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INTEGRAL TRANSFORMATION OF MULTIDIMENSIONAL PHASE CHANGE PROBLEMS: COMPUTATIONAL AND PHYSICAL ANALYSIS

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Abstract. *The Generalized Integral Transform Technique (GITT) is employed in the hybrid numerical-analytical solution of two-dimensional Stefan-type phase change problems. Complementing previous contributions, the present work provides a critical analysis of reordering schemes for eigenfunction expansions in this multidimensional moving boundary problem, when the domain dimensions change with both the space and time variables. In addition, the present work provides a physical analysis of the phase change problem through variation of the corresponding governing parameters, such as Stefan number and the initial shape of the moving boundary.*

Keywords*: Phase change, heat conduction, moving boundary, Stefan problems, hybrid methods, integral transforms.*

1. INTRODUCTION

Transient diffusion problems with moving boundaries are commonly called Stefan problems, in reference to the work of the Slovenian mathematician and physicist Jozef Stefan (Stefan, 1889). Actually, this class of problems was first studied by Lamé and Clapeyron in 1831 but Stefan was the first one to solve it. The original Stefan problem deals with the determination of the rate of ice formation in the polar seas. Along the last few decades, moving boundary problems have received further attention by various research groups around the world, mainly because several other applications were observed and dealt with through this type of mathematical formulation. A characteristic of heat conduction problems with phase change is the existence of a moving interface separating the phases. At this interface, the latent heat associated with the phase change can be absorbed or released, and as a result of the process of heat transfer in two phases the interface position is changing continuously in time (Rubinsky and Cravahlo, 1981). One of the challenges that lead to the study of this class of problems is the lack of a *priori* knowledge of the phase-change interface position, thus making it a nonlinear problem, of remarkable difficulty in obtaining analytical solutions.

The Generalized Integral Transform Technique (GITT) (Cotta, 1990; Cotta, 1993) is an eigenfunction expansion methodology for solving linear or nonlinear convection-diffusion problems, especially those not a priori transformable by the classical approach. It has also found applicability in the context of moving boundary problems, particularly motivated by applications in thermal protection systems of space vehicles, offering high accuracy and mild computational cost. In the early work of Cotta (1986), the GITT approach was used to obtain a hybrid solution of one-dimensional diffusion problems with moving boundaries in finite media with a prescribed evolution of the boundary. In such analysis, the diffusion equation was presented in a very general form, including diffusion in solids of usual geometries such as slabs, cylinders and spheres, while the diffusivity was taken as independent of the concentration field. To illustrate the approach, an application on metals oxidation at high temperatures was dealt with. Results were obtained for the oxidation of β-zircaloy, a typical material used in the cladding of nuclear fuel rods.

Diniz *et al.* (1990) extended the GITT in solving a one-dimensional formulation of thermal ablation in protection systems for space vehicles. The authors illustrate the applicability of the technique obtaining results for the temperature profiles and the moving interface, using different forms of the prescribed heat flux on one side of the plate as a function of time, and comparing with results previously reported in the literature. Ruperti and Cotta (1991) applied the GITT to analyze the one-dimensional multiregion formulation of thermal ablation controlled by conduction heat transfer, again in spacecraft thermal protection systems. The convergence of the hybrid analytical-numerical solution was illustrated by truncating the eigenfunction expansions at increasing orders, and critical comparisons were made with purely numerical solutions for an Aluminum-Teflon system. The GITT methodology and the computational implementation are described in more detail in Ruperti *et al.* (1992), also aimed at the design of thermal shields for spacecraft atmospheric reentry. Cotta (1993) devoted a chapter in this book to compile the procedures and analysis for the integral transformation of moving boundary problems. The author presented the mathematical formulation of this general class of problems in one-dimensional form, and then described the procedure for obtaining the coupled ordinary differential system resulting from the integral transformation. Also supplied is the matrix inversion procedure in order to obtain the formal solution of the problem. A comparison and covalidation of numerical and hybrid techniques was presented by Ruperti *et al.* (1998) for the one-dimensional Stefan problem in transient heat conduction. The GITT was used to solve the problem,

after adopting a transient filter strategy. The study also employed he finite volume method combined with a coordinate transformation in order to fix the moving boundary. The results for the temperature profile and moving boundary position were presented for two different materials, metal and paraffin. An analysis of heat diffusion with moving boundary in cylindrical bodies representing burning wood was provided in Macêdo *et al.* (2000). The GITT methodology was used with the local-instantaneous filtering strategy, in order to obtain homogeneous boundary conditions and enhance convergence behavior of the eigenfunction expansions. The results were compared with those reported in the literature and demonstrated an excellent agreement.

