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Partial Synchronization Groups in the Kuramoto Model: Definitions and Numerical Experiments.

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Abstract. *The focus of this study is to shed more light over the important Kuramoto Model, describing the phase dynamics of a finite set of oscillators, coupled via a network. We take an empirical numerical approach, in order to investigate the influence over some synchronization phenomena of different combination of coupling topologies. The measurement of the synchronization is performed through the usual parameter of order, associated with a metric for partial synchronization over the edges of the graph. This second measure allowed us to study the formation of clusters of oscillators with distinct synchronized behavior. The concepts of Fixed Phase Groups and Almost-synchronization are presented.*

Keywords: *Kuramoto Model, Partial Synchronization, Clusters of Synchronization, DDE.*

1. INTRODUCTION

In the last two decades, the study about relations between collective behavior and the structure of the graph connecting its members has been a very attractive field to scientist from different areas (Strogatz, 2001; Barabási, 2003; Newman, 2003). From ecology to social sciences, from control system to computer graphics (Jadbabaie *et al.*, 2004), we are studying how groups of autonomous agents, like flock of birds, school of fish, people (Low, 2000; Vicsek, 2001), robots and unmanned vehicles (Tanner *et al.*, 2003b,a), can achieve a synchronized motion consensus, without a centralized control.

We study here the Kuramoto Model (KM), a system of differential equations of phase variables describing the dynamics of an ensemble of $N \geq 1$ coupled oscillators. This option was made due to: (1) its simplicity and generality (Pikovsky *et al.*, 2003); (2) a complete analytical theory for its dynamics is still under development (Acebrón *et al.*, 2005; Coutinho *et al.*, 2013; Jadbabaie *et al.*, 2005); (3) the importance of its real applications, such as, (Bageston, 2011), collective motion of agents in space (Paley *et al.*, 2005), synchronization of lasers (Wang and Ghosh, 2007) and a variety of social and biological systems (Gleeson *et al.*, 2012). The concept of groups of synchronization is a feature of special interest, mainly because it is related with communities of members into a system with distinct dynamic behavior.

We assume the following equation for each oscillator $i = 1, \ldots, N$ in our KM

$$\theta_i'(t) = \omega_i - k \sum_{j=1}^{N} A_{ij} \sin(\theta_i(t) - \theta_j(t - \tau)), \qquad (1)$$

where $\theta_i(t)$ is the phase variable of the i-th oscillator, assuming values in the real line $\mathbb{R}$, which can also be seen as angle in the unit circumference, $\theta_i(t) \mod 2\pi$. The prime accent $'$ indicates the differential related to the time variable t. The natural frequencies of the oscillators are given by the parameter $\boldsymbol{w} = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^N$, while $k \in \mathbb{R}$ is the coupling strength, adjusting the intensity of the influence between neighbor oscillators in the coupling graph. The undirected coupling graph[1] is expressed by its adjacency matrix $A_{N \times N} \doteq (A_{ij})$, with $A_{ii} = 0$; $A_{ij} = 1$, if oscillators i, j are connected; and $A_{ij} = 0$, otherwise.

The coupling time delay between oscillators is the parameter $\tau \geq 0$ (Yeung and Strogatz, 1999); if $\tau = 0$, the influence is instantaneous, otherwise, the MK (1) is a delayed differential system, which requires specific methods for its numerical integration as (Wirkus, 1999). We fix $\boldsymbol{\theta}^0 = \left(\theta_1^0, \ldots, \theta_N^0\right) \in S^N$ as the initial condition for the KM (1).

[1]When it is clear from the context, we identity the coupling network with its adjacency matrix. Thus, the terms *graph* and *network* are applied with the same meaning.

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The aim of this work is to provide and illustrate two definitions of partial synchronization groups, with concepts from (Gómez-Gardeñes *et al.*, 2007; Baptista *et al.*, 2012), comparing its meaning with usual definitions of synchronization between two oscillators (Pikovsky *et al.*, 2003).

