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NUMERICAL SOLUTION OF THE TWO DIMENSIONAL TEMPERATURE FIELD ON THE COLD SUBSTRATE OF THERMOELECTRIC COOLERS

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Abstract. *Thermoelectric coolers (TEC) can be used as heat pumps, for heating and cooling processes and as electrical generators. Since the dimensions of the object to be cooled may not coincide with the dimensions of the cold substrate, the knowledge of the temperature distribution on the cold substrate is useful on project and analysis of a TEC for a desired application. This work presents results of temperature distributions on the cold substrate of a single stage TEC using finite difference method. The steady-state heat conduction equation was solved using an algorithm implemented on MATLAB® and results were compared with an approximate analytical solution available in the literature. It was developed an analysis of the mesh size on the average and maximum temperatures. The numerical errors involved and the order of the numerical solution were also studied, in order to establish a consistent solution technique. Results showed small differences on the results of the numerical solution for the average temperature and higher differences for the maximum temperature. Also, analysis regarding the numerical aspects of the numerical solution showed that even when a full second order finite difference formulation is used only a first order solution accuracy is achieved, considering a discrete heat source on the cold substrate. Those conclusions were set after evaluation of several formulations for the boundary condition and also considering also a continuous heat source on the cold substrate.*
Keywords: *thermoelectric cooler, finite difference method, numerical simulation, overall solution order*

1. INTRODUCTION

The thermoelectric phenomenon is associated with Peltier, Seebeck and Thomson effects. Although thermoelectricity was discovered in the early nineteenth century, thermoelectric modules were only applied commercially on last decades.

A thermoelectric module is composed of thermoelectric couples (n and p-type semiconductor legs) connected electrically in series and thermally in parallel and then soldered between two ceramic plates (the hot and cold substrates). The major advantages of TECs are the lack of moving parts and the fact that they are small, highly reliable, and flexible, does not require maintenance and they can be easily combined with other components. The major disadvantage is the low efficiency.

Research, analysis and development of TEC are still major tasks, but lately, it can be noticed an increase on application related to refrigeration of electronic devices. Project methodologies using experimental setup or numerical simulation are also investigated by many researchers and companies.

Drabkin et al. (2004a) presented an analysis of the temperature distribution on intermediate substrates of a multistage TEC. The authors showed that, in order to reduce heat losses from one stage to the next one, modules with pellets cross-section widely varying from stage to stage should be applied. This results in decreasing sizes of intermediate substrate and increasing heat flux density across the substrates, making the TEC more efficient, since the pellets on the edge of the substrates are better involved in cooling.

Drabkin et al. (2004b) analyzed the effect of the temperature distribution on the cold substrate over the TEC performance. The analyses were considered using a approximate analytical solution developed by Dulnev et al. (1975) for the temperature distribution in a single and two stage TEC. The authors presented a criterion based on the temperature difference between the intermediate substrate and the upper substrate. The results showed that sometimes one-dimensional temperature distribution is not enough, especially when the TEC is required to perform intense heat pumping or high temperature difference.

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Besides the evaluation of the temperature distribution on the substrates, methodologies on sizing and selecting TECs for different applications have also been developed. Tan et al. (2008) presented a methodology for energy saving based on a combination of solar cells and a thermoelectric cooler. This hybrid system was used in a building. It is also presented an internet-based process to promote an interaction between TECs developers and fabricants. The authors suggested that this process will make the project, analysis and optimization easier, since it promotes exchange of information and experience.

Newer applications of thermoelectric coolers include the one proposed by Okuma et al. (2012). The authors applied thermoelectric modules in a HVAC system operating under cold climate operation. The main objective was to boost the heating capacity of the HVAC system in an energy-efficient way, especially for cold climate operation. The combination of the two systems increases the overall system efficiency and augments the system capacity through the high COP of the TEC for small temperature. Gao et al. (2012) analyzed the potential of thermoelectric devices applied in a high-temperature polymer electrolyte membrane fuel cell (HTPEMFC). The thermoelectric generators were embedded inside a gas-liquid heat exchanger to form a heat recovery subsystem jointly for electricity production. According to the authors the results demonstrated that the thermoelectric-assisted heat flux regulation and heat-loss reduction can also effectively help solve issues related to the performance of the system. The combination presented is of great value and worthy of further study.