More recently, a simulation approach for the analysis of ablation in thermal protection systems (TPS) with atmospheric reentry in ballistic flights was developed by Ruperti *et al.* (2004). First, the authors proposed a differential model for the analysis of ablative thermal protection, which involves the use of materials with low thermal diffusivity. The results for the one-dimensional problem of thermal ablation in plates were compared with those obtained by the solution of improved lumped formulations also developed in this work. In addition, an integrated symbolic-numerical code (TPS-Nose) was built using the *Mathematica* symbolic computational platform for the derivation and calculation of all quantities related to the flight, and to obtain the transient behavior of the TPS recession and its thermal performance for uniform or variable thickness along the vehicle nose. Gomes *et al.* (2005) also studied the problem of heat conduction with ablation in aerospace thermal protection systems, and the one-dimensional formulation of this process was performed using the GITT, in different coordinates systems. In a recent work, Sias *et al.* (2009), the GITT approach was once more used for the hybrid analytical-numerical solution in the ablation of thermal protection systems for space vehicles during atmospheric reentry. Comparisons were shown against a reference case previously studied, and then the more realistic aerodynamic heating of a ballistic flight was illustrated, using thermophysical properties of actual thermal protection materials. Then, following this development, Sias *et al.* (2010) have generalized their analysis to include a more general heat transfer model of ablation including materials that undergo pyrolysis.

In this context, the present contribution aims at further advancing a hybrid numerical-analytical approach for moving boundary problems, now handling the two-dimensional Stefan-type phase change problems using GITT. In a natural sequence to a previous contribution (Monteiro *et al.*, 2010), the present work deals with the analysis of reordering schemes for the eigenfunction expansions, and also brings a physical analysis of the phase change problem through variation of the governing parameters, such as Stefan number and the initial shape of the moving boundary.

2. ANALYSIS

The problem to be studied is related to phase change of a pure substance, in which there is movement of the boundary from an initial temperature distribution and from an initial shape of the phase-change interface as a function of the x-coordinate. The vertical boundaries are kept insulated, and it is considered that the temperature in the region that has already changed phase is maintained equal to the phase change temperature at the interface throughout the transient process, as shown in Figure 1. Therefore, the mathematical formulation for this problem in dimensionless form is written as (Monteiro *et al.*, 2010):

$$\frac{\partial\theta(X,Y,\tau)}{\partial\tau}=\frac{\partial^2\theta(X,Y,\tau)}{\partial X^2}+\frac{\partial^2\theta(X,Y,\tau)}{\partial Y^2},\quad 0<X<1,\quad 0<Y<S(X,\tau)\quad \tau>0 \tag{1}$$

$$\theta(X,Y,0)=F(X,Y),\quad 0\le X\le 1,\quad 0\le Y\le S(X,\tau) \tag{2}$$

$$\frac{\partial\theta(0,Y,\tau)}{\partial X}=0;\quad \frac{\partial\theta(1,Y,\tau)}{\partial X}=0,\quad 0\le Y\le S(X,\tau)\quad \tau>0 \tag{3,4}$$

$$\frac{\partial\theta(X,0,\tau)}{\partial Y}=0;\quad \theta(X,S(X,\tau),\tau)=0,\quad 0\le X\le 1\quad \tau>0 \tag{5,6}$$

$$\frac{\partial S(X,\tau)}{\partial\tau}+Ste\left[\left(\frac{\partial S(X,\tau)}{\partial X}\right)^2+1\right]\frac{\partial\theta(X,S(X,\tau),\tau)}{\partial Y}=0,\quad 0<X<1,\quad \tau>0 \tag{7}$$

$$S(X,0)=G(X),\quad 0\le X\le 1 \tag{8}$$

Specific expressions of the interface energy balance may be written for the moving boundary extremes as:

$$S(0,\tau)=S_0(\tau);\quad S(1,\tau)=S_1(\tau),\quad \tau>0 \tag{9,10}$$

$$\frac{dS_0(\tau)}{d\tau}+Ste\frac{\partial\theta(0,S_0(\tau),\tau)}{\partial Y}=0;\quad \frac{dS_1(\tau)}{d\tau}+Ste\frac{\partial\theta(1,S_1(\tau),\tau)}{\partial Y}=0\quad \tau>0 \tag{11,12}$$

$$S_0(0)=G(0);\quad S_1(0)=G(1) \tag{13,14}$$

The following dimensionless groups were employed in the above formulation:

$$\theta(X,Y,\tau)=\frac{T(x,y,t)-T_m}{\Delta T_c};\quad X=\frac{x}{L};\quad Y=\frac{y}{L};\quad \tau=\frac{\alpha_s t}{L^2};\quad S(X,\tau)=\frac{s(x,t)}{L};\quad Ste=\frac{C_p}{L_{fs}}\Delta T_c \tag{15-20}$$

where L is the domain length between the two insulated surfaces, α_s is the thermal diffusivity of the phase change material, C_p is the specific heat, L_{fs} is the latent heat of phase change, Ste is the Stefan number and ΔT_c is a reference temperature difference.

