VORTEX DYNAMICS AND THE MOMENTUM TRANSFER EQUATIONS

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Abstract. *A mathematical description of a vortex is aimed by scientists since the early studies in Fluid Mechanics, especially after the recognition of the importance of vortical coherent structures in turbulence dynamics. Since the recognition of a closed theory around the subject is not a fact, the coupling between vortex identification and the equations of fluid dynamics has been putted aside. In the present work, the vortex identification parameters based on the strain acceleration are written as a function of the terms of the specific case of Euler and Navier-Stokes equations. The strain-acceleration parameters are reasoned on the fact that, vortices are regions where the strain rate tensor is orthogonal (out-of-phase) with its acceleration, an objective tensor with respect to the changes in coordinates system. Applying this methodology to the equations of motion, it is possible to identify which terms are in-phase or out-of-phase with respect to the strain rate tensor and its tendency. Those new variables are evaluated for the so-called ABC flow, solution of the Euler equations and the turbulent flow inside a cavity.*

Keywords: *Vortex Dynamics, Coherent Structures, Navier-Stokes Equations*

1. INTRODUCTION

A mathematical description of a vortex, with full acceptance, is one great lacuna in fluid mechanics. Several points distinguish the proposed criteria and its hypothesis limit their application to all kinds of flow. Some of them presents alternative formulations for compressive or non-newtonian flows but fail to identify vortices on regions of high vorticity. Most of proposed criteria are based on kinematic assumptions. Some intuitive criteria, like vorticity magnitude or spiralling streamlines are fundamented on the velocity field and their behavior along space. Even some proposed criteria which are based on dynamic assumptions have their mathematical description as a function of kinematic entities.

Since a vortex definition has not find yet a common description, there is not much to say about the association of dynamic entities on vortex identification in the literature. Although some intuitive ideas and observation can lead us to some basic conclusions, the lack of a mathematical description around the subject limits the comprehensiveness to some simple flows. The description, for example, of a oil-water emulsion formation depends, among other reasons, on how the local mixing is responsible for breaking the oil droplets into small ones and make the mixing happens or how the local mixing is responsible to propagate a flame in turbulent combustion. If a vortex identification criterion could be mathematically inserted into the momentum equations for each phase or associated in the mixture equation, these phenomena could be associated.

The objective of the present work is to derive the momentum equations and associate them to a vortex criterion based on the strain acceleration. The strain-acceleration parameters are reasoned on the fact that, vortices are regions where the strain rate tensor is orthogonal (out-of-phase) with its acceleration, an objective tensor with respect to the changes in coordinates system. The application of a material derivative to the terms of momentum equations results in the description of strain accelerations as a function of some important entities in fluid dynamic description, as the pressure hessian, for example. Evaluating if a specific term contributes to the in-phase or out-of-phase portion of strain acceleration, its possible to evaluate how this portion contributes to vortex evolution, by dissipating vortex motion or enhancing it.

2. VORTEX IDENTIFICATION VERSUS MOMENTUM TRANSFER EQUATIONS

The analysis of vortex identification parameters inside the momentum equations is relatively new. Jeong and Hussain (1995) proposed a criterium with respect to the eigenvalues of pressure Hessian, a tensor composed by the second derivatives of pressure with respect to space coordinates. This criterion is based on a pressure minimum at the vorticity plane. The gradient of the Navier-Stokes equation can be separated into a symmetric and skew-symmetric parts. The pressure hessian can be obtained applying the gradient operator to Navier-Stokes equation

$$\frac{D}{Dt}\mathbf{u} = -\frac{1}{\rho}\nabla\mathbf{p} + \nu\nabla^2\mathbf{u} \tag{1}$$

$$\frac{D}{Dt}(\nabla\mathbf{u}) + \nabla\mathbf{u}\nabla\mathbf{u} = -\frac{1}{\rho}\mathbf{P} + \nu\nabla^2\mathbf{D} \tag{2}$$

