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ANALYSIS OF LAMINAR FLOW OF NON-NEWTONIAN FLUIDS IN ECCENTRIC ANNULAR DUCTS USING THE BIPOLAR COORDINATE SYSTEM

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Abstract. *In this problem a hybrid numerical-analytical solution based on the Generalized Integral Transform Technique (GITT) is obtained for the hydrodynamically fully developed and thermally developing flows in annular ducts for non-Newtonian fluids that follow the power-law rheological model. In this paper, it is employed the bipolar coordinate system to map the eccentric annular duct. It is analyzed the velocity field, the numerical results for the velocity field and the product of the Fanning friction factor-Reynolds number are produced for different values of the governing parameters, eccentricity, radii ratio and power-law indices. These values will be shown in convergences table. Such results will be compared with those of a previous contribution providing critical comparisons in order to illustrate the adequacy of the employed integral transform approach.*

Keywords: *Generalized integral transform technique, non-Newtonian fluids, annular ducts, velocity field, power-law.*

1. INTRODUCTION

Industrial applications in which processing of materials behaving as non-Newtonian fluids are those commonly encountered in the chemical, cosmetics, food processing, polymer and petrochemical industries. The petrochemical industries are in search of solutions for the velocity and temperature field of the fluid flow with characteristics typically non-Newtonian. In these applications, the power-law model can described adequately the rheology of such fluids.

The Generalized Integral Transform Technique (GITT), present in this work is known as a powerful method in solving and manipulation of certain classes of problems of heat and mass diffusion. The GITT allows solution of problems of hybrid form to problems with the Newtonian complexity involved that can not be treated by usual analytical techniques. The basic idea is to transform a system of partial differential equations on an infinite system of ordinary equations, by eliminating spatial dependencies, where these can be solved more simply, with the advantage of producing a more accurate and more economical than to allow for control over the relative error results.

There are several works in which the flow of Newtonian fluids and non-Newtonian fluids are studied, in this work ducts of different geometries and different methods of solving partial differential equations are used. In Chaves et al. (2001) studied the thermally developed laminar flow of non-Newtonian fluids that follow the rheological model of the power law in rectangular ducts using as a method of solution GITT, in Chandrupatla et al. (1977) study was performed in heat transfer by forced convection non-Newtonian fluid in a square duct.

The developing laminar flow and heat transfer in the annular passages have been investigated by Heaton et al. (1964), Feldman et al. (1982), in this latter, it is solved laminar developing flow in eccentric annular ducts using the bipolar coordinate system. Others problems were also solved numerically using bipolar coordinates, such as those in the work of Heyda (1959). The author determined the Green’s function in bipolar coordinates for a potential flow and obtained a solution for the momentum equation. El-Shaarawi et al. (1998) use the bipolar coordinate system for determined developing laminar forced convection in eccentric annuli, the author has based the analysis on the work of El-Saden (1961), where it was studied heat conduction in an eccentrically hollow, infinitely long cylinder.

The objective of the present paper is to obtain a hybrid solution through the GITT approach for the fully developed flow of non-Newtonian fluids in eccentric annular ducts by using a bipolar coordinate system to map the region of such annular duct. Also, it is intend to develop a numerical algorithm to solve the transformed equation. Therefore, the numerical results will be confronted with results from the literature (Monteiro et al., 2010).

2. MATHEMATICAL FORMULATION

We consider fully developed laminar flow in the eccentric doubly connected duct geometry. The transformation equation from the cylinder coordinate system to this bipolar coordinate system is used to map the duct walls. It was

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considered that the two-dimensional flow is laminar and incompressible and stationary, the fluid follows the rheological power-law model, the properties of the fluid are constant and that the duct walls are impermeable and non-slip.

