\mathrm{RK}_\theta}\right)^2\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\left(\frac{\mathrm{RK}_\theta}{\mathrm{r}}\right)^2 + \left(\frac{\mathrm{r}}{\mathrm{RK}_\theta}\right)^2 - 2\right]}\right] - \mathrm{p}(r) \tag{59}$$

using Eq. 52 and substituting;

$$\mathbf{T}_{rr} = \frac{\left(\mathrm{r}^2 - \mathrm{C}_2\right)\mathrm{K}_\theta}{2\mathrm{r}^2}\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\frac{(\mathrm{r}^2 - \mathrm{C}_2)\mathrm{K}_\theta}{\mathrm{r}^2} + \frac{\mathrm{r}^2}{(\mathrm{r}^2 - \mathrm{C}_2)\mathrm{K}_\theta} - 2\right]}\right] - \mathrm{p}(r) \tag{60}$$

$$\mathbf{T}_{\theta\theta} = \frac{\mathrm{r}^2}{2\left(\mathrm{r}^2 - \mathrm{C}_2\right)\mathrm{K}_\theta}\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\frac{(\mathrm{r}^2 - \mathrm{C}_2)\mathrm{K}_\theta}{\mathrm{r}^2} + \frac{\mathrm{r}^2}{(\mathrm{r}^2 - \mathrm{C}_2)\mathrm{K}_\theta} - 2\right]}\right] - \mathrm{p}(r) \tag{61}$$

considering: $f = \mathrm{K}_\theta\left(1 - \frac{\mathrm{C}_2}{\mathrm{r}^2}\right)$ gives

$$\mathbf{T}_{rr} = \frac{f}{2}\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\frac{(f-1)^2}{f}\right]}\right] - \mathrm{p}(r); \quad \mathbf{T}_{\theta\theta} = \frac{1}{2f}\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\frac{(f-1)^2}{f}\right]}\right] - \mathrm{p}(r) \tag{62}$$

from equilibrium equations are obtained:

$$\frac{d\mathbf{T}_{rr}}{d\mathrm{r}} + \frac{\mathbf{T}_{rr} - \mathbf{T}_{\theta\theta}}{\mathrm{r}} = 0; \qquad \frac{d\mathbf{T}_{r\theta}}{d\mathrm{r}} + 2\frac{\mathbf{T}_{r\theta}}{\mathrm{r}} = 0; \qquad \frac{d\mathbf{T}_{rz}}{d\mathrm{r}} + \frac{\mathbf{T}_{rz}}{\mathrm{r}} = 0 \tag{63}$$

Since zero-stress boundary conditions are assumed on the inner and outer walls, just the first of the above equations has to be solved. Integrating this equation and using the expressions of (62):

$$\mathbf{T}_{rr} = \int_{r_2}^{r}\frac{\mathbf{T}_{\theta\theta} - \mathbf{T}_{rr}}{\mathrm{r}}d\mathrm{r} + \mathbf{T}_{rr}|_{r=r_2} = \int_{r_2}^{r}\frac{1}{2}\left(\frac{1}{f} - f\right)\left[\mathrm{Cb}_1\mathrm{e}^{\mathrm{b}_1\left[\frac{(f-1)^2}{f}\right]}\right]d\mathrm{r} \tag{64}$$

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where; $\mathbf{T}_{rr}$ in $r = r_2$ is the radial stress at the outer grown wall and is specified as zero, the internal pressure is $-\mathbf{T}_{rr}$ in $r = r_1$. The model was solved numerically using MathCAD version 15.0 because not has analytical solution, by specifying the growth outer radius r_2 and solving for final inner radius value that gives a zero transmural pressure. The processing scheme is the following: an initial value of external radius r_2 is specified. With Eq. (52), C_2 and r_1 are obtained. Then; the roots of $\mathbf{T}_{rr}$ are calculated to obtain new values of r_2 and r_1, which enter again in the scheme to calculate a new value of C_2 and new values of r_2 and r_1, the scheme loops until the difference $(r_{2i+1} - r_{2i})$ approaching to zero (tolerance value assumed was 10^{-12})

The results are shown in the graphics of Fig. (6) for different values of K_θ. Model results shown how circunferencial growth can give rise to a transmural distribution of residual stress that cause the cylinder change shape when cut; stretch (growth) or shorten (reabsorption).