where $\mathbf{D}$ represent the strain rate tensor, ν the molecular viscosity and ρ the density. Splitting the velocity gradient, $\nabla\mathbf{u}$ into a symmetric ($\mathbf{D}$) and antisymmetric part ($\mathbf{W}$), the left side of Eq. (2) can be rewritten as

$$\frac{D}{Dt}(\nabla\mathbf{u}) + \nabla\mathbf{u}\nabla\mathbf{u} = \left[\frac{D}{Dt}(\mathbf{D}) + \mathbf{WW} + \mathbf{DD}\right] + \left[\frac{D}{Dt}(\mathbf{W}) + \mathbf{WD} + \mathbf{DW}\right] \tag{3}$$

The right side of Eq. (3) can be also slitted into a symmetric and antisymmetric parts, respectively. The antisymmetric part correspond to the vorticity equation

$$\frac{D}{Dt}(\mathbf{W}) + \mathbf{DW} + \mathbf{WD} = 0 \tag{4}$$

and the symmetric part of Eq. (3) is equal to the left side of Navier-Stokes gradient, resulting into the relation

$$\frac{D}{Dt}(\mathbf{D}) - \nu\nabla^2\mathbf{D} + \mathbf{WW} + \mathbf{DD} = -\frac{1}{\rho}\mathbf{P} \tag{5}$$

The first two terms in Eq. (6) are not considered by the authors. They represent the influence, on the pressure hessian, of local instantaneous material derivative and the viscous effects.

The occurrence of a pressure minimum in a plane requires two positive eigenvalues for pressure hessian. Jeong and Hussain (1995) defined vortices as compact regions where the tensor $\mathbf{D^2} + \mathbf{W^2}$, which balances the pressure term, presents two negative eigenvalues. Since $\mathbf{D}^2 + \mathbf{W}^2$ is symmetrical, real eigenvalues are guaranteed. By rearranging the eigenvalues, $\lambda_i^{\mathbf{D}^2+\mathbf{W^2}}$, so that $\lambda_1^{\mathbf{D}^2+\mathbf{W^2}} \geq \lambda_2^{\mathbf{D}^2+\mathbf{W^2}} \geq \lambda_3^{\mathbf{D}^2+\mathbf{W^2}}$, its possible to conclude that $\lambda_1^{\mathbf{D}^2+\mathbf{W^2}}$ is always positive and $\lambda_3^{\mathbf{D}^2+\mathbf{W^2}}$ is always negative. So, the criterium defines vortical structures as compact regions where $\lambda_2^{\mathbf{D}^2+\mathbf{W^2}} < 0$.

The criterium proposed by Jeong and Hussain (1995) are based on dynamic assumptions, but are restricted by its hypothesis. In the Stokes regime, for example, the pressure gradient in the flow is caused solely by viscous effects, and the notion of vortices can be found in this regime (Dombre *et al.* (1986)).

The work of Haller (2005) establish a relation between his vortex identification criterium, which is based on Lagrangian operators, and define vortices in the region where the tensor

$$\nu\nabla^2\mathbf{D} + \mathbf{WW} + \mathbf{DD} - \frac{1}{\rho}\mathbf{P} \tag{6}$$

is positive indefinite.

3. NEW SET OF VORTEX IDENTIFICATION PARAMETERS

It is very common, in many physical and mathematical situations, the identification of the necessity to compare the diagonal components of a matrix with its off-diagonal ones. One simple idea is to measure this competition by an overall ratio index. A parameter which has in the numerator and the denominator, the intensities of one and other sides of this balance: diagonal and off-diagonal components of the strain acceleration tensor, evaluated in the strain basis.