In this work we assume zero delay time between the influence among oscillators, $\tau = 0$. We are already developing further studies in order numerically explore the onset of synchronization groups, considering different distributions for the natural frequency; the presence of a not null delay time; and coupling graphs as complex networks like Erdös-Rényi (ER), Barabási-Albert (BA), Watts-Strogatz (WS) (Barabási, 2003).

1.1 Division of sections

In the next section we included some common definitions and metrics related to synchronization, for the sake of establishing a clear terminology. Then, in following section, we define and discuss two types of groups of synchronization. In both sections, numeric experiments are presented to illustrate ideas.

2. METRICS AND TYPES OF SYNCHRONIZATION

2.1 Synchronization metrics

We begin presenting measures that will help us to recast some usual concepts of phase synchronization. The first one, the well known *order parameter*, is a complex number related with the mean field of all oscillators in the system. It is defined as

$$\rho(t) \doteq \frac{1}{N}\sum_{j=1}^{N} e^{\imath\theta_j(t)}.$$

We denote the norm of the order parameter $|\rho|$ as r and its argument as ψ, so $re^{\imath\psi} \doteq \rho$. The term $\psi = \psi(t) \in [0, 2\pi)$ gives the direction of the mean field of the system, while $r = r(t) \in [0, 1]$ is an index of the spreading of the system. If $r = 0$, then all oscillator are uniformly distributed over the unit circumference, and, if $r = 1$, then $\theta_1 = \ldots = \theta_N$. So, if the norm of the order parameter $r(t)$ converges, it means that the phase difference between every two oscillators $\theta_i(t) - \theta_j(t)$ also converges. On the other hand, even after some large transient time t_r, $r(t)$ may not converge. So, $\bar{r}(t)$ and $\Delta r(t)$ will respectively denote the mean value and the maximum variation of $r(t)$, for $t \in I \doteq (t_t, t_r + \Delta t)$ and some large Δt.

For every two oscillators i, j in the network, the second metric $r_{\text{link}(ij)} \in [0, 1]$ indicates how much the *mean phase difference* between θ_i and θ_j varies. For some large enough transient time t_r, we define

$$r_{\text{link}(ij)} \doteq \left\| \lim_{\Delta t\to\infty} \frac{1}{\Delta t}\int_{t_r}^{t_r+\Delta t} e^{\imath(\theta_i(t)-\theta_j(t))}\,\mathrm{d}t \right\|.$$

One can easily check that if $\theta_i(t) \equiv \theta_j(t) + \eta$, then the exponent in the previous integral is constant and $r_{\text{link}(i,j)} = 1$. Nevertheless, if even after the transient $\theta_i(t) - \theta_j(t) \mod 2\pi$ assumes every possible value over the unit circumference, then $r_{\text{link}(ij)}$ is close to zero (Gómez-Gardeñes *et al.*, 2007). Now, we average the contributions of all neighbor oscillators i, j under the graph A to write

$$r_{\text{link}} = \frac{1}{E}\sum_{i,j=1}^{N} A_{ij} r_{\text{link}(ij)},$$

where E is the quantity of undirected edges of the graph. Observe that $r_{\text{link}} \in [0, 1]$. Besides, $r_{\text{link}} = 1$ if and only if, the phase difference between every two oscillators in the network is constant, so, r also converges.

Furthermore, if r_{link} gradually decreases from 1, the system starts to display weaker forms of synchronization. When r_{link} is close to zero, the orbits of every neighbor oscillators are not related. For this reason, r_{link} is considered a measure of partial synchronization.

2.2 Synchronization Types

There are many types of phase synchronization and they may be called under different names in the literature. This section highlights the definitions of synchronization studied throughout this paper, from the stronger one to the weaker one. Here, $\theta(t)$ e $\tilde{\theta}(t)$ denote the phase of two arbitrary oscillators.