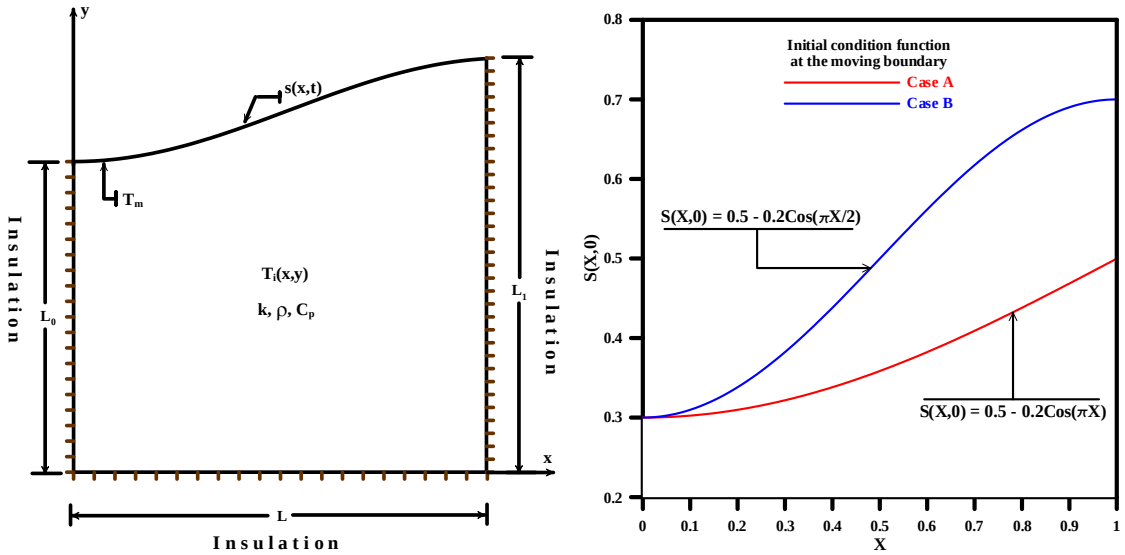

Figure 1. Description of the phase-change heat conduction problem with a moving interface as function of position and time.

The problem defined by Eqs. (1-8) presents non homogeneous boundary conditions for the moving boundary heat balance equation. Therefore, before applying the integral transformation process, it is advisable to filter the original problem in order to make the boundary conditions homogeneous, and thus to improve the convergence of the eigenfunction expansions. Here, a quite straightforward filter expression is proposed as (Monteiro *et al.*, 2010):

$$S(X,\tau)=(1-X)S_0(\tau)+XS_1(\tau)+H(X,\tau) \tag{21}$$

where $S_0(\tau)$ and $S_1(\tau)$ are, respectively, the positions of the moving boundary at the borders X=0 and 1. The resulting homogeneous problem for the filtered surface position, $H(X,\tau)$, may then be integral transformed in the X direction, to yield an ODE system for the transformed boundary positions, to be numerically solved together with the transformed temperatures, as a function of time. After application of the filter, the nonlinear moving boundary equation is given by:

$$\frac{\partial H(X,\tau)}{\partial \tau}+(1-X)\frac{dS_0(\tau)}{d\tau}+X\frac{dS_1(\tau)}{d\tau}+Ste\left[\left(S_1(\tau)-S_0(\tau)+\frac{\partial H(X,\tau)}{\partial X}\right)^2+1\right]\frac{\partial \theta(X,S(X,\tau),\tau)}{\partial Y}=0, \tag{22}$$

$$0<X<1,\quad \tau>0$$

$$H(X,0)=G(X)-(1-X)S_0(0)-XS_1(0),\quad 0\le X\le 1 \tag{23}$$

$$H(0,\tau)=0;\quad H(1,\tau)=0,\quad \tau>0 \tag{24,25}$$

An analytical solution for the problem defined by Eqs. (1) to (4) and Eqs. (22) to (25) is unlikely to be obtainable, in light of the nonlinear nature of the moving boundary formulation. A hybrid analytical-numerical solution of the problem is then here developed by extending the ideas of the generalized integral transform technique (GITT) as described and reviewed in Cotta (1990), Cotta (1993), Cotta and Mikhailov (1997), Cotta (1998) and Cotta and Mikhailov (2006). Here, we propose to perform a double integral transformation of the temperature field, in both space coordinates, and a single transformation in the X-coordinate for the filtered moving boundary position. The integral transform pairs for the temperature field and the filtered moving boundary position are given as, respectively:

$$\tilde{\bar{\theta}}_{ik}(\tau)=\int_0^1\int_0^{S(X,\tau)}\frac{\psi_i(X,Y,\tau)\Gamma_k(X)}{L_i(X,\tau)M_k}\theta(X,Y,\tau)dYdX \quad ; \quad \theta(X,Y,\tau)=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\psi_i(X,Y,\tau)\Gamma_k(X)\tilde{\bar{\theta}}_{ik}(\tau) \tag{26,27}$$

(transform) (inverse)

$$\bar{H}_m(\tau)=\frac{1}{N_m}\int_0^1\Omega_m(X)H(X,\tau)dX \quad ; \quad H(X,\tau)=\sum_{m=1}^{\infty}\Omega_m(X)\bar{H}_m(\tau) \tag{28,29}$$

(transform) (inverse)