Here, we have developed two methods for an anisotropic comparison between the diagonal and off-diagonal components of a matrix. Following Haller (2005) and Thompson (2008) we use the matrix associated with the second Rivlin-Ericksen tensor. The first method which will be called here line-method is to compare, in the diagonal components of the tensor $\mathbf{A}_2^{\mathbf{A}_1}$, acceleration tensor on the basis of the strain tensor, $\mathbf{L}$, the part of each component that comes from the diagonal and off-diagonal component of tensor $\mathbf{A}_2^{\mathbf{A}_1}$

$$AR_i^A = \frac{(A|_{ii})^2}{(A^2)|_{ii}} \tag{7}$$

where $\mathbf{A} = \mathbf{A}_2^{\mathbf{A}_1}$ represents the second Rivlin-Ericksen tensor (strain acceleration) on the basis of the first one (strain rate tensor). An isotropic version was also formulated, based on the same idea provided in above relations

$$IR = \frac{A_{ii}A_{ii}}{[AA]_{jj}} \tag{8}$$

4. STRAIN ACCELERATION VERSUS MOMENTUM TRANSFER EQUATIONS

This work is intended to present the evaluation of strain acceleration in the momentum equations. Although the parameters proposed in this section are written as the parts of Euler equations, it can be simply applied to the Navier-Stokes equations as well. The procedure will be demonstrated for the Euler equations, but it can be easily applied to

Navier-Stokes flow as wel. Applying the gradient operator to the Euler equations, its possible to establish, in tensorial notation,

$$\frac{d\mathbf{D}}{dt} = -(\mathbf{D}^2 + \mathbf{W}^2) - \frac{1}{\rho}\mathbf{P} \tag{9}$$

Applying the the tensor $\mathbf{A}_2 = d\mathbf{D}/dt + \mathbf{A}_1 \nabla \mathbf{v} + \nabla \mathbf{v}^T \mathbf{A}_1$ in Eq. (9), results in the following relation between the strain acceleration and the pressure Hessian:

$$\frac{1}{2}\mathbf{A}_2 = (\mathbf{D} + \mathbf{W})(\mathbf{D} - \mathbf{W}) - \frac{1}{\rho}\mathbf{P} \tag{10}$$

The Eq. (10) permits an direct analysis of how the terms of Euler equations can influence in the in-phase or out-of-phase parts of $\mathbf{A}_2$ tensor with respect to $\mathbf{A}_1 = 2\mathbf{D}$ tensor. In other words, it turns possible to establish how a specific term tends to excite or dissipates vortices locally. It is possible now to indicate parameters that can measure the influence of each part. The first one, $IR_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$, evaluates how much the tensor $\mathbf{N} = (\mathbf{D} - \mathbf{W})(\mathbf{D} + \mathbf{W})$, in the base of $\mathbf{A}_1$ eigenvectors, is in-phase or out-of-phase with $\mathbf{A}_1$ tensor and can be represented by the following relation

$$IR_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})} = 1 - \frac{2}{\pi}\cos^{-1}\left(\frac{\left[\mathbf{N}^{\mathbf{A}_1}\right]_{ii}\left[\mathbf{N}^{\mathbf{A}_1}\right]_{ii}}{\left[\mathbf{N}^{\mathbf{A}_1}\mathbf{N}^{\mathbf{A}_1}\right]_{jj}}\right) \tag{11}$$

In the case when the operator $IR_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$ presents values between 0 and 0.5, it is possible to conclude that the tensor $(\mathbf{D} - \mathbf{W})(\mathbf{D} + \mathbf{W})$ is out-of-phase with respect to $\mathbf{A}_1$ and contributes to the vortex formation. When the same parameters presents values between 0.5 and 1, the same tensor is in-phase to $\mathbf{A}_1$ and tends to impose a stretch behavior in material elements in the respective region. The operator IR_P evaluates how the pressure Hessian, in the strain base is in-phase or out-of-phase with the same tensor and can be represented by the following relation

$$IR_{\mathbf{P}} = 1 - \frac{2}{\pi}\cos^{-1}\left(\frac{\left[\mathbf{P}^{\mathbf{A}_1}\right]_{ii}\left[\mathbf{P}^{\mathbf{A}_1}\right]_{ii}}{\left[\mathbf{P}^{\mathbf{A}_1}\mathbf{P}^{\mathbf{A}_1}\right]_{jj}}\right) \tag{12}$$