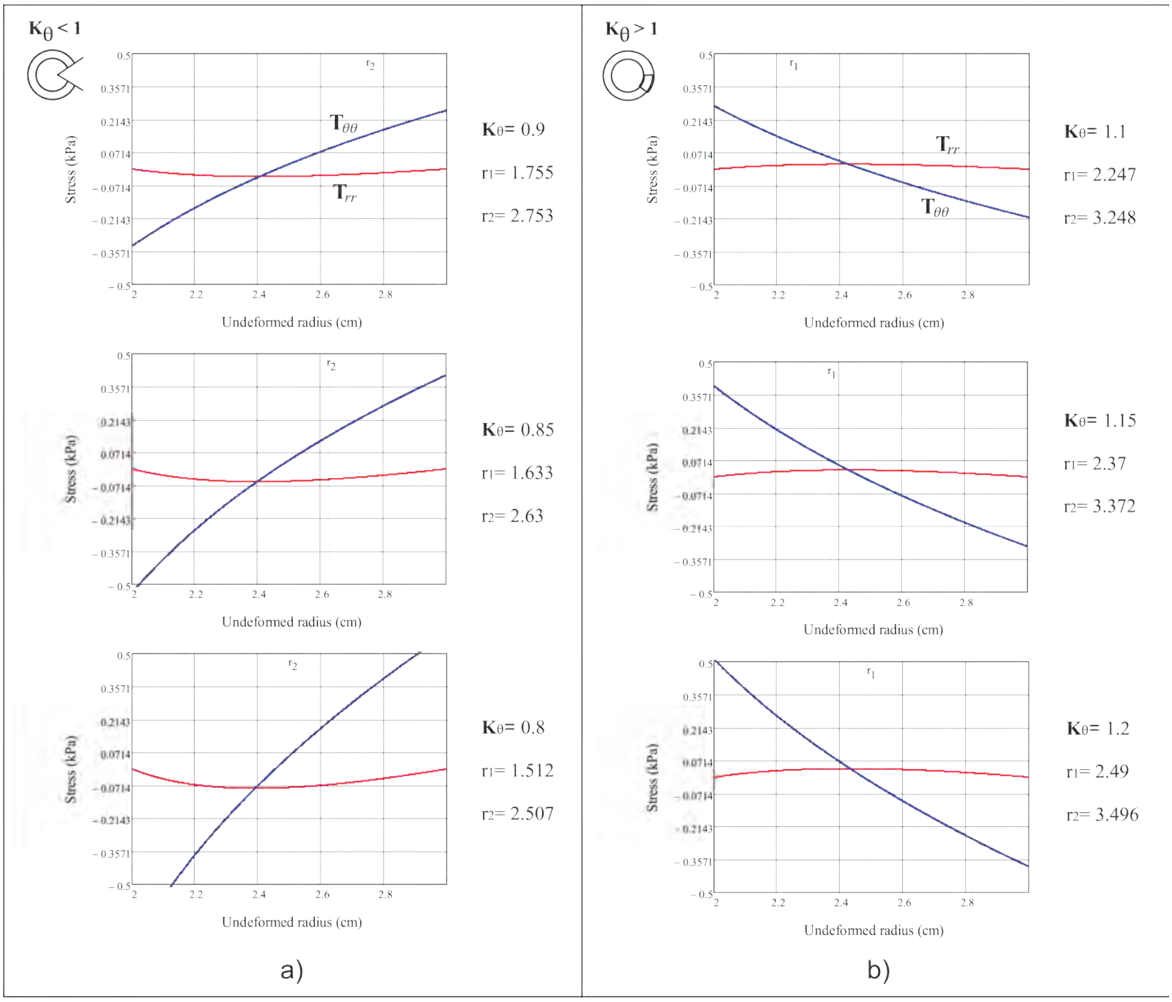

Figure 6. Radial and circunferencial stresses for different K_θ values; a) $K_\theta < 1$ Reabsorption b) $K_\theta > 1$ Growth

Considering $\epsilon = 1$ and initial values of internal and external radius of 2 cm and 3 cm respectively; three values of K_θ for the case of reabsorption were analyzed: 0.9, 0.85 and 0.8 where the grown radius values, internals and externals, satisfying the equilibrium were: 1.755 and 2.753 cm, 1.633 and 2.63 cm and 1.512 and 2.507 cm as shown in Fig. (6.a)

Residual stress shown zero values of radial stress $\mathbf{T}_{rr}$ at inner (endocardium) and outer (epicardium) walls since the cylinder was unloaded. Circunferencial stress $\mathbf{T}_{\theta\theta}$ shows nonlinear behavior from compression at the endocardium to tension at the epicardium.

Also, three values of K_θ for the case of growth were analyzed: 1.1, 1.15 and 1.2 where the grown radius values, internals and externals, satisfying the equilibrium were: 2.247 and 3.248 cm, 2.37 and 3.372 cm and 2.49 and 3.496 cm as shown in Fig. (6.b)

As in reabsorption, residual stress shown zero values of radial stress $\mathbf{T}_{rr}$ at inner and outer walls since the cylinder was unloaded but stress gradient for growth case were reversed from those of reabsorption and circunferencial stress $\mathbf{T}_{\theta\theta}$ were compressive at the epicardium and tensile at the endocardium.

Finally, with this study and revision of growth theory using continuum mechanics and motivated by previous works of the authors in hip prostheses biomechanics and in dental biomechanics O'Connor *et al.* (2008), O'Connor *et al.* (2010), this work can be considered as a first step for future works in the implementation of material models using FEM with the capability to grow or reabsorb as a response of different stress levels, the results could help medicine specialists in order to predict zones of bone formation associated to growth and bone mass loose zones associated to reabsorption in the case of orthopaedics, or growth or reabsorption in other hard or soft tissues under different load levels.

7. Conclusions

It was presented an study and a revision of the theory for thermo-elastic materials with growth and its treatment using continuum mechanics involving balance laws; mass, linear and angular momentum and energy balances, multiplicative decomposition of the deformation gradient, the constitutive equations and thermodynamics and entropy balance.

An example of finite adaptive heart growth due to abnormal loads using the multiplicative decomposition of the deformation gradient for a thermo-elastic material was analyzed, and results obtained were discussed. Also results obtained matched with previous results presented by other authors showing agreement.

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