The quantities $\psi_i(X,Y,\tau)$ and $\Omega_m(X)$ are eigenfunctions of appropriately chosen eigenvalue problems, and details in obtaining these eigenvalue problems and respective solutions are shown in the work of Monteiro *et al.* (2010). The resulting transformed system, after operating on both the energy and interface equations, Eqs. (1, 22), is given by:

$$\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}A_{ijkl}(\tau)\frac{d\,\tilde{\bar{\theta}}_{jl}(\tau)}{d\tau}+B_{ik}(\tau)\frac{dS_0(\tau)}{d\tau}+C_{ik}(\tau)\frac{dS_1(\tau)}{d\tau}+\sum_{m=1}^{\infty}D_{ikm}(\tau)\frac{d\bar{H}_m(\tau)}{d\tau}=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}E_{ijkl}(\tau)\tilde{\bar{\theta}}_{jl}(\tau) \tag{30}$$

$$\sum_{n=1}^{\infty}F_{mn}(\tau)\frac{d\bar{H}_n(\tau)}{d\tau}+G_m(\tau)\frac{dS_0(\tau)}{d\tau}+F_m(\tau)\frac{dS_1(\tau)}{d\tau}=Ste\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}I_{ikm}(\tau)\tilde{\bar{\theta}}_{ik}(\tau) \tag{31}$$

$$\frac{dS_0(\tau)}{d\tau}=\frac{Ste}{S_0(\tau)}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\beta_i Sin(\beta_i)\tilde{\bar{\theta}}_{ik}(\tau)\,; \quad \frac{dS_1(\tau)}{d\tau}=\frac{Ste}{S_1(\tau)}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\beta_i Sin(\beta_i)Cos(\lambda_k)\tilde{\bar{\theta}}_{ik}(\tau) \tag{32,33}$$

The same operations can be performed over the initial conditions given by Eqs. (2) and (23), to furnish

$$\tilde{\bar{\theta}}_{ik}(0)=a_i b_k \; ; \quad \bar{H}_m(0)=a_m \tag{34,35}$$

All coefficients of the transformed system shown above are analytically derived using a platform of symbolic computation, *Mathematica v.7.0* (Wolfram, 2005).

In this two-dimensional application, the final integral transform solution for the temperature field is expressed as a double summation according to the inverse formula, Eq.(27). From a computational point of view, only a truncated version of such nested summations can be actually evaluated. However, if one just truncates each individual summation to a certain prescribed finite order, computations become quite ineffective, and even risky. By following this path some still important information to the final result can be disregarded due to the fixed summations limits, while other terms are accounted for that have essentially no contribution to convergence of the potential in the relative accuracy required. Therefore, for an efficient computation of these expansions, the infinite multiple summations should first be converted to a single sum representation with the appropriate reordering of terms according to their individual contribution to the final numerical result. Then, it would be possible to evaluate the minimal number of eigenvalues and related derived quantities required to reach the user-prescribed accuracy target (Correa et al., 1997; Cotta & Mikhailov, 1997). Since the final solution is not, of course, known *a priori*, the parameter which shall govern this reordering scheme must be chosen with care. The most common choice of reordering strategy is based on the argument of the dominating exponential term, which offers a good compromise between the overall convergence enhancement and simplicity in use. In the case of a two-dimensional conduction problem in a fixed domain, the reordering criterion based on eigenvalues summation, results in a new set of eigenvalues and a single index given by $\mu_p{}^2=\mu_{ik}{}^2=\beta_i{}^2+\lambda_k{}^2$, where β_i and α_k are the respective eigenvalues of the problem in each space coordinate (Mikhailov and Cotta, 1996). The criterion for reordering of terms in this work is based on the proportionality associated with the eigenvalues in the form $\mu_p(X,\tau)^2=\mu_{ik}(X,\tau)^2=\beta_i{}^2/S(X,\tau)^2+\lambda_k{}^2$. However, it is clear that this relationship now depends on position and time. For convergence analysis and comparisons of different reordering criteria, following the same rule and scheme described by Mikhailov and Cotta (1996), the criterium is first evaluated with S(X,0) at the leftmost position (X=0), then at the average value ($X=X_m$) and finally at the rightmost position (X=1), for the two cases of initial conditions considered in this work. Therefore, the employed reordering criteria are: $\mu_p(0,0)^2=\beta_i{}^2/S(0,0)^2+\lambda_k{}^2$; $\mu_p(X_m,0)^2=\beta_i{}^2/S(X_m,0)^2+\lambda_k{}^2$ and $\mu_p(1,0)^2=\beta_i{}^2/S(1,0)^2+\lambda_k{}^2$.

According to the explanation above, for the construction of the transformed differential system, we set a new index p, for the reordered indices i and k, and an index q, for the reordered indices j and l, i.e., (i,k)→p and (j,l)→q. Thus, the sums are reduced to the following form:

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\rightarrow\sum_{p=1}^{\infty} \; ; \quad \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\rightarrow\sum_{q=1}^{\infty} \tag{36,37}$$

Therefore, Eqs. (3.18) and (3.19) are written in terms of simple summations as:

$$\sum_{q=1}^{\infty} A_{pq}(\tau)\frac{d\bar{\theta}_q(\tau)}{d\tau}+B_p(\tau)\frac{dS_0(\tau)}{d\tau}+C_p(\tau)\frac{dS_1(\tau)}{d\tau}+\sum_{m=1}^{\infty} D_{pm}(\tau)\frac{d\bar{H}_m(\tau)}{d\tau}=\sum_{q=1}^{\infty} E_{pq}(\tau)\bar{\theta}_q(\tau) \tag{38}$$

$$\sum_{n=1}^{\infty} F_{mn}(\tau)\frac{d\bar{H}_n(\tau)}{d\tau}+G_m(\tau)\frac{dS_0(\tau)}{d\tau}+F_m(\tau)\frac{dS_1(\tau)}{d\tau}=Ste\sum_{p=1}^{\infty} I_{pm}(\tau)\bar{\theta}_p(\tau) \tag{39}$$