Many studies on methods for solving partial differential equation using finite difference method are available in the literature. Beyond those, emphasis should be taken to the ones that analyze the solution of second order elliptical equation with insulated conditions on the boundaries. Dai (2010) proposed a kind of new accurate second order difference scheme for one-dimensional elliptical differential equations with insulated boundaries. The author showed that this scheme proved to be unconditionally stable and the solution is always second order accurate. In addition, there are several ways to mathematically represent an insulated boundaries, differing generally from the other of the approximation, usually first or second order.

The main objectives of this paper is to present a numerical solution of the temperature distribution on the cold substrate using finite difference method and analyze the order of the numerical solution for several formulations of the method and for different formulations of the heat source on the cold substrate. Validation of the numerical solution will be done with an approximate analytical solution available in the literature.

2. MATHEMATICAL MODEL FOR A SINGLE STAGE THERMOELECTRIC COOLER

Consider a single stage TEC with N pellets and a heat source Q_0 located in the middle of the cold substrate. A scheme with the TEC configuration and heat rates are shown in Fig. 1.

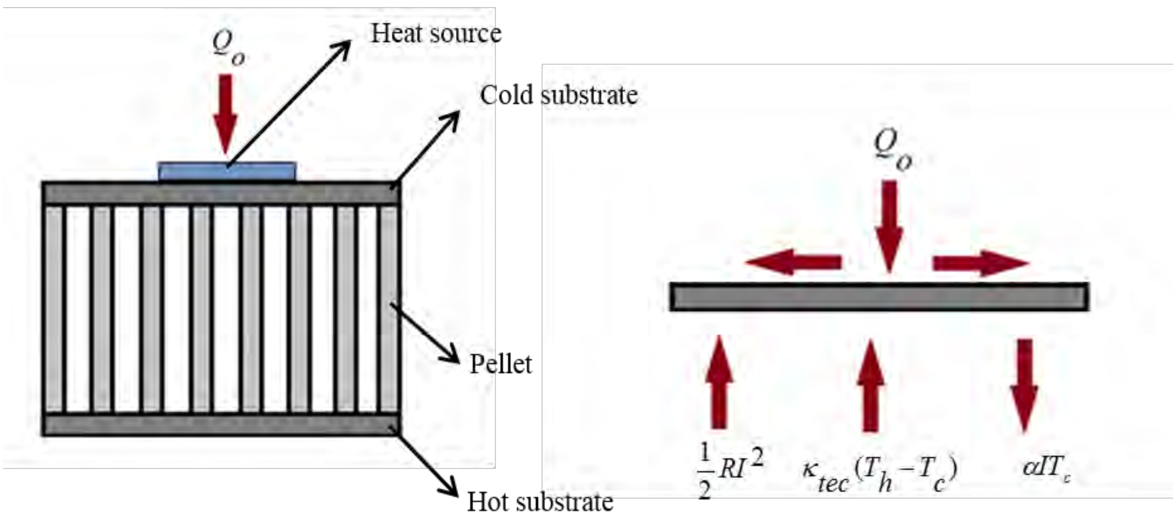

Figure 1. Thermoelectric cooler and heat fluxes on the cold substrate

According to Lee (2010), when an electric current passes through the pellet, heat is transferred from the hot substrate to the cold substrate. The heat transfer rate through one pellet is given by Eq. (1):

$$Q_{pellet} = 0.5RI^2 + \kappa_{tec}\left(T_h - T_c\right) - I\alpha T_c \tag{1}$$

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Where R is the pellet electrical resistance, I is the electrical current, κ_{tec} is the pellet thermal conductance, T_h is the hot substrate temperature, T_c is the cold substrate temperature and α is the Seebeck coefficient. In Eq. (1) the first term is due to Joule effect, the second is due to the pellet conductance and the last is due to the Peltier effect (see Fig. 1).

