A NOVEL MATHEMATICAL MODEL FOR FLEXURE BEARINGS APPLIED IN PRIMARY TORQUE STANDARD MACHINES

Antonio Carlos Peixoto Bitencourt, antonio.carlos@ifba.edu.br
Instituto Federal de Educação, Ciência e Tecnologia da Bahia, Brasil

Luis Antonio Gonçalves Junior, jr_lag@hotmail.com
Universidade Federal da Bahia, Brasil

Alexander Wagner, a.wagner@tu-ilmenau.de
René Theska, rene.theska@tu-ilmenau.de
Technische Universität Ilmenau, Germany

Herman A. Lepikson, herman@ufba.br
Universidade Federal da Bahia, Brazil

Walter L. Weingaertner, wlw@emc.ufsc.br
Universidade Federal de Santa Catarina, Brazil

Abstract. *There are some mathematical models for flexure bearing in the literature. Most models are simplifications of bending or bulking behavior of leaf-springs. These approaches aim to supply a parametric design guideline for leaf-spring bearings. The present work adopts an intermediary solution, taking advantage of both parametric and analytical approaches to cope with this limitation. The parametric approach provides the optimal value of geometric parameters of flexure bearings. The analytical model, in turn, provides accurate measurements of the influence of flexure bearings, e.g. retro torque and rotary deviation, which behave systematically. The developed mathematical model is able to determine the main characteristic of flexure bearings made of cross leaf-spring. It takes into account the effect of moment, horizontal and vertical loads. The mathematical model developed is used to find the optimal value of geometric and form parameters to minimize the retro torque and rotary deviation.*

Keywords*: Cross Leaf-Spring Bearing, Torque Standard Machine, Mathematical Model*

1. INTRODUCTION

A flexure bearing is an elastic element that provides the relative rotation between two adjacent rigid members through flexing (bending). It can be applied as a bearing with limited rotation capability. Flexure bearings are widely used in precision engineering. They show almost no friction, but only internal bounding forces (Krause, 2004). A main drawback of flexure bearings is given by the highly limited angle of rotation. Furthermore, flexure bearings do not provide a pure rotation around a fixed rotary axis because of the complex elastic behavior (Lobontiu, 2002). On the other hand, the position of the rotary axis dependent on the distortion of the bearing is highly repeatable. The requirements of the lever bearing in a torque standard machine (TSM) are low friction and the repeatable position of the rotary axis, so that flexure bearings seem to fit best (Bitencourt *et al.* 2008; 2010). With a detailed knowledge of the overall behavior of the bearing failure compensation can be applied. The characteristics of flexure bearings can be predictable if they are working in the elastic range.

The best configuration of a flexure bearing is two crossed leaf-springs, shown in Figure 1. They are known as corner filled hinges when cross leaf-springs are made in a monolithic form (Krause, 2004). This configuration gives the smallest center shift, the highest stiffness in the other axis and the highest compliance in the rotary axis. The problem is that mathematical models for this configuration lead to overdeterminated beams, which makes the model more complex. Most of the models to predict flexure behavior presented in the literature are simplifications based on the bending or bulking behavior. They can be classified into three approaches: cinematic (Wuest, 1950), parametric (Hongzhe and Shusheng, 2010) and analytical (Zelenika and De Bona, 2002).

Cinematic approaches use the principle of relative movement between two pole paths. One curve is fixed and the other one unrolls on the fixed curve. The bearing behavior is modeled as cinematic mechanism of relative movement between the curves (Wuest, 1950). However, cinematic approaches only account for the effect of pure moment, which is a limitation. The parametric approaches aim to supply guidelines for the bearing design (Hongzhe and Shusheng, 2010). Therefore, they make simplifications of the equation which provides the parametric set that helps the designer to choose a better configuration of the bearing. However, these simplifications are not enough to accurately predict the bearing behavior in order to compensate system deviations during operation. The analytical approaches support more precise behavior models of cross leaf-springs. However, they require high computational effort and produce complex results which cannot be applied directly. This complexity limits he analytical approach restricted to symmetric bearing configuration (Zelenika and De Bona, 2002).

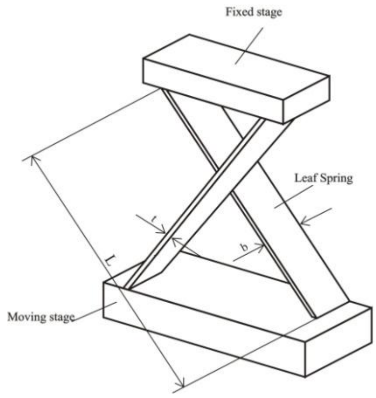

Figure 1. Crossed leaf springs.