$$\frac{dS_0(\tau)}{d\tau}=\frac{Ste}{S_0(\tau)}\sum_{p=1}^{\infty} I0_p\bar{\theta}_p(\tau)\,;\quad I0_p=\beta_{i(p)}\sin(\beta_{i(p)})\delta_{i(p)k(p)} \tag{40}$$

$$\frac{dS_1(\tau)}{d\tau}=\frac{Ste}{S_1(\tau)}\sum_{p=1}^{\infty} I1_p\bar{\theta}_p(\tau)\,;\quad I1_p=\beta_{i(p)}\sin(\beta_{i(p)})\cos(\lambda_{k(p)}) \tag{41}$$

$$\bar{\theta}_p(0)=a_{i(p)k(p)}\,;\quad \bar{H}_m(0)=a_m\,;\quad S_0(0)=G(0);\quad S_1(0)=G(1) \tag{42,43}$$

The ordinary differential equations system Eqs. (38) to (43) constitutes an initial value problem of coupled nonlinear equations, which should be truncated to an order of terms NT (temperature) and NS (boundary position), both large enough to ensure convergence of the transformed temperature field, $\bar{\theta}_p(\tau)$, and to the transformed moving boundary position, $\bar{H}_m(\tau)$, within the error control criteria established by the user. It is advisable to make use of special subroutines for stiff problems, such as subroutine DIVPAG from the IMSL Library (1991). Therefore, once the transformed potentials are numerically obtained, the temperature field and the moving boundary position distributions are recovered from the inverse analytical formulae given by Eqs. (27) and (29). The reader is referred to (Monteiro *et al.*, 2010) for the complete specification of the transformed system coefficients in Eqs.(38-43).

3. RESULTS AND DISCUSSION

Numerical results for the moving boundary position and for the temperature field were obtained from a computational code developed in the programming language FORTRAN 95/2003. The code was implemented on a PENTIUM DUAL CORE 1.73 GHz microcomputer, and the system given by Eqs. (38) to (43) was handled through the subroutine DIVPAG from the IMSL Library (1991). The hybrid solution was then computed using up to sixty terms (NT = 60) in the reordered eigenfunction expansions, and results were obtained for two different initial conditions functions of the moving boundary position, as presented by Gupta and Kumar (1986), for Stefan number equal to unity. These two cases were here named Cases A and B, corresponding to the initial positions $S(X,0)=0.5–0.2\text{Cos}(\pi X/2)$ and $S(X,0)=0.5–0.2\text{Cos}(\pi X)$. In addition, it is presented results for Stefan numbers equal to 0.1 and 0.01 for these test-cases.

Figures 2.a,b show the validation of the GITT solution in comparison to the 1D limiting case, for two initial positions of the uniform boundary, S = 0.3 and 0.5, and Stefan number equal to unity. The 2D solution tends to these limiting cases as the number of terms increases, showing no significant differences in the graphic scale for NT=40.

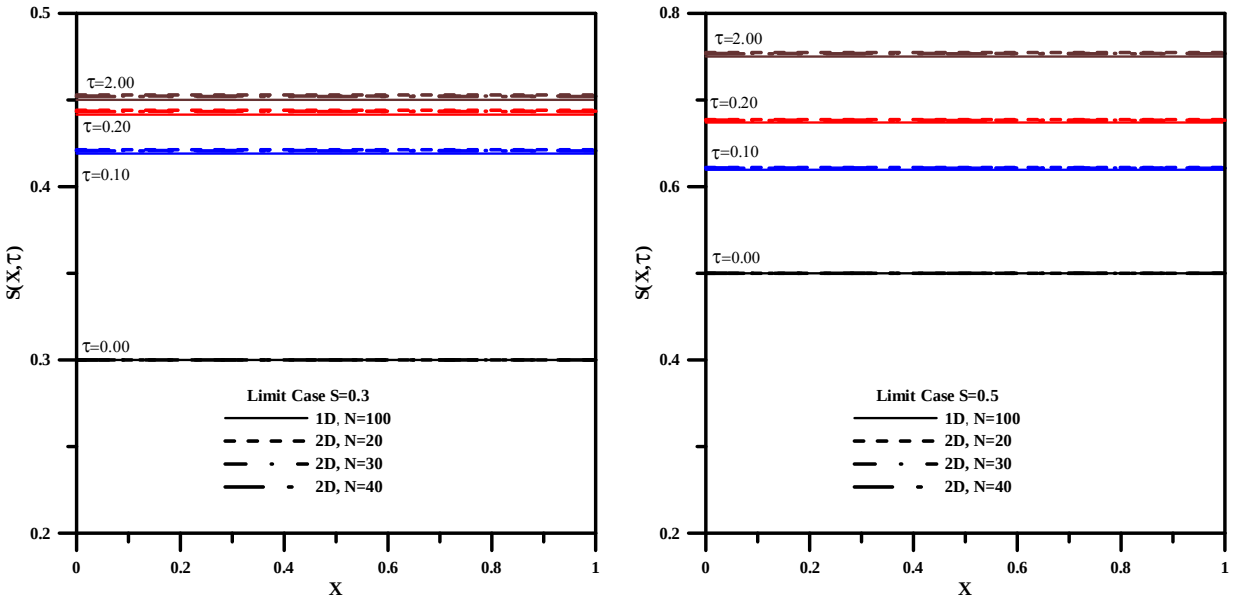

Figure 2. Verification of the phase-change moving boundary position obtained by GITT from the two-dimensional model, against the one-dimensional solution at different times: a) S(X,0)=0.3. b) S(X,0)=0.5.