Hence, as shown in Drabkin et al. (2004b), the steady-state two dimensional heat diffusion on the cold substrate governing equation is given by Eq. (2):

$$k_{sub} d\left(\frac{\partial^2 T_c}{\partial x_1^2} + \frac{\partial^2 T_c}{\partial x_2^2}\right) - \frac{N_{pellets}(\alpha I + \kappa_{tec})}{S_{sub}} T_c + \frac{N_{pellets}\left(0.5RI^2 + \kappa_{tec} T_h\right)}{S_{sub}} + \frac{\{Q_0\}}{S_{source}} = 0 \tag{2}$$

In Eq. (2), k_{sub} is the substrate thermal conductivity, d is the substrate thickness, S_{sub} is the substrate area, S_{source} is the heat source area and $\{Q_0\}$ denotes the discrete heat source power. The TEC is insulated on the boundaries, hence the boundary conditions can be represented by Eq. (3):

$$\left.\frac{\partial T_c}{\partial x_1}\right|_{x_1=0,x_1=L_{sub}} = \left.\frac{\partial T_c}{\partial x_2}\right|_{x_2=0,x_2=L_{sub}} = 0 \tag{3}$$

Where L_{sub} is the substrate length. Different discrete formulations can be applied to represent the Eq. (3). One of the objectives of this paper is to analyze those formulations and compare the results of order of the numerical solution.

3. NUMERICAL METHODOLOGY

A finite difference method (Ferziger and Peric, 2002) was applied to solve the differential equation that represents the heat diffusion on the cold substrate of a thermoelectric cooler (TEC) with a square cold substrate discretized into a MxN mesh. Considering the energy balance on the cold substrate, a general finite difference equation, considering internal nodes, edges and corners, is given by Eq. (4):

$$\begin{aligned} & k_{sub} d\left[\left(\frac{a_1 T_{i-1,j} - a_2 T_{i,j} + a_3 T_{i+1,j}}{\frac{\Delta x_1}{A}}\right)\frac{\Delta x_2}{B} + \left(\frac{b_1 T_{i,j-1} - b_2 T_{i,j} + b_3 T_{i,j+1}}{\frac{\Delta x_2}{B}}\right)\frac{\Delta x_1}{A}\right] + \\ & - \frac{\Delta x_1}{A}\frac{\Delta x_2}{B}\frac{N_{pellets}(\alpha I + \kappa_{tec})}{S_{sub}} T_{i,j} + \frac{\Delta x_1}{A}\frac{\Delta x_2}{B}\frac{N_{pellets}\left(0.5RI^2 + \kappa_{tec} T_h\right)}{S_{sub}} + \frac{\Delta x_1}{A}\frac{\Delta x_2}{B}\frac{\{Q_0\}}{S_{source}} = 0 \end{aligned} \tag{4}$$

Constants $a_1, a_2, a_3, b_1, b_2, b_3, A, B$ in Eq. (4) are given in Table 1, depending on the portion of the domain the nodes are located:

Table 1. Constants of Eq. (4)

Domain portion \ Constants	a_1	a_2	a_3	b_1	b_2	b_3	A	B
Interior nodes	1	2	1	1	2	1	1	1
Upper boundary	1	2	1	1	1	0	1	2
Lower boundary	1	2	1	0	1	1	1	2
Right boundary	1	1	0	1	2	1	2	1
Left boundary	0	1	1	1	2	1	2	1
Upper right corner	1	1	0	1	1	0	2	2
Lower right corner	1	1	0	0	1	1	2	2