The operator $IR^{A2}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$ do the same evaluation with respect to the tensor $\mathbf{A}_2^{\mathbf{A}_1}$ corresponde ao tensor $(\mathbf{D} - \mathbf{W})(\mathbf{D} + \mathbf{W})$ and is represented by the equation bellow

$$IR^{A2}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})} = 1 - \frac{2}{\pi}\cos^{-1}\left(\frac{\|\Phi^{\mathbf{A}_1}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}\|}{\|\Phi^{\mathbf{A}_1}_{\mathbf{A}_2}\|}\right) \tag{13}$$

Based on these relations, as the operator $IR^{A2}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$ tends to unity, greater is the share in the out-of-phase parte of tensor $\mathbf{A}_2$ with $\mathbf{A}_1$ and greater will be its influence in the formation and excitation of vortices. IR^{A2}_P do the same analysis with respect to the previously mentioned tensor, but with the pressure hessian in the in-phase part of $\mathbf{A}_2$ with $\mathbf{A}_1$, following the relation

$$IR^{A2}_{\mathbf{P}} = 1 - \frac{2}{\pi}\cos^{-1}\left(\frac{\|\Phi^{\mathbf{A}_1}_{\mathbf{P}}\|}{\|\Phi^{\mathbf{A}_1}_{\mathbf{A}_2}\|}\right) \tag{14}$$

As much as the IR^{A2}_P operator tends to unity, greater will be its participation in the in-phase part of $\mathbf{A}_2$ with $\mathbf{A}_1$. Finally, the operators $IR^{A2'}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$ and $IR^{A2'}_P$ do the same analysis, but with respect to the out-of-phase parte of $\mathbf{A}_2$ with respect to $\mathbf{A}_1$. These parameters can be written by the following relations

$$IR^{A2'}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})} = 1 - \frac{2}{\pi}\cos^{-1}\left(\frac{\|\tilde{\Phi}^{\mathbf{A}_1}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}\|}{\|\tilde{\Phi}^{\mathbf{A}_1}_{\mathbf{A}_2}\|}\right) \tag{15}$$

$$IR^{A2'}_{\mathbf{P}} = 1 - \frac{2}{\pi}\cos^{-1}\left(\frac{\|\tilde{\Phi}^{\mathbf{A}_1}_{\mathbf{P}}\|}{\|\tilde{\Phi}^{\mathbf{A}_1}_{\mathbf{A}_2}\|}\right) \tag{16}$$

5. ABC FLOW

The ABC flow is a classical flow due to its chaotic behavior even for laminar flows (Dombre et al., 1986).

5.1 Lamb vector and helicity density

The local geometrically orthogonal decomposition of the velocity vector $\mathbf{v}$ with respect to the vorticity vector $\mathbf{w}$ introduces two quantities of crucial importance in vorticity dynamics: the vector $\mathbf{w} \times \mathbf{v}$, known as Lamb vector, and the scalar $\mathbf{w} \cdot \mathbf{v}$, known as helicity density. The two interesting non-trivial cases are when the helicity density or the Lamb vector vanishes. When $\mathbf{w} \cdot \mathbf{v} = 0$ and $\mathbf{w} \times \mathbf{v} \neq 0$, the flow is called *complex lamellar flow*. It exists if and only if

$$\mathbf{v} = \lambda \nabla \xi \tag{17}$$

where ξ = const are equi-potential surfaces orthogonal to the streamlines everywhere (potential flow, also called lamellar flow, is obtained when $\lambda = 1$). When $\mathbf{w} \cdot \mathbf{v} = 0$ and $\mathbf{w} \times \mathbf{v} = 0$ the streamlines are parallel to the vorticity lines, or

$$\mathbf{w} = \zeta \mathbf{v} \tag{18}$$

which implies that the velocity is an eigenvector of the curl operator. This kind of flow is called Beltrami (or helical) flow. If ζ is constant, the flow is specifically called Trkalian.