Table 1 shows a comparison of three different reordering criteria in the convergence of the moving boundary position, for case A and Ste=1, for values at the two boundaries, X=0 and 1 and for different times. The truncation orders are varied from N=5 up to 40, to allow for a brief convergence analysis. The first row in each set of results corresponds to the criterium taking the initial value of the boundary position at X=0, i.e., S(0,0)=0.3, while the second row corresponds to taking the initial value of the boundary position at X=1, S(1,0)=0.5, and the third and last row is the merging of these two sets of reordered eigenvalues. Clearly, the first criterium provides the lowest convergence rates, though it already provides fairly accurate results, with up to three converged digits, at the order N=40. The third criterium, which is in fact the combination of the required terms as obtained from the two other choices, provides the best convergence rates, and has then be preferred throughout the computations.

Table 1. Convergence of the moving boundary position $S(X,\tau)$ for case A and Ste = 1.0

NT	Case A									
	X=0					X=1				
	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00
5	0.4055[a]	0.4462	0.5103	0.5139	0.5140	0.5864	0.6302	0.7012	0.7048	0.7049
	0.3957[b]	0.4352	0.4957	0.4985	0.4985	0.5727	0.6133	0.6776	0.6804	0.6804
	0.3939[c]	0.4323	0.4917	0.4945	0.4945	0.5718	0.6119	0.6760	0.6788	0.6789
10	0.3950	0.4342	0.4942	0.4969	0.4970	0.5715	0.6109	0.6728	0.6754	0.6755
	0.3946	0.4333	0.4927	0.4953	0.4954	0.5711	0.6101	0.6717	0.6742	0.6743
	0.3941	0.4333	0.4930	0.4957	0.4957	0.5701	0.6091	0.6703	0.6728	0.6729
20	0.3945	0.4335	0.4933	0.4959	0.4960	0.5694	0.6065	0.6653	0.6677	0.6678
	0.3928	0.4314	0.4903	0.4929	0.4929	0.5683	0.6051	0.6634	0.6657	0.6657
	0.3933	0.4324	0.4922	0.4947	0.4948	0.5676	0.6043	0.6621	0.6644	0.6645
30	0.3946	0.4338	0.4935	0.4961	0.4962	0.5680	0.6043	0.6621	0.6644	0.6645
	0.3933	0.4322	0.4915	0.4940	0.4941	0.5664	0.6021	0.6590	0.6613	0.6613
	0.3930	0.4318	0.4909	0.4934	0.4935	0.5660	0.6017	0.6582	0.6605	0.6605
40	0.3935	0.4325	0.4918	0.4943	0.4944	0.5653	0.6006	0.6568	0.6590	0.6591
	0.3934	0.4323	0.4916	0.4941	0.4942	0.5648	0.5998	0.6556	0.6578	0.6579
	0.3933	0.4323	0.4915	0.4940	0.4941	0.5647	0.5997	0.6555	0.6576	0.6577

a)Reordering with S(0,0)=0.3 b)Reordering with S(1,0)=0.5 c)Reordering with S(0,0)=0.3 and S(0,0)=0.5

Tables 2 to 7 illustrate the convergence behavior of the present results for the moving boundary position at different times and for X=0 and 1, with different Stefan numbers. Clearly, the larger value of Ste results in slower convergence, due to the more severe variations promoted by the wider movement of the phase change front. One can observe that all four significant digits shown in Tables 5 and 6 are fully converged, while in Tables 3 and 4, in a few cases, full convergence at the fourth significant digit might not yet be achieved. In Tables 1 and 2 for Ste=1, in the worst situation, at least two significant digits are fully converged to N<60, where it becomes evident that the behavior at the boundary X=1 requires a few extra terms for convergence.

Table 2. Convergence of the moving boundary position $S(X,\tau)$ for case A and Ste = 1.0

NT	Case A									
	X=0					X=1				
	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00
5	0.3939	0.4323	0.4917	0.4945	0.4945	0.5718	0.6119	0.6760	0.6788	0.6789
15	0.3932	0.4320	0.4910	0.4936	0.4936	0.5689	0.6071	0.6670	0.6694	0.6694
30	0.3930	0.4318	0.4909	0.4934	0.4935	0.5660	0.6017	0.6582	0.6605	0.6605
45	0.3929	0.4317	0.4907	0.4931	0.4932	0.5642	0.5990	0.6545	0.6567	0.6567
60	0.3928	0.4317	0.4907	0.4931	0.4931	0.5624	0.5964	0.6509	0.6530	0.6530

Table 3. Convergence of the moving boundary position $S(X,\tau)$ for case B and Ste = 1.0