We say that two oscillators achieve *Perfect Phase Synchronization* when $\left|\theta(t) - \tilde{\theta}(t)\right| \to 0$. It is immediate that this conditions is equivalent to $r_{\text{link}} = 1$ and $r \to 1$. Higher synchronization order $n : m$ is characterized by $|n\theta(t) - m\theta(t)| \to 0$, with $n, m \in \mathbb{Z}$. In this work, we consider only Perfect Phase Synchronization, *i.e.*, synchronization of order $1 : 1$.

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The regime *Fixed Phase Synchronization* occurs when $\theta(t) \to \tilde{\theta}(t) + \eta$, with $\eta \in [0, 2\pi)$. In other words, the phase difference converges to a constant angle η. If $\eta = \pi$, it is said that the oscillator are under *Anti-Phase Synchronization*. Of course, If $\eta = 0$, these oscillators are perfectly phase synchronized. For this type of synchronization, $r_{\text{link}} = 1$ and $r \to c$, with $c \in [0, 1]$.

Phase Synchronization happens when we no longer require convergence, like before. It is said that two oscillator exhibit this behavior if

$$\left|\theta(t) - \left(\tilde{\theta}(t) + \nu\right)\right| \leq \varepsilon, \tag{2}$$

for some $\varepsilon > 0$ and for every t after a transient. The metric r doesn't converge now, but Δr is low. The smaller is the maximum variation ε, the closer to one is r_{link}.

If we weaken the later definition, allowing (2) to be valid only for *most of the time*, instead of all the time t after transient t_r, we say that oscillators present *Almost-Synchronization*. The typical case of Almost-Synchronization is known as *Phase Slip*. This phenomenon is characterized by long time intervals where the phase difference between oscillators is constant (or with slow variation), intermingled by one fast additional cycle in the phase difference, which practically appears as a jump (Pikovsky *et al.*, 2003). Again, r doesn't converge, but Δr can be maximal, in the presence of Phase Slips. The lesser is the proportion of time that the phase difference between oscillators still constant, the smaller is the metric r_{link}.

When none of these types of synchronization takes place, we say that oscillators are *Not Synchronized*.

2.3 Synchronization Types in the Space Parameter

The purpose of this subsection is to exemplify the synchronization concepts previously introduced. We perform numerical integration of the KM (1) with $N = 2$, mutually coupling and $\boldsymbol{\theta}^0 = (0, \pi)$, for different values of dissonance $\nu \doteq \omega_2 - \omega_1$ (we kept $\omega_1 = 1$) and coupling strength k.