Upper left corner	0	1	1	1	1	0	2	2
Lower left corner	0	1	1	0	1	1	2	2

Another formulation for the governing equation previously presented is to consider a central difference formulation for the second derivative coupled with a discrete equation to represent the boundary conditions in Eq. (3). This discrete formulation, following a first order discretization scheme is expressed by the following equations (Özisik, 1994):

$$T_{1,j} - T_{2,j} = 0 \tag{5}$$

$$T_{M-1,j} - T_{M,j} = 0 \tag{6}$$

$$T_{i,1} - T_{i,2} = 0 \tag{7}$$

$$T_{i,N-1} - T_{i,N} = 0 \tag{8}$$

Similarly, another formulation, following a second order formulation for the boundary conditions, is shown in Eq. (9), Eq. (10), Eq. (11) and Eq. (12) (Özisik, 1994):

$$-3T_{i,1} + 4T_{i+1,1} - T_{i+2,1} = 0 \tag{9}$$

$$-3T_{i,N} + 4T_{i-1,N} - T_{i-2,N} = 0 \tag{10}$$

$$-3T_{1,j} + 4T_{1,j+1} - T_{1,j+2} = 0 \tag{11}$$

$$-3T_{M,j} + 4T_{M,j-1} - T_{M,j-2} = 0 \tag{12}$$

Besides the formulations presented before, the finite difference scheme proposed by Dai (2010) was also implemented.

To solve the system of equations by using finite difference method, a code was developed in MATLAB®. Since the resulting system of equations is sparse, the authors used the sparse matrix function available in the software to store the elements of each non-zero diagonal in a separate array, resulting in computer memory savings, when compared to storing in a two-dimensional array. In order to ensure the validation of the numerical solution, the approximate analytical solution proposed by Dulnev *et al.* (1975) was also implemented.

The different numerical formulations presented were used to analyze the effect on the overall order of the numerical solution. The overall order of the scheme can be computed considering 3 meshes, with systematically reduction of the element size, according to Equation 13 (Ferziger and Peric, 2002):

$$p = \frac{\ln\left(\dfrac{\Phi_{2h} - \Phi_{4h}}{\Phi_{h} - \Phi_{2h}}\right)}{\ln(2)} \tag{13}$$

In Eq. (13), Φ_h is the solution provided by the fine mesh, Φ_{2h} is the solution provided by the intermediate mesh (with double node spacing of the fine mesh) and Φ_{4h} is the solution provided by the coarse mesh (with double node spacing of the intermediate mesh).

4. RESULTS AND DISCUSSION

In this section will be shown the validation of the numerical solution using the approximate analytical solution, application using the numerical solution and an analysis of the order of the numerical solution.

4.1 Validation of numerical solution

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The main parameters of the TEC used in this paper are shown in Table 3.

Table 3. TEC main parameters

Parameter	Value
Hot substrate temperature [K]	300
Substrate thermal conductivity [W/m K]	30
Substrate thickness [m]	0.001
Substrate length [m]	0.04
Heat Source length [m]	0.02
Pellet conductance [W/K]	1.392
Pellet cross section area [m^2]	1.96
Number of pellets	16
Heat input [W]	40
Electrical resistance [Ω]	1.857
Seebeck coefficient [V/K]	0.0566

Based on the methodology presented by Roache (1998), several meshes were created with a known refinement factor in order to analyze the convergence of the numerical solution. Table 4 presents results for average and maximum temperatures considering the formulation presented in Eq. (4). All simulation were carried in an Intel® Core™ i7 CPU @ 2.20GHz with 8GB RAM computer. For the finest mesh, simulation time was about 30 minutes.

Table 4. Validation of the numerical solution

<table>
<tr><th rowspan="2">Mesh</th><th rowspan="2">Nodes</th><th colspan="2">Numerical solution</th><th colspan="2">Approximate analytical solution</th></tr>
<tr><th>T_{ave} (K)</th><th>$T_{\max}$ (K)</th><th>T_{ave} (K)</th><th>$T_{\max}$ (K)</th></tr>
<tr><td>1</td><td>11x11</td><td>271.6423</td><td>276.6282</td><td rowspan="7">271.8865</td><td rowspan="7">272.2495</td></tr>
<tr><td>2</td><td>21x21</td><td>272.0697</td><td>276.6427</td></tr>
<tr><td>3</td><td>41x41</td><td>271.9934</td><td>276.6421</td></tr>
<tr><td>4</td><td>81x81</td><td>271.9546</td><td>276.6414</td></tr>
<tr><td>5</td><td>161x161</td><td>271.9350</td><td>276.6408</td></tr>
<tr><td>6</td><td>321x321</td><td>271.9251</td><td>276.6405</td></tr>
<tr><td>7</td><td>641x641</td><td>271.9202</td><td>276.6403</td></tr>
</table>

Results show small differences for the average temperature on the cold substrate for the numerical and analytical solution and a difference of approximate 4.4K for the maximum temperature. The temperature profiles for both solutions (numerical and analytical) are shown on Fig. 2 and Fig. 3, respectively. Also, Fig. 4 shows the relative error between both solutions.