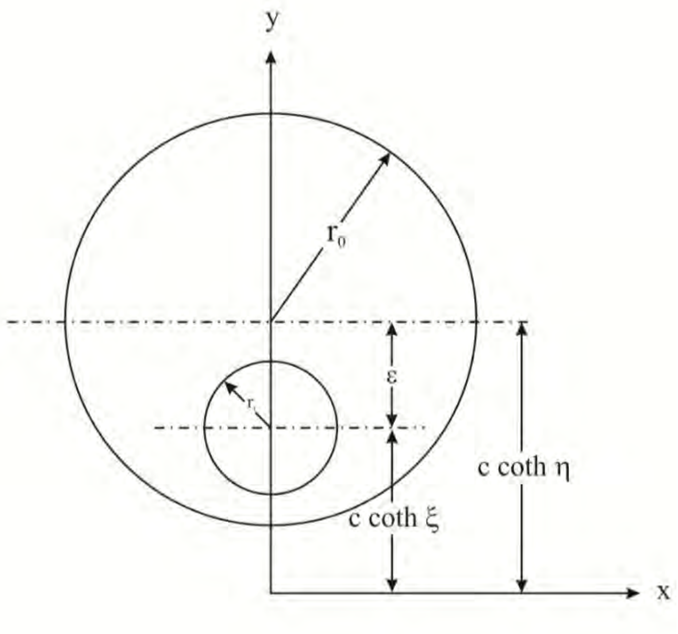

Figure 1 - Type geometry doubly connected ducts with angular symmetry eccentric.

The mathematical formulation of the flow problem is given by the momentum conservation equation in the axial direction, in dimensionless form, as follows:

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial \mathrm{V}_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial \mathrm{V}_z}{\partial y}\right)=-c \tag{1}$$

where,

$$\mu=\left[\left(\frac{\partial \mathrm{V}_z}{\partial x}\right)^2+\left(\frac{\partial \mathrm{V}_z}{\partial y}\right)^2\right]^{\frac{n-1}{2}} \tag{2}$$

The boundary condition for the present problem is $\mathrm{V}_z=0$ on the surface. Also, the velocity distribution must be symmetrical about the x-axis.

Where, in Eqs. (1) and (2) above the following dimensionless groups were employed:

$$x=\frac{x^*}{r_0};\quad y=\frac{y^*}{r_0};\quad \gamma=\frac{r_i}{r_0};\quad c=\left(\frac{r_0}{D_h}\right)^{n+1}=\frac{1}{\left[2\left(1-\gamma\right)\right]^{n+1}} \tag{3-6}$$

$$\mathrm{V}_z=\mathrm{V}_z^*\left[\left(-\frac{dp}{dz}\right)\frac{\mathrm{D}_h^{n+1}}{\mathrm{K}}\right]^{-1/n};\quad \mu=\mu^* r_0^{-1+n}\left[\left(-\frac{dp}{dz}\right)\frac{D_h^{n+1}}{K}\right]^{-n+1/n} \tag{7-8}$$

The related transformation equations from the Cartesian coordinate system to this bipolar coordinate system are given below:

$$x=a\frac{\sinh(\eta)}{\cosh(\eta)-\cos(\xi)} \tag{9}$$

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$$y = a\frac{\sin(\xi)}{\cosh(\eta)-\cos(\xi)} \tag{10}$$

Where a is a positive constant give by:

$$a = \gamma\sinh(\eta_i) = \sinh(\eta_0) \tag{11}$$

Making the transformation of coordinate systems by using Eqs. (9) to (11) above, we obtain the following equations:

$$\frac{\partial}{\partial\xi}\left(\mu\frac{\partial V_z}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\mu\frac{\partial V_z}{\partial\eta}\right) = -\frac{c\,a^{n+1}}{\left(\cosh(\eta)-\cos(\xi)\right)^2} \tag{12}$$

$$V_z(\xi,\eta_i)=0;\quad V_z(\xi,\eta_0)=0;\quad \frac{\partial V_z(0,\eta)}{\partial\xi}=0;\quad \frac{\partial V_z(\pi,\eta)}{\partial\xi}=0 \tag{13-16}$$

Where,

$$\mu = \left\{\left[\cosh(\eta)-\cos(\xi)\right]^2\left[\left(\frac{\partial V_z}{\partial\xi}\right)^2+\left(\frac{\partial V_z}{\partial\eta}\right)^2\right]\right\}^{n-1/2} \tag{17}$$

2.1 Solution methodology

In order to obtain the solution of Eq. (12), we rewritten such equation as:

$$P(\xi,\eta)\times\frac{\partial^2 V_z}{\partial\eta^2} = -\frac{c\,a^{n+1}}{A^2(\xi,\eta)} - G(\xi,\eta)\times E(\xi,\eta) - (n-1)\,F^{n-3/2}(\xi,\eta)\left\{K(\xi,\eta)\times O(\xi,\eta)+N(\xi,\eta)\right\} \tag{18}$$