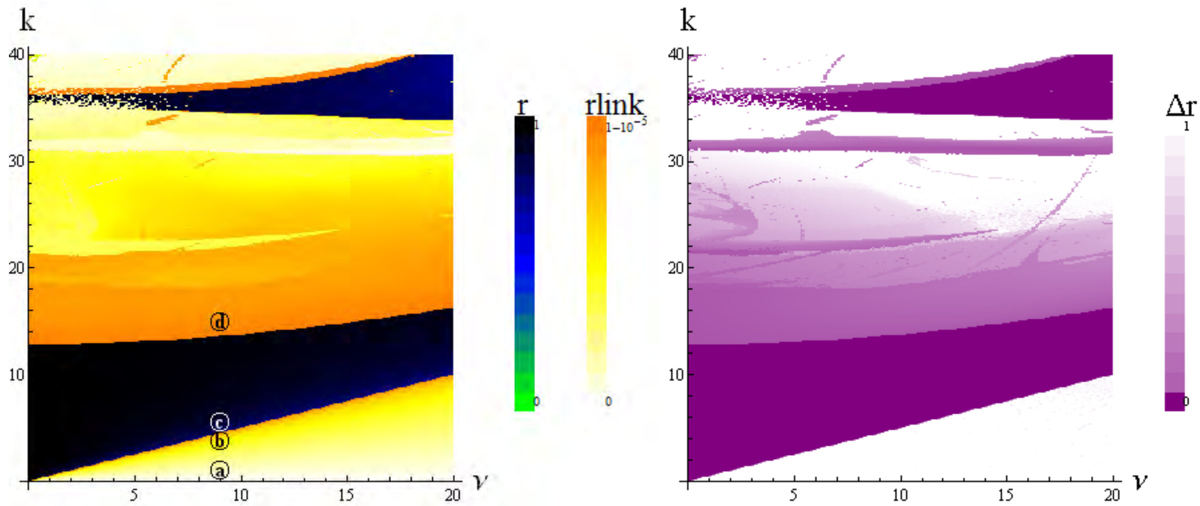

Figure 1. Numerical Integration of KM for two mutually coupled oscillators: synchronization metrics in the parameter space dissonance ν versus coupling strength k. Both figures represent the same experiments. The left side figure shows metrics r, r_{link} and the right side one shows Δr. See remark 1 for color coding explanation. Letter marks represent typical synchronization types: (a) Non-Synchronization,(b) Phase Slips, (c) Fixed Phase Synchronization and (d) Phase Synchronization.

Remark 1 *In the left side diagram of Figure 1, the first (black-blue-green) color coding range corresponds to experiments whose order parameter norm,* r*, converges,* i.e.*, there is Fixed Phase Synchronization (*$\Delta r = 0$ *e* $r_{link} = 1$*). In this case, the lesser is the phase difference* η*, the closer is the color to black. Fixed Phase Synchronization near to Anti-Phase,* $\eta = \pi$*, receives green tones.*

The second (orange-yellow-white) color coding range corresponds to experiments where r *doesn't converge,* i.e.*,* $\Delta r > 0$ *and* $r_{link} < 1$*. Here, the less synchronized oscillators are, the closer is the color to white. Weaker forms of synchronization are presented with colors between orange and yellow.*

In the second diagram, the (white-magenta) color coding range corresponds to maximum variation of r*, given by* Δr*. The area in white means maximal* Δr*: one oscillator spins around the other even after the transient. The areas coded in darkest magenta represent* $\Delta r = 0$*, so,* r *converges.*

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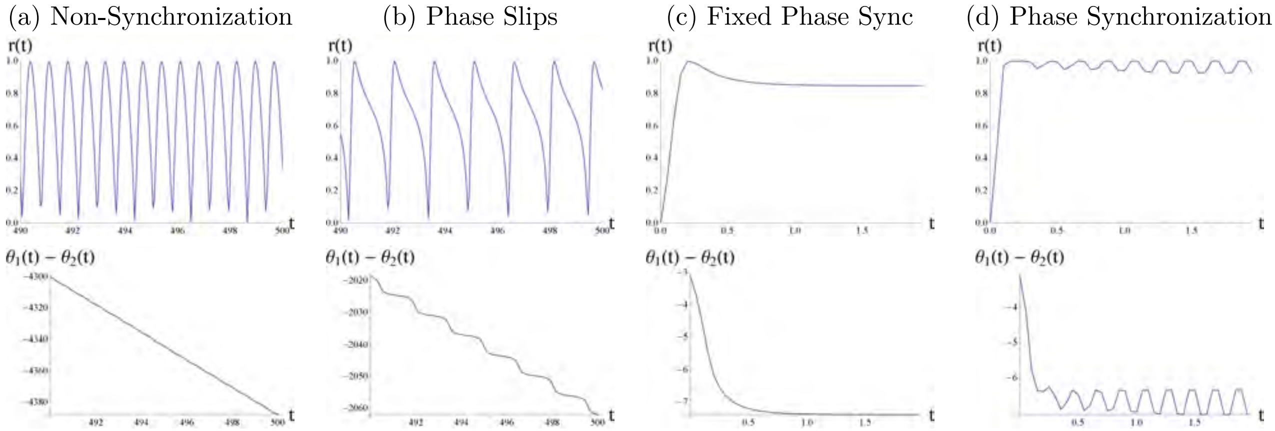

Figure 2. Synchronization Types marked as letters in Figure 1. For the each type of sync, upper plots show the norm of the order parameter $r(t)$ and bottom plots the phase difference $\theta_1(t) - \theta_2(t)$.