The present work adopts an intermediary solution, taking advantage of both the parametric and analytical approaches. The parametric approach provides optimal results for the geometric parameters of cross leaf-springs. The analytical model, in turn, provides accurate measurements of the systematic behavior of flexure bearings. In the following section, the mathematical model developed is shown.

2. DEVELOPED MATHEMATICAL MODEL

The main aim of the mathematical model is to determine the center shift OO' of the cross leaf-spring, see Figure 2. The rotary axis of an unloaded bearing is the cross point of the leaf-springs, point O. Otherwise, if any loads are applied on the bearing, the center will change position to the point O'. In both cases the position of the rotary center is determined by the crossing point of two tangents of the leaf-spring, as shown in Figure 2(b), which is the particularity of the approach presented in this article. Other authors have calculated the rotary axis by the deflection of each leaf-spring in whole length (Haringx, 1949; Zelenika and De Bona, 2002).

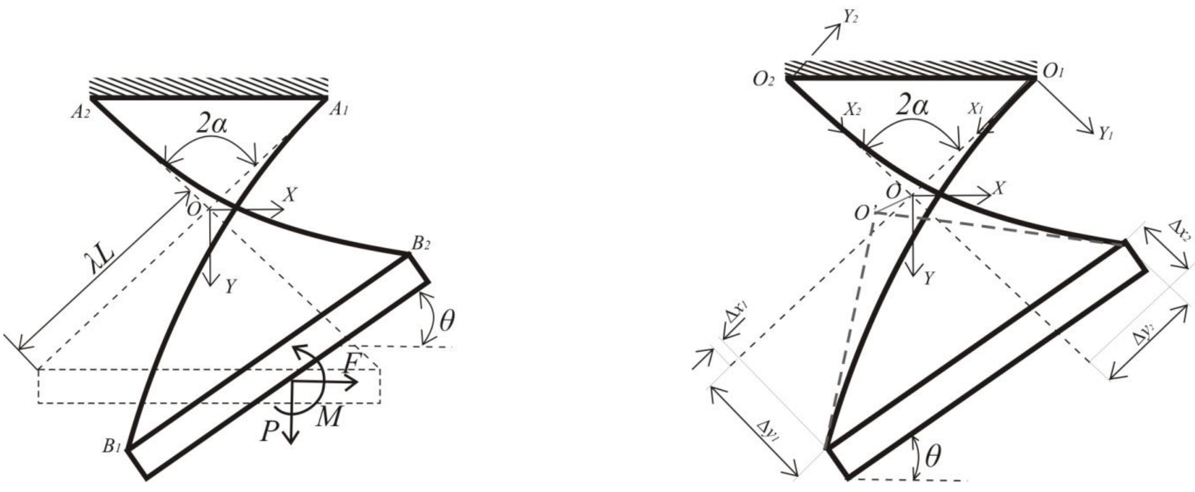

Figure 2. Shift of the rotary axis of the cross leaf-spring.

The tangents can be determined by the end point of the leaf spring and the rotary angle. The Equation 1 shows the tangent for each leaf spring. They are relative to each particular reference system (OXY1 and OXY2) of each leaf spring. When the rotary angle (θ) is smaller than 10° (Wittrick, 1948), the end point of each leaf (Δx_i, Δy_i) bearing can be determined by the elastic theory of a cantilever beam.

$$y_i - \Delta y_i = \tan(\theta)(x_i - \Delta x_i),\ \text{i=1,2} \tag{1}$$

The bend of the beam can be described with the approximate expression for its curvature. In this way it is possible to determine the deflection of each leaf spring, considering that the springs are fixed in its stages (fixed and moving ones). The beams are then constrained to the geometric arrangement of the bearing. The above assumption and the equilibrium conditions of the bearing are shown in Figure 3. The bearing behavior is defined by eleven variables (Δx_1, Δx_2, Δy_1,

Δy_2, P_1, P_2, F_1, F_2, M_{B1}, M_{B2} and θ), as shown in Figure 3. The index refers to each leaf-spring. Therefore, eleven equations are necessary to determine these variables. A similar approach was taken by Zelenika and De Bona (2002), but they used only the symmetric configuration (λ=0.50).