NT	Case B									
	X=0					X=1				
	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00
5	0.3026	0.3026	0.3026	0.3026	0.3026	0.7009	0.7009	0.7009	0.7009	0.7009
15	0.4025	0.4546	0.5742	0.5910	0.5929	0.7491	0.7839	0.8701	0.8814	0.8826
30	0.4042	0.4564	0.5799	0.5978	0.5997	0.7488	0.7828	0.8673	0.8784	0.8795
45	0.4042	0.4566	0.5816	0.5998	0.6018	0.7486	0.7823	0.8667	0.8778	0.8790
60	0.4043	0.4570	0.5827	0.6011	0.6031	0.7481	0.7815	0.8652	0.8761	0.8773

Table 4. Convergence of the moving boundary position $S(X,\tau)$ for case A and Ste = 0.1

NT	Case A									
	X=0					X=1				
	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00
5	0.3114	0.3153	0.3180	0.3180	0.3180	0.5085	0.5132	0.5177	0.5177	0.5177
10	0.3114	0.3152	0.3178	0.3178	0.3178	0.5085	0.5132	0.5175	0.5175	0.5175
20	0.3115	0.3153	0.3180	0.3180	0.3180	0.5084	0.5129	0.5171	0.5171	0.5171
30	0.3115	0.3153	0.3180	0.3180	0.3180	0.5083	0.5127	0.5168	0.5169	0.5169
40	0.3115	0.3153	0.3180	0.3180	0.3180	0.5083	0.5126	0.5166	0.5166	0.5166

Table 5. Convergence of the moving boundary position $S(X,\tau)$ for case B and Ste = 0.1

NT	Case B									
	X=0					X=1				
	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=4.00	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00
5	0.3124	0.3180	0.3255	0.3257	0.3257	0.7059	0.7104	0.7184	0.7186	0.7186
10	0.3129	0.3186	0.3259	0.3261	0.3261	0.7060	0.7102	0.7176	0.7178	0.7178
20	0.3130	0.3188	0.3262	0.3264	0.3264	0.7057	0.7097	0.7168	0.7170	0.7170
30	0.3131	0.3189	0.3263	0.3265	0.3265	0.7056	0.7096	0.7166	0.7167	0.7167
40	0.3132	0.3190	0.3265	0.3266	0.3266	0.7056	0.7095	0.7164	0.7166	0.7166

Table 6. Convergence of the moving boundary position $S(X,\tau)$ for case A and Ste = 0.01

NT	Case A									
	X=0					X=1				
	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=4.00	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00
5	0.3012	0.3015	0.3018	0.3018	0.3018	0.5009	0.5013	0.5018	0.5018	0.5018
10	0.3012	0.3015	0.3018	0.3018	0.3018	0.5009	0.5013	0.5018	0.5018	0.5018
20	0.3012	0.3016	0.3018	0.3018	0.3018	0.5009	0.5013	0.5017	0.5017	0.5017
30	0.3012	0.3016	0.3018	0.3018	0.3018	0.5009	0.5013	0.5017	0.5017	0.5017

Table 7. Convergence of the moving boundary position $S(X,\tau)$ for case B and Ste = 0.01

NT	Case B									
	X=0					X=1				
	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=4.00	τ=0.05	τ=0.10	τ=0.50	τ=1.00	τ=2.00
5	0.3013	0.3021	0.3025	0.3026	0.3025	0.7006	0.7014	0.7018	0.7018	0.7018
10	0.3013	0.3019	0.3025	0.3025	0.3026	0.7006	0.7010	0.7017	0.7018	0.7018
20	0.3013	0.3019	0.3026	0.3026	0.3026	0.7006	0.7010	0.7017	0.7017	0.7017
30	0.3013	0.3019	0.3026	0.3026	0.3026	0.7006	0.7010	0.7017	0.7017	0.7017

Based on the results obtained for the highest number of terms considered in the present simulation (NT=60), some physical discussions are now undertaken. Figures 3.a,b describe the evolution of the phase-change boundary for cases A and B at X=0 and X=1, and for Stefan numbers Ste=0.01, 0.1 and 1.0. It is observed that for lower Stefan numbers, which implies larger latent heats with respect to sensible heats, the boundaries have mild advancement from the original positions. For the case Ste=1.0, since a relatively smaller latent heat is required, for a given provided sensible heat, the front movement is much more pronounced.

Figures 4.a,b bring the evolution of the moving boundary profile at different dimensionless times τ = 0.4 to 4.0, along the longitudinal coordinate, for cases A and B, with different Stefan numbers Ste=0.01, 0.1 and 1.0. Again, the larger Stefan number case illustrates a more significant moving boundary advancement, which is clearly more pronounced at the boundary X=0, where less material is initially present from the initial condition shape.