We describe now different synchronization types that are found when we fix a value for dissonance ν and raise the coupling strength starting from $k = 0$. The choice of parameters follows the letter marks in Figure 1.

- Non-Synchronization: letter mark (a) in Figure 1. The metric r_{link} is close to zero and Δr is one. We can see in Figure 2 (a) that the phase difference between oscillators decreases monotonically. When both oscillators have the same angle in the unit circle, *i.e.*, $(\theta_1(t) - \theta_2(t)) \mod 2\pi = 0$, then the value of $r(t)$ is one, while $(\theta_1(t) - \theta_2(t)) \mod 2\pi = \pi$, yields $r(t) = 0$.
- Phase Slips: letter mark (b) in Figure 1. The metric r_{link} is almost one, but Δr still equals to one. The closer ν, k is to the dark region, the more time oscillators present almost constant phase difference. We see in Figure 2 (b) the jumps between phase synchronization periods occur approximately every two units of time.
- Fixed Phase Synchronization: letter mark (c) in Figure 1. Inside the larger dark region, r_{link} is one and Δr is zero, so, r converges. If we raise k, without leaving this dark region, the phase difference between oscillators decreases and achieve Perfect Phase Synchronization. Figure 2 (c) reveals that the second oscillators give one spin around the first one, before their phase difference in the unit circumference converges to $-7.4 \mod 2\pi = \approx -1.1$.
- Phase Synchronization: letter mark (d) in Figure 1. Just outside the larger dark region, the measures r_{link} and Δr are no longer 1 and 0, respectively, but still close to these values. So, $|\theta_1(t) - (\theta_2(t) + \nu)| \leq \varepsilon$ holds for every time t after a transient. If we raise even more the parameter k, without entering the second dark region,the distance ε between oscillators also raises. We can see in Figure 2 (d) that, again after the second oscillators gives one spin around the first one, their phase difference assumes a regular behavior, ranging in the unit circle from approximately from $-7 \mod 2\pi \approx -0.71$ to $-7.6 \mod 2\pi \approx -1.31$.

This numerical experiment also illustrates an analytical result showing a universal characteristic (Schimansky-Geier, 1986; Strogatz, 2000; Pikovsky *et al.*, 2003), valid to any type of coupled oscillators. Considering small dissonance, the result says that oscillator's phase will synchronize with a critical coupling strength k_c, which is a crescent function of the dissonance. The critical k_c can be seen as the lower border of the greater dark region in the Figure 1.

3. NETWORKS AND GROUPS OF SYNCHRONIZATION

3.1 Definition of Synchronization Groups

In the same way that distinct types of synchronization between two oscillators may exist, different groups of synchronization can also be defined. In this work, we consider two types of synchronization groups based on the following procedure. First, we build a weighted undirected graph $D_{N\times N}$ with the same edges than $A_{N\times N}$ and weights given by $r_{\text{link}ij}$, *i.e.*, $D_{ij} = D_{ji} = \max\{A_{ij}, A_{ji}\} \cdot r_{\text{link}(ij)}$, where the maximum between A_{ij}, A_{ji} is taken to deal with direct or undirected graphs. Then, we define a non-weighted undirected graph $\tilde{D}_{N\times N}$ using only the edges from D greater or equal then a constant threshold T, more precisely

$$\tilde{\mathcal{D}}_{ij} = \begin{cases} 1 & \text{if } D_{ij} \geq T; \\ 0 & \text{otherwise} \end{cases} .$$

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Our synchronization groups are defined as the connected components of graph $\tilde{D}$, so every vertex of $\tilde{D}$ belongs to one and only one group, even if there is only this single vertex in the group.

We define *Fixed Phase Synchronization Groups* by choosing $T = 1$. So, every two oscillators in each group of this type present Fixed Phase Synchronization between them. It is immediate that, considering only the oscillators of each Fixed Phase Synchronization Group, the norm of parameter of order converges.