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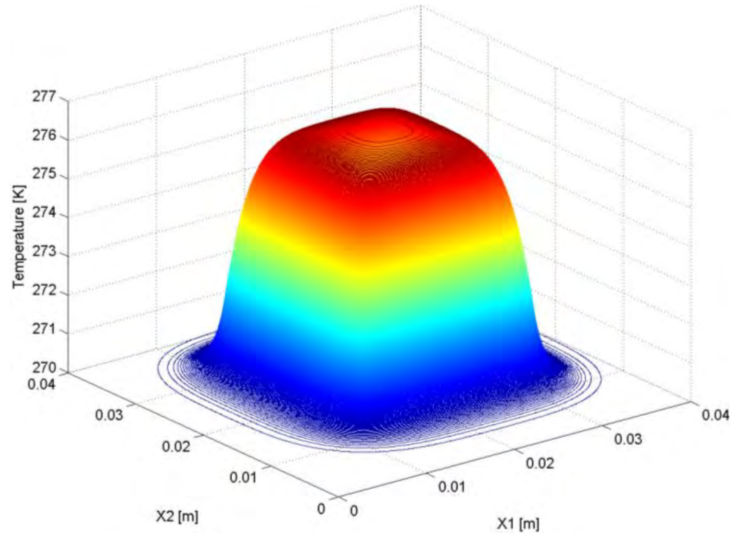