When the Lamb vector is a complex lamellar field or

$$\mathbf{w} \times \mathbf{v} = g \nabla h \tag{19}$$

there exist a set of surfaces h =const, called Lamb surfaces which are orthogonal to the Lamb vector everywhere. It can be shown that the existence of the Lamb surfaces imply the integrability of the system and therefore this kind of flow cannot be chaotic. Therefore, A Beltramian flow is a candidate of a chaotic flow. However, if $\nabla \zeta \neq 0$ the velocity is still integrable, since the velocity will be on the surfaces normal to $\nabla \zeta$. Therefore, the only possibility of an incompressible chaotic steady flow is when $\nabla \zeta = 0$, where the flow is Trkalian.

Arnold (1965), seeking steady inviscid chaotic flow, proposed a Trkalian flow where $\zeta = 1$ or $\mathbf{w} = \mathbf{v}$. The ABC flow in cartesian coordinates is given by

$$u = A \sin z + C \cos y \tag{20}$$

$$v = B \sin x + A \cos z \tag{21}$$

$$u = C \sin y + B \cos x \tag{22}$$

6. RESULTS

Figure 1 shows the values of the isotropic normalized ratio that compares linear acceleration deformation, in the sense provided by the covariant convected time derivative, to angular acceleration gradient (in the same sense). Higher values correspond to hyperbolic-like behavior.

Also all three fields of anisotropic index associated to a line-method are shown in Fig. 1. Since the orientation of the index is different depending on the point considered, we have decided to produce indexes based on the comparison between the three anisotropic indexes of each method. What is shown in the first row of the second column is related to the highest (among three) value of the tendency to evolve persistently the same material line. Figure 1 show the contours of the Q-criterion, proposed by $Hunt et al.$ (1988) and Qs-criterion, proposed by $Tabor and Klapper$ (1994), which is an Euclidian invariant version of Q.

All the criteria are normalized in order to obtain the same basis for comparison so if a certain region presents values below 0.5, this region remains in a vortical region, according to the criterion.

In order to evaluate the Euler equation terms with respect to the in-phase or out-of-phase parts of $\mathbf{A}_2$ with respect to $\mathbf{A}_1$ tensor, its possible to evaluate those parameters in Fig. 2. The left side images evaluates how the tensor $(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})$ is in-phase with $\mathbf{A}_2$, its share of $\mathbf{A}_2$ tensor and the share of the in-phase part of $\mathbf{A}_2$ with $\mathbf{A}_1$ tensor, respectively. In the right side, the same analysis is performed to the pressure Hessian $\mathbf{P}$. Its possible to notice that the tensor $(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})$ works with a big share of the out-of-phase part of $\mathbf{A}_2$ tensor in the vortical region along y-direction identified by the the parameters IR and $AR^A_{1,2,3}$ in Fig. 1.

7. FINAL REMARKS

Therefore, its possible to conclude that the application of the parameters discussed in the present work can lead to important conclusion, with respect to the evolution of vortices or coherent structures in turbulent flows. This analysis, by the way, can lead to the development to new theory based on the strain acceleration. For example, the same knowledge can be applied to the homogeneous theory in multiphase flows, mixing, combustion and other phenomena described also by the momentum equations. The isotropic and anisotropic parameters must be used coupled with the discussed entities in order to measure the importance of evaluating the in-phase or out-of-phase part of an specific term of Euler or Navier-Stokes equation locally.