Where:

$$K(\xi,\eta) = A(\xi,\eta)\left[B^2(\xi,\eta)+C^2(\xi,\eta)\right] \tag{19}$$

$$M(\xi,\eta) = \left\{\mu+(n-1)\,F^{n-3/2}(\xi,\eta)A^2(\xi,\eta)C^2(\xi,\eta)\right\} \tag{20}$$

$$N(\xi,\eta) = 2A^2(\xi,\eta)B(\xi,\eta)\,C(\xi,\eta)\,D(\xi,\eta) \tag{21}$$

$$O(\xi,\eta) = \sinh(\eta)B(\xi,\eta)+\sin(\xi)C(\xi,\eta) \tag{22}$$

$$P(\xi,\eta) = \mu+\left(n-1\right)F^{n-3/2}A^2(\xi,\eta)B^2(\xi,\eta) \tag{23}$$

Where the coefficients are defined by:

$$A(\xi,\eta) = \cosh(\eta)-\cos(\xi);\quad B(\xi,\eta) = \frac{\partial V_z}{\partial\eta}; \tag{24-25}$$

$$C(\xi,\eta) = \frac{\partial V_z}{\partial\xi};\quad D(\xi,\eta) = \frac{\partial^2 V_z}{\partial\xi\,\partial\eta};\quad E(\xi,\eta) = \frac{\partial^2 V_z}{\partial\xi^2}; \tag{26-28}$$

$$\mathrm{F}(\xi,\eta) = \left\{\left[\cosh(\eta)-\cos(\xi)\right]^2\left[\left(\frac{\partial V_z}{\partial\xi}\right)^2+\left(\frac{\partial V_z}{\partial\eta}\right)^2\right]\right\} = \left\{A^2(\xi,\eta)\left[C^2(\xi,\eta)+B^2(\xi,\eta)\right]\right\} \tag{29}$$

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2.2 Eigenvalue problem

The Generalized Integral Transform Technique (GITT) is then employed in the hybrid numerical-analytical solution of the problem (Cotta, 1993). For this purpose, the following auxiliary eigenvalue problem is chosen:

$$\frac{d^2\psi_i}{d\xi^2} + \mu_i^2\psi_i = 0 \tag{30}$$

$$\frac{d\psi_i(0)}{d\xi} = 0; \quad \frac{d\psi_i(\pi)}{d\xi} = 0 \tag{31-32}$$

Equation above can be analytically solved, to yield the eigenfunctions and eigenvalues, respectively as:

$$\psi_i(\xi) = \cos(\mu_i \times \xi); \quad \mu_i = i-1, \quad i = 1,2,3,... \tag{33-34}$$

It can be shown that the eigenfunction, $\psi_i(\xi)$, obey the following orthogonality property, where N_i is the normalization integral:

$$\int_0^{\pi} \psi_i\, \psi_j\, d\xi = \begin{cases} 0, & i \neq j \\ N_i, & i = j \end{cases} \tag{35}$$

$$N_i = \int_0^{\pi} \psi_i^2\, d\xi = \begin{cases} \pi, & i = 1 \\ \dfrac{\pi}{2}, & i > 1 \end{cases} \tag{36}$$

2.3 Inverse-transform pair

Equations (30) to (32) together with the respective orthogonality properties allow the definition of the integral transform pair for the velocity field as:

$$\bar{V}_{z,i}(\eta) = \int_0^{\pi} \psi_i(\xi) V_z(\xi,\eta)\, d\xi, \quad \text{transform} \tag{37}$$

$$V_z(\xi,\eta) = \sum_{i=1}^{\infty} \frac{\psi_i(\xi)}{N_i} \bar{V}_{z,i}(\eta), \qquad \text{inverse} \tag{38}$$

2.4 Integral transformation

To obtain the resulting system of differential equations for the transformed potentials, $\bar{V}_{z,i}$, the partial differential equation, Eq. (18), is multiplied by $\psi_i(\xi)$, integrated over the domain $[\pi, 0]$ in the η-direction, and the inverse formula is employed in place of the velocity distribution $V_z(\xi,\eta)$, resulting in the following transformed ordinary differential system:

$$\sum_{j=1}^{\infty} G_{ij}(\eta) \frac{d^2\bar{V}_{z,j}}{d\eta^2} = H_i(\eta) \tag{39}$$

$$\bar{V}_{z,i}(\eta_0) = 0; \quad \bar{V}_{z,i}(\eta_i) = 0 \tag{40-41}$$

Where $G_{ij}(\eta)$ and $H_i(\eta)$ are given by:

$$G_{ij}(\eta) = \int_0^{\pi} \frac{\psi_i(\xi)\, \psi_j(\xi)\, I(\xi,\eta)}{N_j} d\xi; \tag{42}$$