Three of the equations come from the equilibrium between the moving stages and leaf-spring, Eq. 2-4. The other two come from the geometric restriction of each leaf spring, Eq. 5-6. They must remain fixed in both stages (moved and fixed).

$$F = (P_2 - P_1)\sin(\alpha) + (F_1 + F_2)\cos(\alpha) \tag{2}$$

$$P = (P_1 + P_2)\cos(\alpha) + (F_1 - F_2)\sin(\alpha) \tag{3}$$

$$\begin{aligned} M = {} & M_1 + M_2 + \left[(P_1 - P_2)\cos(\alpha) + (F_1 + F_2)\sin(\alpha)\right]\lambda L\sin(\alpha)\cos(\theta) \\ & - \left[(P_1 + P_2)\sin(\alpha) - (F_1 - F_2)\cos(\alpha)\right]\lambda L\sin(\alpha)\sin(\theta) \end{aligned} \tag{4}$$

$$(\Delta y_1 - \Delta y_2)\cos(\alpha) + (\Delta x_1 + \Delta x_2)\sin(\alpha) = 2\left(\lambda L\sin(\alpha) - \lambda L\sin(\alpha)\cos(\theta)\right) \tag{5}$$

$$(\Delta y_1 + \Delta y_2)\sin(\alpha) - (\Delta x_1 - \Delta x_2)\cos(\alpha) = 2\lambda L\sin(\alpha)\sin(\theta) \tag{6}$$

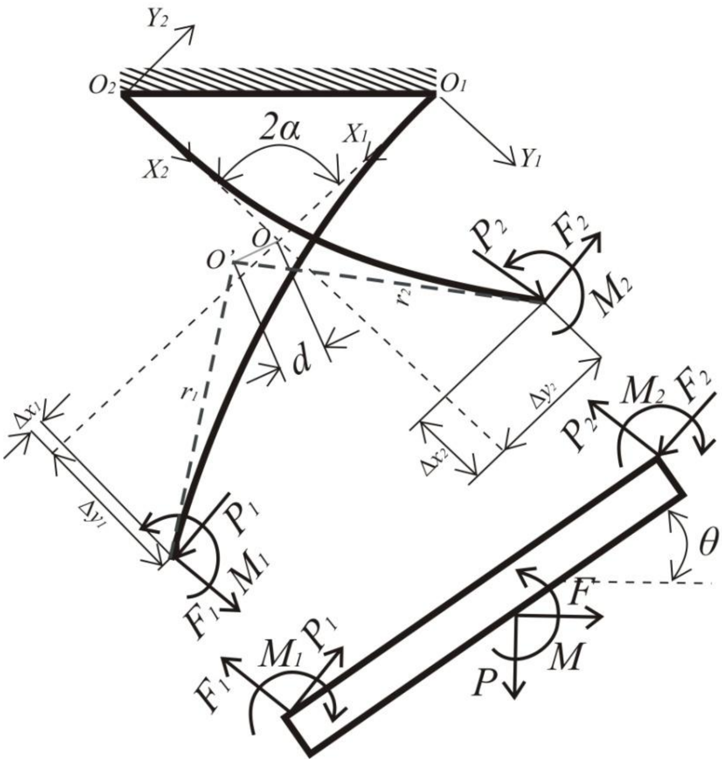

Figure 3. Variables of cross spring-leafs.

The other six equations come from the force equilibrium and deformation of each leaf spring, see Figure 4. These equations are obtained through the differential equation of a bending beam, Eq. 7, and elastic theory, Eq.8.

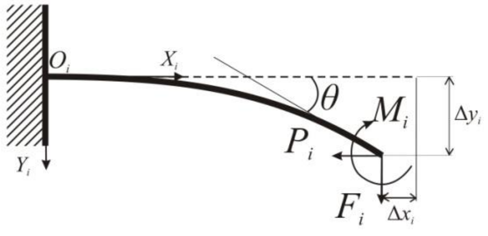

Figure 4. Deflection of each leaf springs.

$$EI \frac{d^2}{dx^2} y_i(x) = -P_i(\Delta y_i - y_i(x)) + F_i(L - x) + M_i, \text{ i=1,2} \tag{7}$$

$$\Delta x_i = \frac{1}{2} \int_0^L \left(\frac{\mathrm{d}}{\mathrm{d}\,x} y_i(x) \right)^2 \mathrm{d}x_i, \text{ i=1,2} \tag{8}$$

These eleven equations together come to a non linear system, which makes its analytical resolution complex. Most authors make some kind of approximation or simplification of the load on the bearing (Zelenika and De Bona, 2002) or use adimensional parameters (Hongzhe and Shusheng, 2010). Both approaches are not appropriate to determine the bearing behavior for online compensation. Therefore numeric optimization was chosen as the method to solve the equation system, because this approach provides more accurate results. The numeric optimization was made carried out the mathematical software MatLab™. The equations were written as a function of the rotary angle. Thus the center shift and the rotation compliance were obtained.