We define *Almost-Synchronization Groups* by choosing $T = r_{\text{link}}$. So, every two neighbors oscillators i, j in each group of this type will have partial synchronization metric $r_{\text{link}(i,j)}$ greater or equal than the mean value of this metric, given by r_{link}. As a result, oscillators in the same group of this kind can exhibit all different synchronization types described in Section 2.2

It is clear that every Fixed Phase Synchronization Group must belong to one and only one Almost-Synchronization Group, since $r_{\text{link}} \leq 1$. On the other hand, there may exist none or any number of distinct Fixed Phase Synchronization Groups inside an Almost-Synchronization Group.

3.2 Examples of Synchronization Groups

We illustrate here how those two different groups of synchronization may appear. We consider a KM (1) with $N = 8$ and $\tau = 0$. The coupling graph and natural frequencies are show in Figure 3 (a).

(a) Coupling Graph and Natural Frequencies

(b) Synchronization Groups: Coupling Strength $k = 0.11$

(c) Synchronization Groups: Coupling Strength $k = 0.21$

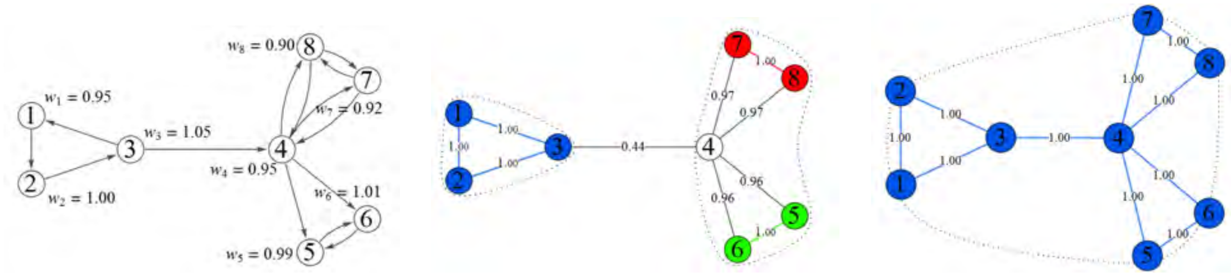

Figure 3. (on-line color) Synchronization groups found in the KM (1), with Coupling Graph and Natural Frequencies as Figure (a). An arrow from oscillator i to oscillator j indicates the direction of influence, so $A_{ij} = 1$. For Figures (c,d), the metric $r_{\text{link}(i,j)}$ is shown on its own undirected edge $\{i, j\}$. Almost-Synchronization Groups are shown inside dotted lines, while vertexes in different Fixed Phase Synchronization Groups are marked with different colors.

Figures 3 (b) present the two Almost-Synchronization Groups, with oscillators $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8\}$, found in the KM for coupling strength $k = 0.11$. Observe that the biggest Almost-Synchronization Group contains two distinct Fixed Phase Synchronization Group with more than one oscillator, $\{5, 6\}$ (green) and $\{7, 8\}$ (red). The smaller Almost-Synchronization Group has the same oscillators than the biggest Fixed Phase Synchronization Group, $\{1, 2, 3\}$ (blue.).

Figures 3 (c) shows a fixed synchronized network for coupling strength $k = 0.21$. There is only one group of both types, with all oscillators inside.

4. CONCLUSION

This work showed how common types of phase synchronization can be defined (and numerically detected) in terms of the partial synchronization metric r_{link} and the maximum variation of the order parameter norm Δr. A diagram in the space parameters dissonance, ν, versus coupling strength, k, was analyzed.

Another contribution of this paper was the definition of two meaningful types of synchronization groups, based on undirected graphs. The stronger one, called Fixed Phase Synchronization Groups, is related with Fixed Phase Synchronization. The second one, Groups of Almost Synchronization, is easier to appear, but can be related to any kind of synchronization. A simple example of these groups emphasizing their construction is also provided.

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