Figure 2. Temperature profile from numerical solution

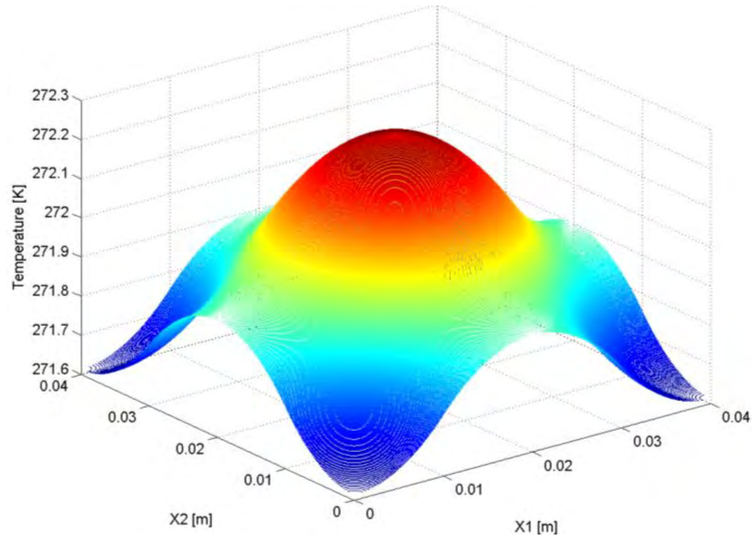

Figure 3. Temperature profile from approximate analytical solution

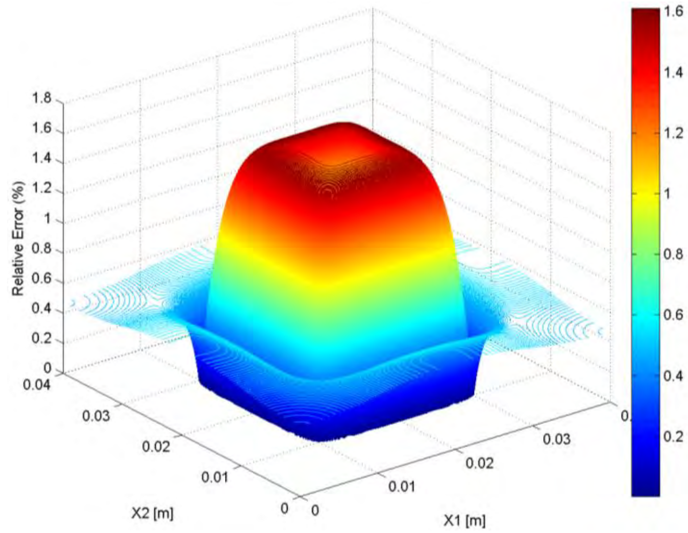

Figure 4. Relative error between numerical and analytical solutions

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It can be noticed that the temperature profile obtained by the analytical solution is nearly uniform on the entire cold substrate. The temperature profile provided by the numerical solution describes a more reliable result considering the presence of the discrete heat source on the center of the substrate. The behavior of the uniform profile provided by the analytical solution can be probably justified by the fact this solution is approximated (Dulnev *et al*., 1975).

Figure 4 shows that the maximum relative error is found in the center of the substrate since in this position where the temperature difference is higher (see Table 4). It can be also notice that the relative error is smaller on the transition between the substrate and the discrete heat source.

4.2 Applications of the numerical solution

The numerical solution of Eq. (2) gives more reliable results, especially for maximum temperature (see Table 4), since when mesh refinement is increased the numerical solution approaches the exact solution. In this section, the numerical solution is used to perform numerical simulations with different configurations of the discrete heat source.

Figure 5 shows the temperature field on the cold substrate for a non-centered smaller discrete heat source with center on (0.03,0.03) and length of 10 mm, for a 641x641 mesh:

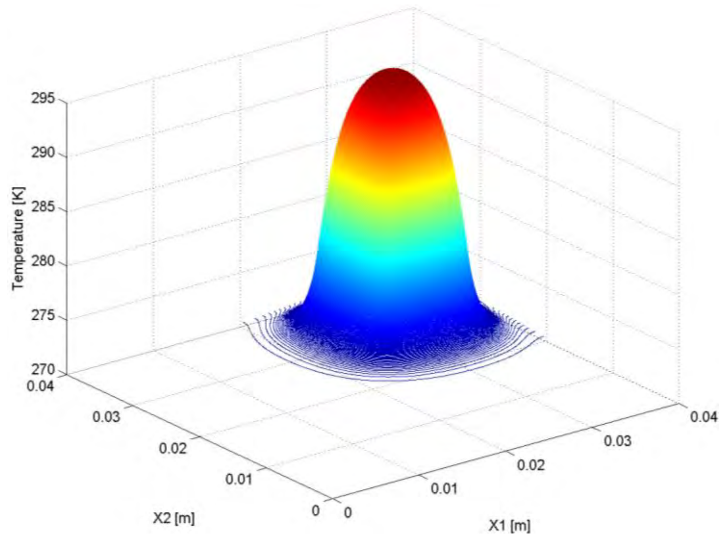

Figure 5 – Temperature distribution on the cold substrate for non-centered smaller heat source

It can be noticed that the numerical solution was able to describe correctly the behavior of the temperature distribution when the heat source is not place in the center of the domain, as expected. It can also be noticed that, in this case, since the heat source area is smaller, higher temperatures are achieved in the cold substrate, since a higher heat flux is present and neither of the operational parameters of the TEC were changed.

Figure 6 shows the influence of the number of pellets between hot and cold substrates of the TEC used in this work. All simulations were carried out in a 641x641 mesh and operational parameters of Table 3, excepting number of pellets.