Routines were written in the MatLab™ environment as shown in Figure 5. The `mancal_principal.m` is responsible for calling the other ones and for the inputs. The `mancal_rigidez.m` determines the dimension of the leaf spring that gives the smallest stiffness and satisfies the failure criteria (Maximum-Shear-Stress Theory and Bulking). The non linear equation system in function of the moment applied to the bearing must be written. The routines `mancal_sistema_momento.m` and `solucao_inicial_momento.m` are responsible for solving this equation system. With the dimension of the leaf-spring it is possible to determine the center shift. The routine `mancal_desvio_centro.m` is responsible for determining the center shift by calculating the tangents equations and their cross point. It is also necessary to calculate the deformation of the end of each leaf spring for each rotary angle in order to write the tangent equations. This task is done by the routines `mancal_sistema_theta.m` and `solucao_inicial_theta.m`.

These routines and the mathematical approach provide the best configuration of the bearing for TSM application as well as to determine the center shift and the compliance which can be used to do online compensation. These results are shown in the following section.

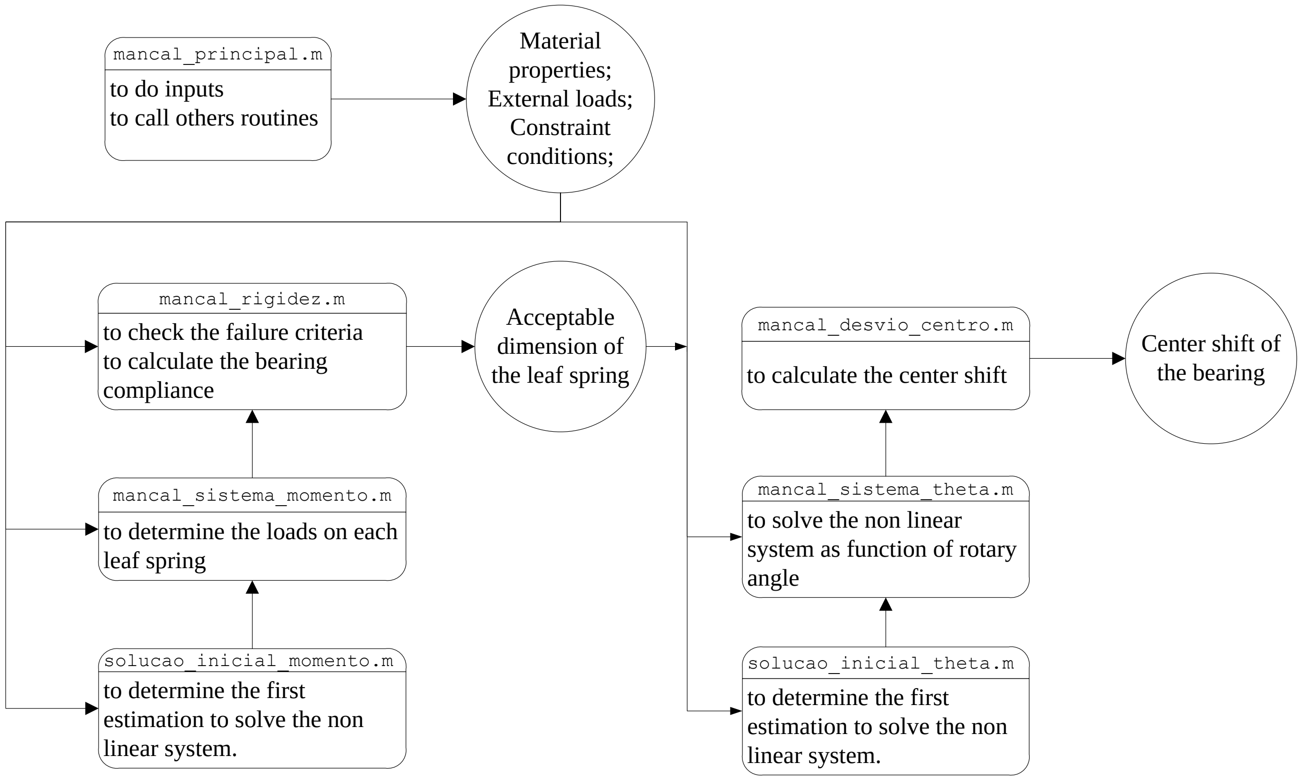

Figure 5. Developed routines

3. RESULTS AND DISCUSSION