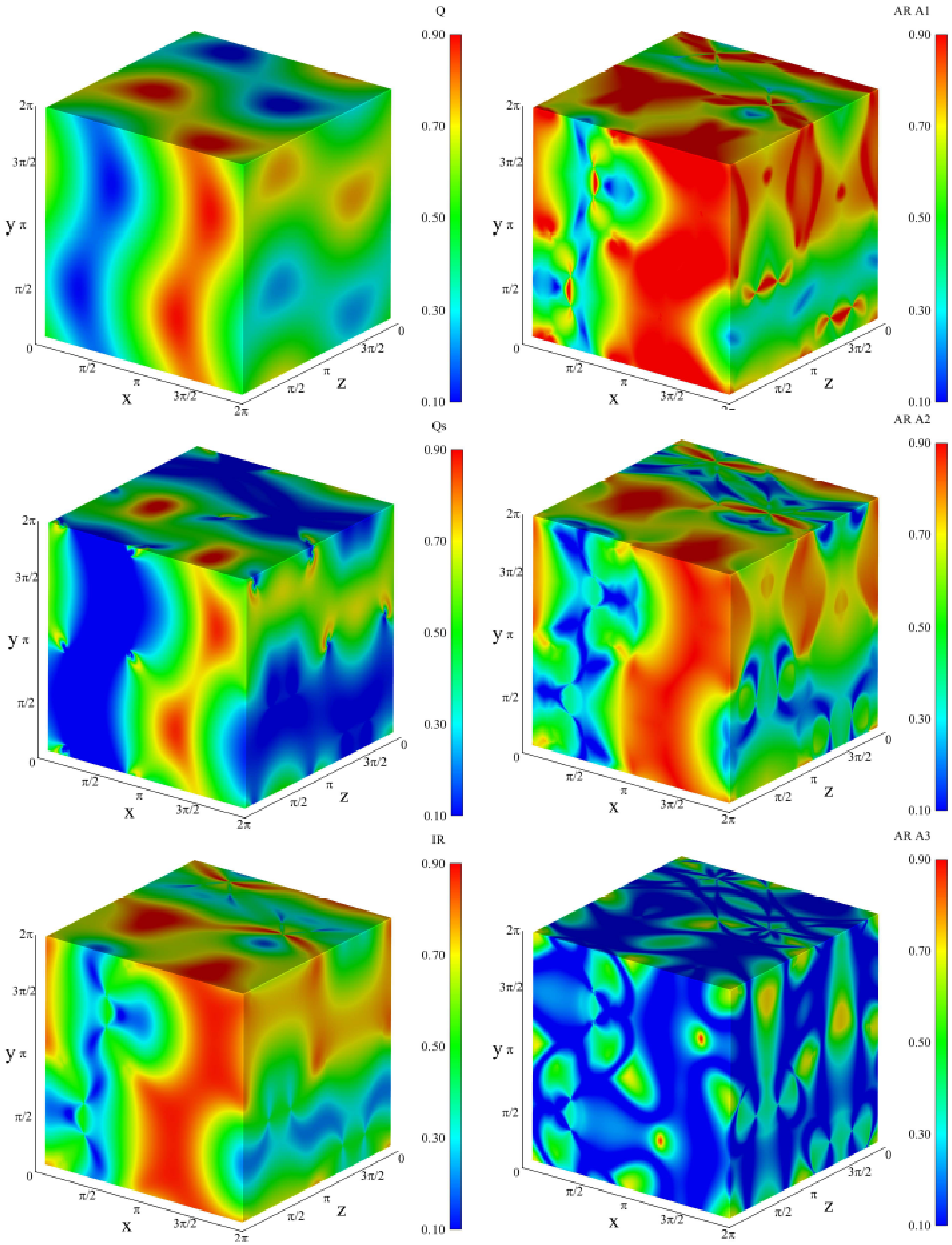

Figure 1. Vortex identification criteria evaluated in ABC flow.