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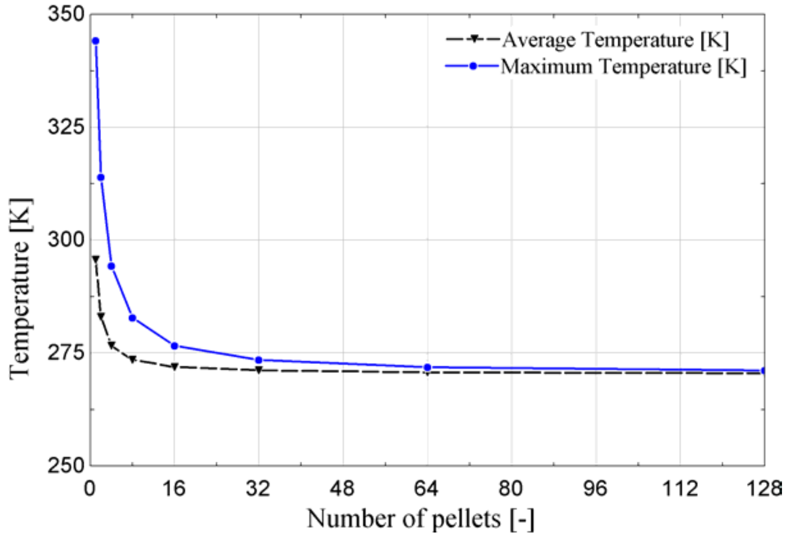
Figure 6 – Influence of the number of pellets on average and maximum temperatures



It can be noticed that increasing the number of pellets, the difference between maximum and average temperatures decreases, since the heat flux removed from the cold substrate is increased, as expected.

4.3 Analysis of the order of the numerical solution

In order to analyze the order of the numerical solution on the boundaries, Table 5 shows results of the average and maximum temperatures for the formulations presented is this paper. It should be pointed that general formulation is the one provided by Eq. (4), first and second order were obtained by using centered-finite difference scheme inside the domain and Eqs. (5), (6), (7) and (8) for the boundaries and Eqs. (9), (10), (11) and (12), respectively.

Table 5. Temperatures for different formulations with discrete heat source

Mesh	First order formulation		General formulation	
	T_{ave} (K)	$T_{\max}$ (K)	T_{ave} (K)	$T_{\max}$ (K)
5	271.935016800240	276.640810261590	271.934980434790	276.640810260746
6	271.925165029875	276.640477592444	271.925149229840	276.640477592068
7	271.920230009652	276.640296412730	271.920222638314	276.640296412551
p	1	1	1	1
Mesh	Second order formulation		Weizhong Dai (2010) formulation	
	T_{ave} (K)	$T_{\max}$ (K)	T_{ave} (K)	$T_{\max}$ (K)
5	271.934979878220	276.640810260733	271.9349458727581	276.6408102599459
6	271.925149166837	276.640477592066	271.9251326848812	276.6404775916631
7	271.920222630747	276.640296412550	271.9202145335037	276.6402964123452
p	1	1	1	1

According to Ferziger and Peric (2002), the rate at which the error is reduced when the grid is refined is the most important information, instead of the formal order of the scheme as defined by the leading term in the truncation error. Also, one way to consider this feature is to apply Eq. (13) for the three meshes, as showed in Table 5.

However, applying Eq. (13) for the meshes considered in Table 5, results showed that even when a second order finite difference formulation is applied for the entire domain, the solution, which is expected to be second order

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accurate, actually is first order accurate. This behavior was observed for a local temperature, represented by the maximum temperature, which happened to be on the center of the domain and an average value, represented by the average temperature. It is noteworthy that the aforementioned behavior was found for all formulations.

In order to investigate the influence of the discrete heat source on this behavior of the numerical solution, it was also considered a continuous heat source, represented by Eq. (14):

$$Q_{0,cont} = Q_0^2\left[-x_1\left(x_1 - L_{sub}\right) + x_2\left(x_2 - L_{sub}\right)\right] \quad (14)$$

Where Q_0 is the same value considered in Table 3 but representing a heat flux in Eq. (14). Tables 6 and 7 shows a comparison for average and maximum temperatures considering the same formulations presented in Table 5.