The results of the proposed model were compared to the literature. The first comparison was made with those accepted to theoretically give the best accuracy for cross leaf-spring with pure moment (Zelenika and De Bona, 2002). The results of the proposed model fit better for rotary angles smaller than 5°. Another comparison was made with Hongzhe and Shusheng (2010). The authors found the values of λ that give the smallest center shift are 13% and 87%. The developed model came up with similar results, as shown in Figure 7.

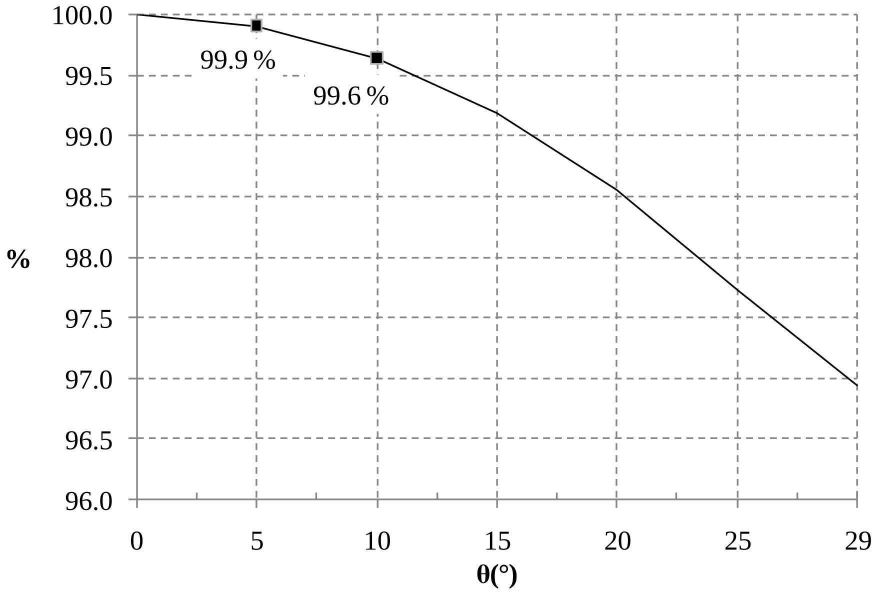

Figure 6. Relative deviations of proposed model from Zelenika and De Bona (2002) in function of θ(°)

Once the model had been validated, we applied it to determine the optimal configuration for the cross leaf-spring bearing. The first step is to determine the material of the leaf spring. Ashby's method was adopted to determine the best material for this function (high strength and low stiffness) (Ashby, 1994). The selected material was beryllium-copper alloy (E = 131 GPa and σ_R = 1,251 MPa). The maximum loads are moment of 2 Nm and vertical load 100 N. For safety factor of 1.5 it was found the leaf dimensions of length=20.0mm, width=31.0mm and thickness=0.4mm. These dimensions give a bearing stiffness of 5.866 Nm/rad. The center shift of the bearing as a function of the geometric parameter (α and λ) is shown in Figure 8.