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$$H_i(\eta) = \int_0^{\pi} \psi_i(\xi)\, J(\xi,\eta)\, d\xi; \tag{43}$$

$$I(\xi,\eta) = \left[\mu + (n-1) F^{n-3/2}(\xi,\eta)\; A^2(\xi,\eta)\, B^2(\xi,\eta) \right]; \tag{44}$$

$$\begin{aligned} J(\xi,\eta) = &-\frac{c\,a^{n+1}}{A^2(\xi,\eta)} - \left\{ \mu + (n-1)\, F^{n-3/2}(\xi,\eta) A^2(\xi,\eta) C^2(\xi,\eta) \right\} E(\xi,\eta) \\ &-(n-1)\, F^{n-3/2}(\xi,\eta) \left\{ A(\xi,\eta) \left[B^2(\xi,\eta) + C^2(\xi,\eta) \right] \times \left[\sinh(\eta) B(\xi,\eta) + \sin(\xi) C(\xi,\eta) \right] \right. \\ &\left. + 2A^2(\xi,\eta) B(\xi,\eta)\, C(\xi,\eta)\, D(\xi,\eta) \right\} \end{aligned} \tag{45}$$

The coefficients $G_{ij}(\eta)$ and $H_i(\eta)$ depend on the transformed potentials and vary along η, the Eqs. (39-41) form an infinite nonlinear boundary value problem, which has to be truncated in a sufficiently high order NT, followed by computation of the transformed potentials of the velocity field, $\bar{V}_{z,i}(\eta)$, to within a user prescribed precision goal. For the solution of such a system, due to the expected stiff characteristics, specialized subroutines have to be employed such as the DVPFD from the IMSL Library (1991).

In order to compute the product of the friction factor by the Reynolds number, first it is necessary to calculate the average velocity, and then from the introduction of the inverse formula, Eq. (38), into its usual definition, one obtains:

$$V_{z,m} = \frac{2a^2}{\pi(1-\gamma^2)} \sum_{i=1}^{\infty} \sum_{m=1}^{\infty} K_{i,m}\, L_{i,m} \tag{46}$$

The coefficients in Eq. (46) above are defined as follows:

$$K_{i,m} = \int_0^{\pi} \frac{\psi_i(\xi)\cos(\lambda_m \xi)}{N_i M_m}\, d\xi; \tag{47}$$

$$L_{i,m} = \int_{\eta_0}^{\eta_i} \frac{e^{-\lambda_m \eta} (\lambda_m + \coth(\eta)) \bar{V}_{z,i}(\eta)}{\sinh^2(\eta)}\, d\eta \tag{48}$$

From the definition of the friction factor and Reynolds number, it is concluded that the product fRe is given by:

$$f\,\mathrm{Re} = \frac{1}{2V_{z,m}^n} \tag{49}$$

3. RESULTS AND DISCUSSIONS

To validate the method used is necessary to make a convergence analysis of the results obtained for comparison with the literature of the same. To solve this system has been developed a computer code of programming language FORTRAN 90/95 using the calculation routine DBVPFD of the IMSL Library (1991).

A convergence analysis of the numerical results obtained for the product fRe, in fully developed laminar flow for Newtonian fluids, which flow through doubly connected ducts is made in this report. To perform of computer simulations was prescribed a relative error of 10^{-12} in the solution of the system of ordinary differential equations infinite and coupled.

In Table 1 we show the convergence analysis of the results of the product fRe of non-Newtonian fluid in eccentric annular ducts for n = 0.5, the values were calculated for different values of aspect ratios (γ = 0.2; 0.5 e 0.8) and for different values of eccentricity (ε = 0.1; 0.5 e 0.9) depending on the number of terms NT. There is a good convergence of results still in low numbers of terms, with the gradual increase of the eccentricity there is the convergence in number values higher, it can be seen clearly in the amount γ = 0.2, in which in ε = 0.1 convergence occurs with 9 number of terms, in ε = 0.5 the values converge with 15 number of terms, while in ε = 0.9 is observed the convergence with 19 number of terms, the most critical cases are those with ε = 0.9, in which convergence is between 15 and 21 terms. In the convergence analysis was performed to validate the work that the comparison between the values obtained in the

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present paper and the results of the work of Monteiro et al. (2010), there is a good agreement between both work concluding the validation of results.