Table 6. Temperatures for different formulations with continuous heat source

Mesh	First order formulation		General formulation	
	T_{ave} (K)	$T_{\max}$ (K)	T_{ave} (K)	$T_{\max}$ (K)
5	275.7035685077582	278.3395290728516	275.6959180022027	278.3395289709745
6	275.7130717140622	278.3395290061418	275.7095498461964	278.3395289580220
7	275.7180422193125	278.3395289782642	275.7163540874693	278.3395289548096
p	1	1	1	2
Mesh	Second order formulation		Dai (2010) formulation	
	T_{ave} (K)	$T_{\max}$ (K)	T_{ave} (K)	$T_{\max}$ (K)
5	275.6971570812142	278.3395289877392	275.6887263647021	278.3395288741153
6	275.7098595109389	278.3395289623196	275.7058836328478	278.3395289076100
7	275.7164315759339	278.3395289559050	275.7145036822558	278.3395289290215
p	1	2	1	1

For the continuous heat source and applying Eq. (13) for the same meshes previously considered, it was possible to achieve second order solution for maximum temperature for general and second order formulations only. The order of the numerical solution for the average temperature was always 1 for all formulations considered, meaning that some portions of the domain are still achieving only first order accurate solution, including for the formulation proposed by Dai (2010), in which the author ensured second order solution accuracy.

5. CONCLUSIONS

This paper presented the numerical solution of the temperature distribution on the cold substrate of thermoelectric coolers using the finite difference method. Results were validated and compared with an approximate analytical solution presented by Dulnev *et al*., (1975). An analysis regarding the order of the solution for some formulations on the boundaries were carried out. Results showed good agreement for the average temperature on the cold substrate and a maximum 4.4K difference for the maximum temperature. Results also showed that a second order finite difference formulation on the entire domain did not lead to a second order solution on all points of the domain. In order to confirm those findings, a continuous heat source was also considered in order to investigate if the discrete heat source is influencing the order of the numerical solution. In this case, results showed that there are still some portions of the domain where only a first order solution was achieved. However, in the center of the domain, where the maximum temperature occurs, a second order solution was achieved for two second order formulations. More investigations are needed to correctly describe the order of the numerical solution, for example by application of a more robust and efficient method to solve the differential equation that represents the temperature distribution on the cold substrate of a thermoelectric cooler.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

Dai, W., 2010. “A new accurate finite difference scheme for Neumann (insulated) boundary condition of heat conduction”. *International Journal of Thermal Sciences*, Vol. 49, p. 571-579.

Drabkin, I. A., Dashevsky, Z.M., Yershova, L.B., Gromov, G.G., 2004a. “Methods of Reducing Temperature Losses on the Intermediate Substrates of Multistage TE Modules”. *Proceedings of Annual Seminar on Thermoelectricity.*

Drabkin, I.A., Yershova, L.B., Kondratiev, D.A., Gromov, G.G., 2004b. “The Effect of the Substrates Two-Dimensional Temperatura Distribution on the TEC Performance”. *Proceedings of 8th European Workshop on Thermoelectrics.*

Ferziger, J. H., Peric, M., 2002. *Computational Methods for Fluid Dynamics*. Springer, New York, 3rd edition.

Dulnev G.N., Polschikov B.V., 1975. “Temperature Field of a Plate with a Discrete Energy Source”. *Engineering Physics Journal*, vol. XXIX, n. 4, p.722 – 727.

Gao, X., Chen, M., Adreasen, S. J., Kaer, S.K., 2012. “Potential Usage of Thermoelectric in a High-Temperature Polymer Electrolyte Membrane (PEM) Fuel Cell System: Two Case Studies”. *Journal of Electronic Materials*, Vol. 41, No. 6.

Lee, H. S., 2010. *Thermal design: heat sinks, thermoelectrics, heat pipes, compact heat exchangers and solar cells*. Wiley, New Jersey, 1st edition.

Okuma, T., Radermacher, R., Hwang, Y., 2012. “A Novel Application of Thermoelectric Modules in an HVAC System Under Cold Climate Operation”. *Jornal of Electronic Materials*, Vol. 41, No. 6.

Roache, P. J., 1998. *Fundamentals of Computational Fluid Dynamics*. Hermosa Publishers, New Mexico, 1st edition.

Tan, F.L., Fok, S.C., 2008. “Methodology on sizing and selecting thermoelectric cooler from different TEC manufacturers in cooling system design”. *Energy Conversion and Management*, Vol. 49, p. 1715–1723.

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