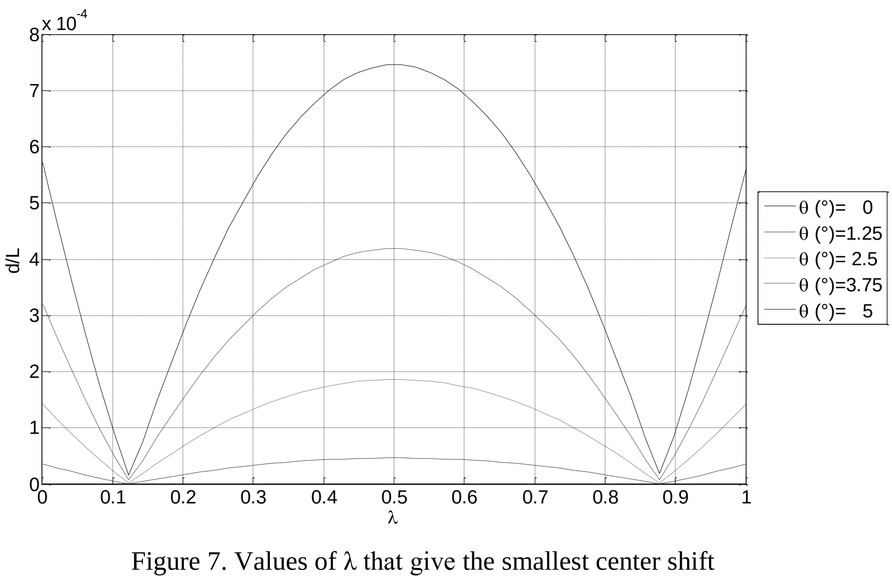

Figure 7. Values of λ that give the smallest center shift

There are specific values of α and λ that give minimal center shift (λ_1 = 0.13 and λ_2= 0.87), Figure 9. The corresponding values for α can be found using λ_1 and λ_2 calculated for different rotary angle (θ), Figure 10. 30° was selected, which gives small center shift and small fabrication complexity.