Table 1 - Convergence analysis of the product fRe eccentric annular ducts for non- Newtonian fluids with n = 0, 5. b - Monteiro et al. (2010)

NT	γ = 0,2			γ = 0,5			γ = 0,8		
	ε = 0,1	ε = 0,5	ε = 0,9	ε = 0,1	ε = 0,5	ε = 0,9	ε = 0,1	ε = 0,5	ε = 0,9
	fRe								
3	7.675	6.665	6.295	7.852	6.290	5.662	7.881	6.111	5.488
5	7.677	6.637	5.608	7.852	6.257	4.876	7.881	6.051	4.635
7	7.676	6.636	5.420	7.852	6.256	4.662	7.881	6.051	4.365
9	7.675	6.636	5.372	7.852	6.256	4.615	7.881	6.051	4.362
11	7.675	6.636	5.365	7.852	6.256	4.597	7.881	6.051	4.362
15	7.675	6.637	5.334	7.852	6.256	4.595	7.881	6.051	4.278
19	7.675	6.637	5.333	7.852	6.256	4.592	7.881	6.051	4.249
21	7.675	6.637	5.333	7.852	6.256	4.592	7.881	6.051	4.249
b	7.676	6.634	5.334	7.854	6.255	4.591	7.879	6.051	4.247

In Table 2 we show the convergence analysis of the results of the product fRe of non-Newtonian fluid in eccentric annular ducts, the values were calculated for different values of aspect ratios (γ = 0.2; 0.5 e 0.8) and for different values of eccentricity (ε = 0.1; 0.5 e 0.9) depending on the number of terms NT. There is a good convergence of results still in low numbers of terms, with the gradual increase of the eccentricity there is the convergence in number values higher, it can be seen clearly in the amount γ = 0.2, in which in ε = 0.1 convergence occurs with 9 number of terms, in ε = 0.5 the values converge with 11 number of terms, while in ε = 0.9 is observed the convergence with 27 number of terms, the most critical cases are those with ε = 0.9, in which convergence is between 13 and 27 terms. In the convergence analysis was performed to validate the work that the comparison between the values obtained in the present paper and the results of the work of Monteiro et al. (2010), there is a good agreement between both work concluding the validation of results.

Table 2 - Convergence analysis of the product fRe eccentric annular ducts for non- Newtonian fluids with n = 1, 5. b - Monteiro et al. (2010)

NT	γ = 0,2			γ = 0,5			γ = 0,8		
	ε = 0,1	ε = 0,5	ε = 0,9	ε = 0,1	ε = 0,5	ε = 0,9	ε = 0,1	ε = 0,5	ε = 0,9
	fRe								
3	66.056	49.949	57.249	68.024	48.931	42.906	68.476	48.6445	40.111
5	66.059	49.299	37.904	68.024	48.557	30.585	68.476	48.325	29.092
9	66.060	49.279	32.501	68.024	48.552	28.072	68.476	48.322	27.038
11	66.060	49.279	32.056	68.024	48.552	27.964	68.476	48.322	26.968
13	66.060	49.279	31.892	68.024	48.552	27.938	68.476	48.322	26.954
17	66.060	49.279	31.805	68.024	48.552	27.938	68.476	48.322	26.951
21	66.060	49.279	31.793	68.024	48.552	27.938	68.476	48.322	26.951
27	66.060	49.279	31.790	68.024	48.552	27.938	68.476	48.322	26.951
29	66.060	49.279	31.790	68.024	48.552	27.938	68.476	48.322	26.951
b	66.063	49.280	31.788	68.025	48.553	27.929	68.477	48.323	26.952

4. CONCLUSIONS

A solution based on the Generalized Integral Transform Technique (GITT) was developed to predict fully developed laminar flow of non-Newtonian power-law fluids in eccentric annular ducts. The proposed integral transform approach provided reliable and cost effective simulations for the considered cases by employing a bipolar coordinate representation of the solution domain. Benchmark results for the product of the Fanning friction factor-Reynolds number were systematically tabulated for different values of the governing geometric parameters, demonstrating the usefulness and robustness of the GITT alternative solution procedure.

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