8. ACKNOWLEDGEMENTS

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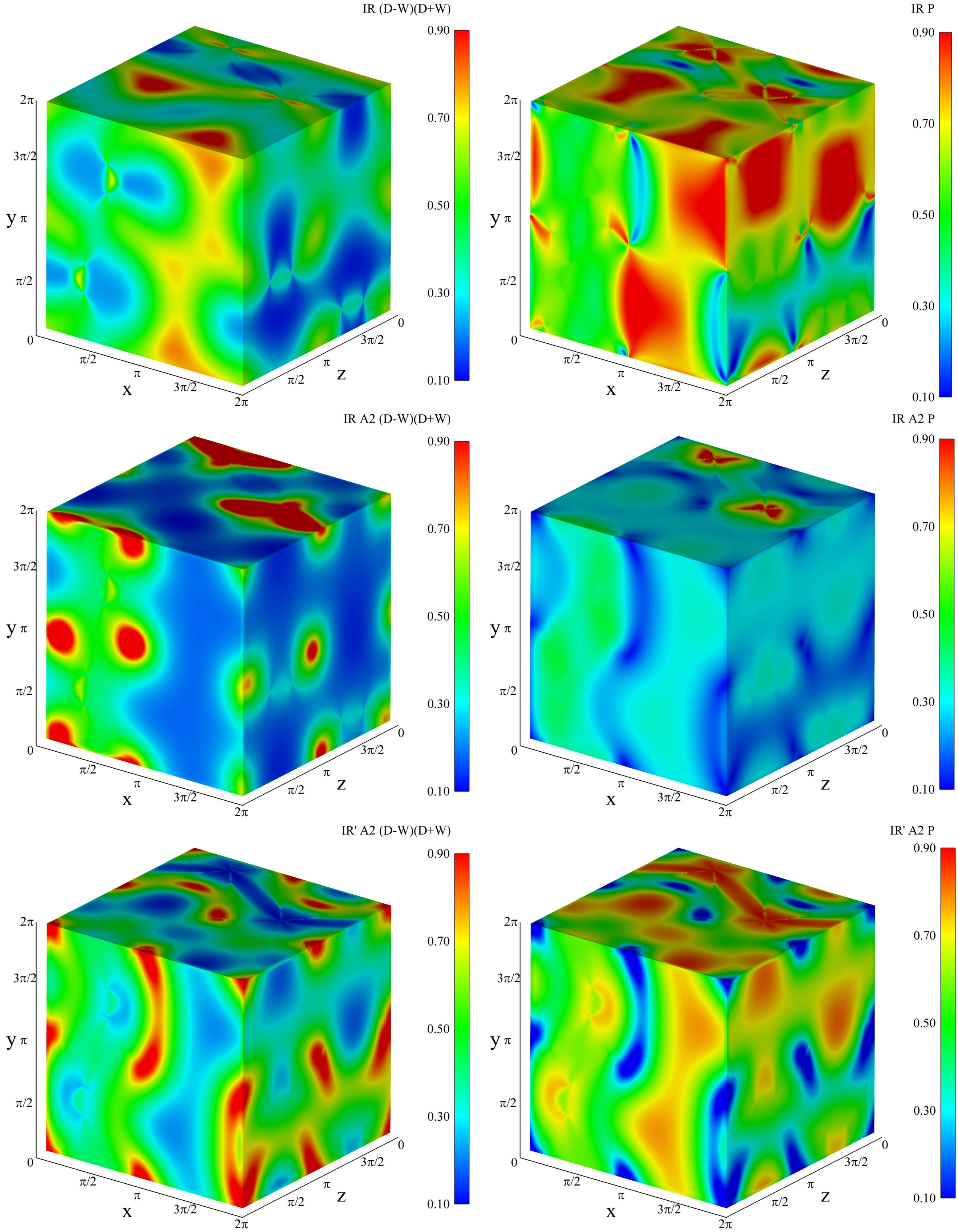

Figure 2. $IR_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$, IR_P, $IR^{A2}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$, IR^{A2}_P, $IR^{A2'}_{(\mathbf{D}-\mathbf{W})(\mathbf{D}+\mathbf{W})}$ e $IR^{A2'}_P$ operators in ABC-flow.

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