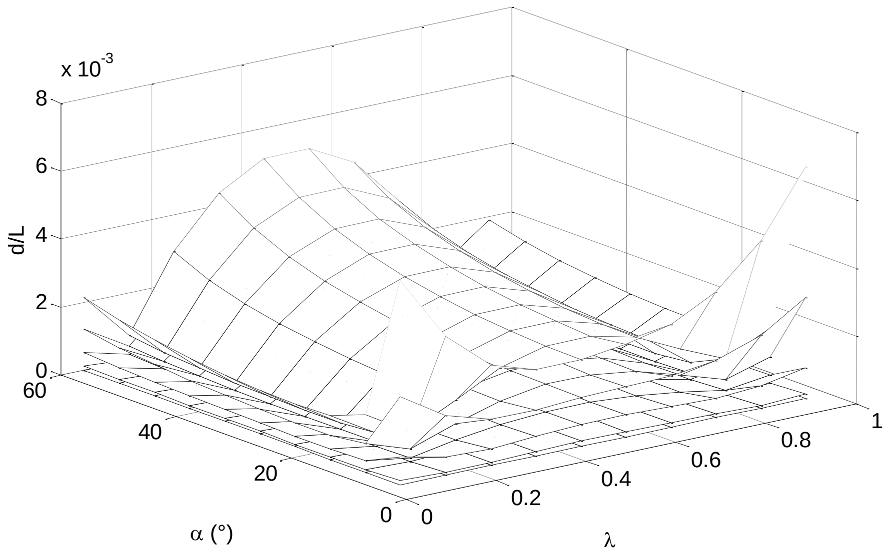

Figure 8. Center point behavior as function of α and λ

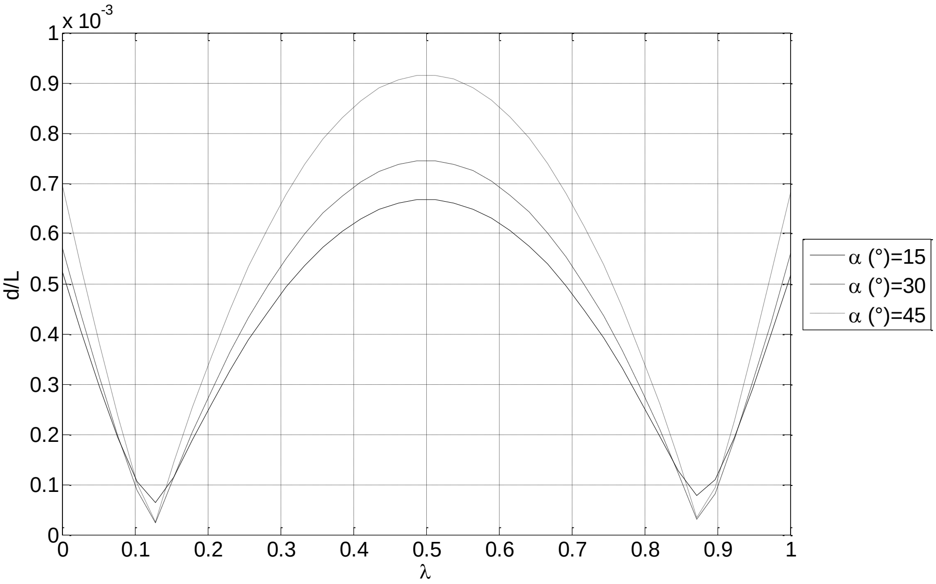

Figure 9. Behavior of center shift as function of α and λ, θ=5°

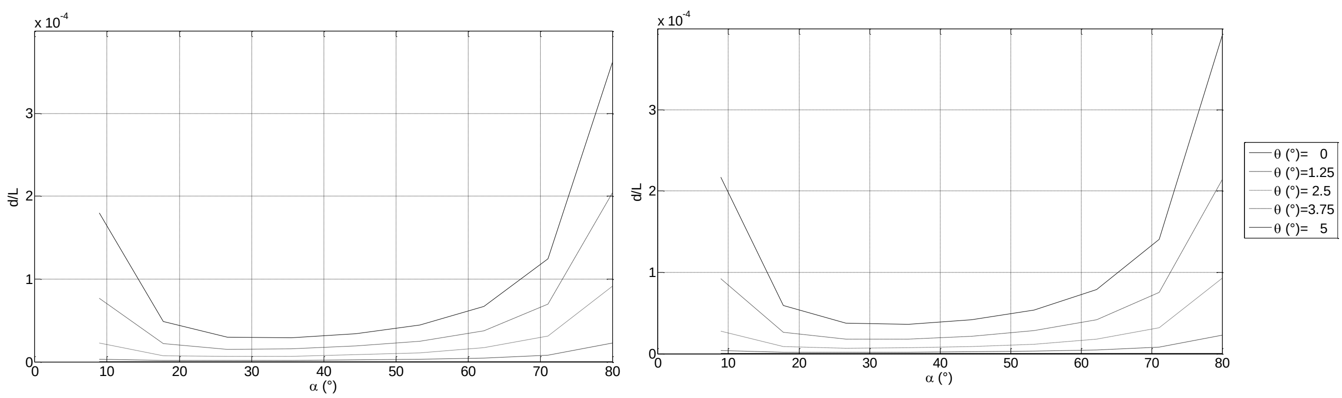

Figure 10. Behavior of center shift as function of α, λ_1 =0.13 and λ_2 =0.87

Figure 11 shows the center shift for the chosen configuration α=30° and λ=0.13. With these results it is possible to compensate the parasitic movement of the cross leaf-spring. Figure 12 shows a sketch of the cross leaf-spring bearing.

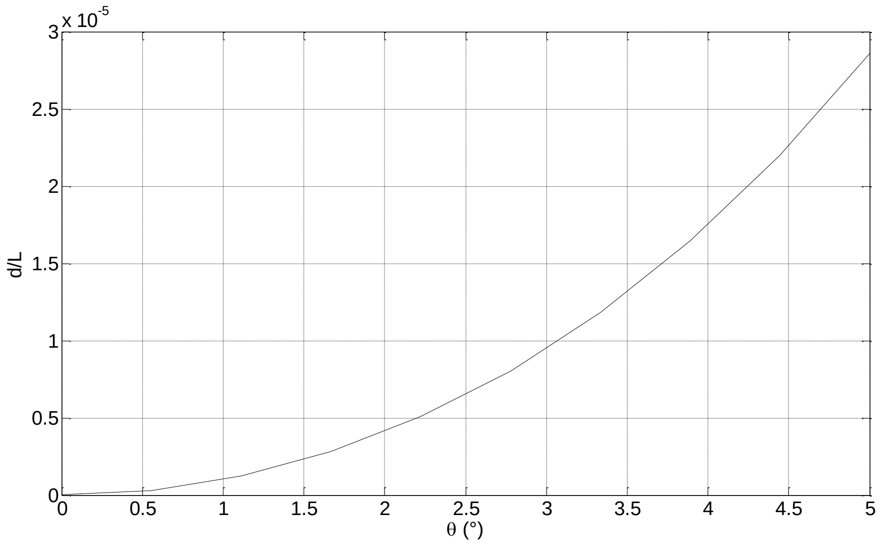

Figure 11. Center shift of cross leaf spring bearing, α=30° and λ =0.13

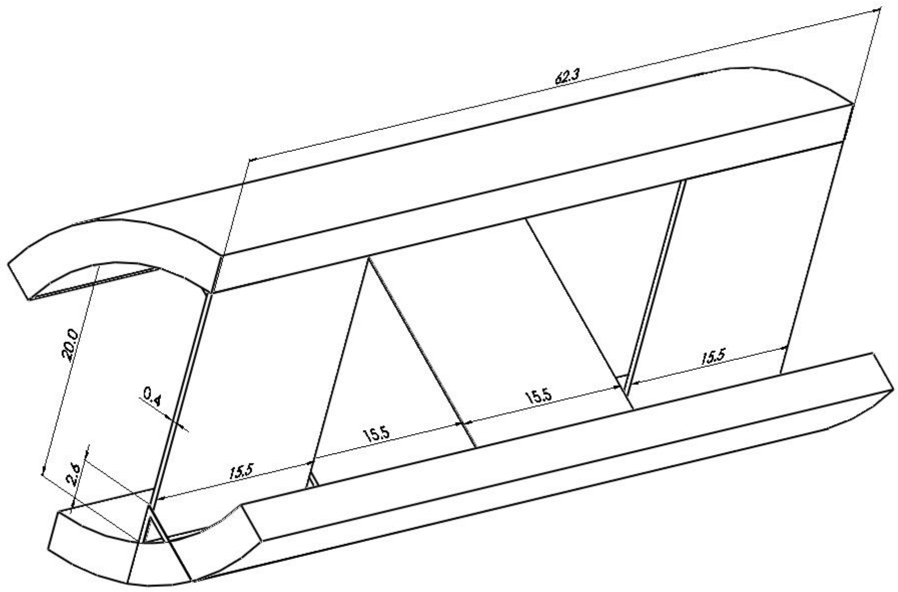

Figure 12. Cross leaf-spring bearing, α=30° and λ =0.13

4. CONCLUSION

This article presented an alternative approach to determine the main characteristic of a cross leaf-spring bearing using numerical and optimization methods. The main bearing characteristics considered were stiffness and center shift. The mathematical model is solved through a numerical and optimization approach. Routines in Matlab™ were written to help solve a non linear system with eleven equations. The proposed model was compared to results available in the literature to show its feasibility. In this way, we demonstrate that it is possible to determine the optimal leaf spring form and the optimal configuration of the cross spring bearing.

5. ACKNOWLEDGEMENTS

These results were from the research project DEBRATOR, a sub-project of the BRAGECRIM Research Initiative, funded by DFG (German Research Foundation) and CAPES (Brazilian Research Foundation).

6. REFERENCES

Ashby, M., 1994. "Materials selection in mechanical design". Metallurgia Italiana, Vol. 86, No. 10, pp. 475–480. ISSN 0026-0843.

Bitencourt, A.C.P., Weingärtner, W.L., Lepikson, H.A. and Theska, R., 2010. "Padrão primário para baixo torque e esforços para melhoria da sua incerteza." *Produto & Produção*, Vol. 11, pp. p. 41–55

Bitencourt, A.C., Theska, R., Weingaertner, W.L., Stoeterau, R.L. and Lepikson, H.A., 2008. "Solutions principles for small torque standard machine". In 53rd *Internationales Wissenschaftliches Kolloquium*. Technische Universität Ilmenau, Ilmenau

Haringx, J., 1949. "The cross-spring pivot as a constructional element". *Applied Scientific Research*, Vol. 1, No. 1, pp. 313–332.

Hongzhe, Z. and Shusheng, B., 2010. "Accuracy characteristics of the generalized cross-spring pivot". *Mechanism and Machine Theory*, Vol. 45, No. 10, pp. 1434–1448. ISSN 0094-114X.

Krause, W., 2004. *Konstruktionselemente der Feinmechanik*. München, Wien, 3rd edition. ISBN: 3-446-22336-3.

Lobontiu, N., 2002. Compliant mechanisms: design of flexure hinges. London. ISBN 0-8493-1367-8

Wittrick, W.H., 1948. "The theory of symmetrical crossed flexure pivots". *Australian Journal of Scientific Research, Series A: Physical Sciences*, Vol. 1, p. 121.

Wuest, W., 1950. "Blattfedergelenke für meßgeräte". *Feinwerktechnik*, Vol. 7, No. 54, pp. 167–171.

Zelenika, S. and De Bona, F., 2002. "Analytical and experimental characterisation of high-precision flexural pivots subjected to lateral loads". *Precision Engineering*, Vol. 26, No. 4, p. 381 388. ISSN 0141-6359.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.