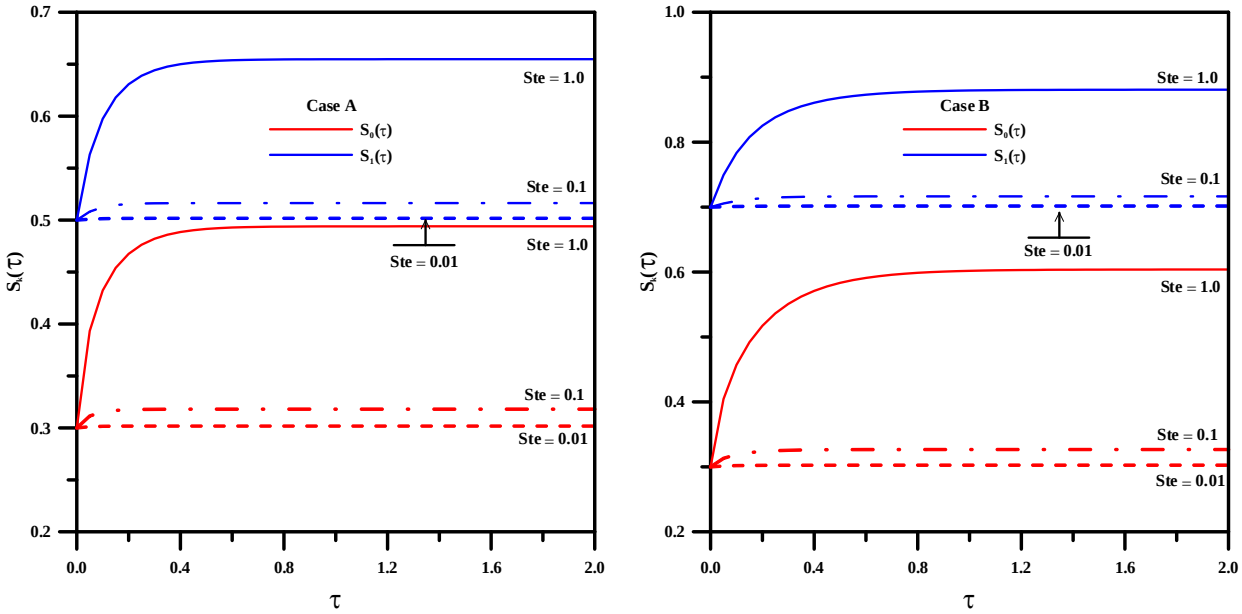

Figure 3. Evolution of the phase-change boundary with time for different Stefan numbers at X = 0 and X = 1: a) Case A. b) Case B.

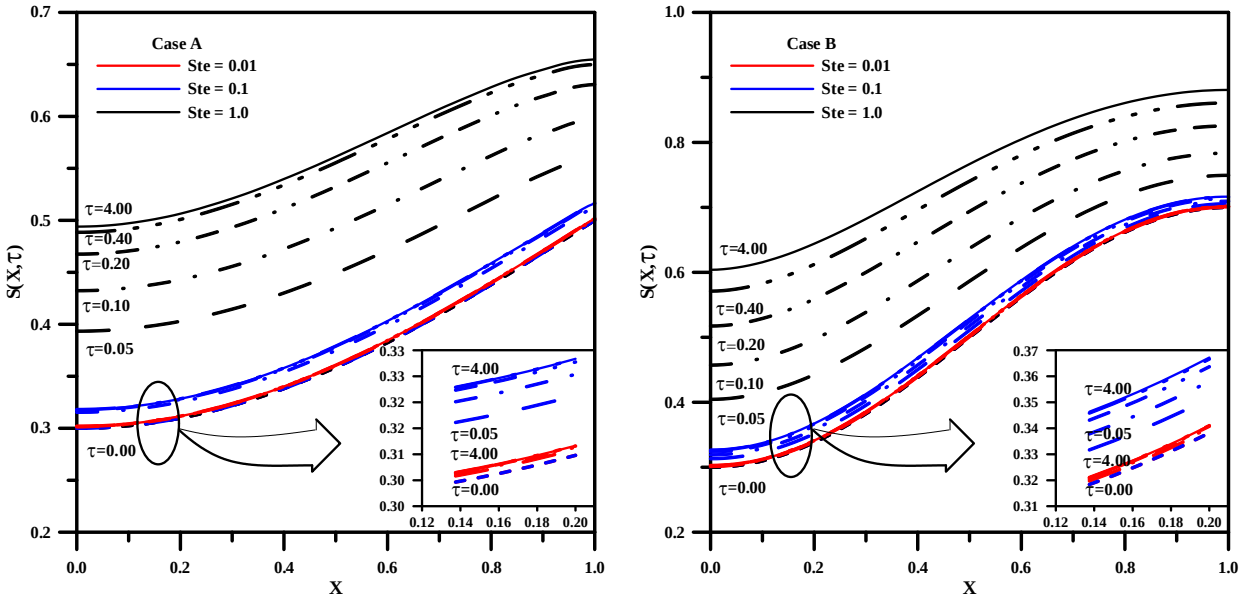

Figure 4. Evolution of the moving boundary at different times and Stefan numbers: a) Case A. b) Case B.

Figures 5.a,b show the evolution of dimensionless temperature at the two insulated contours (X=0 and X=1), on the lower plane Y=0, for cases A and B, and Stefan numbers Ste=0.01, 0.1 and 1.0. In both cases, it is observed that the temperatures at X=0 (dashed lines) have a faster transient behavior, as expected, since initially the medium width is smaller at this position, providing higher temperature gradients and higher rates of heat transfer. One may also observe the larger temperature differences that occur, at the same dimensionless time, for the lower values of Ste.

Figures 6.a,b show the temperature distributions along the horizontal direction X, again at position Y=0, for the cases A and B and Stefan numbers, Ste=0.01, 0.1 and 1.0. It can be observed that the temperature distributions progressively deviate from the initial uniform profile, due to the pronounced transients and variable domain shape, with an evolution to a new uniform profile at larger times.

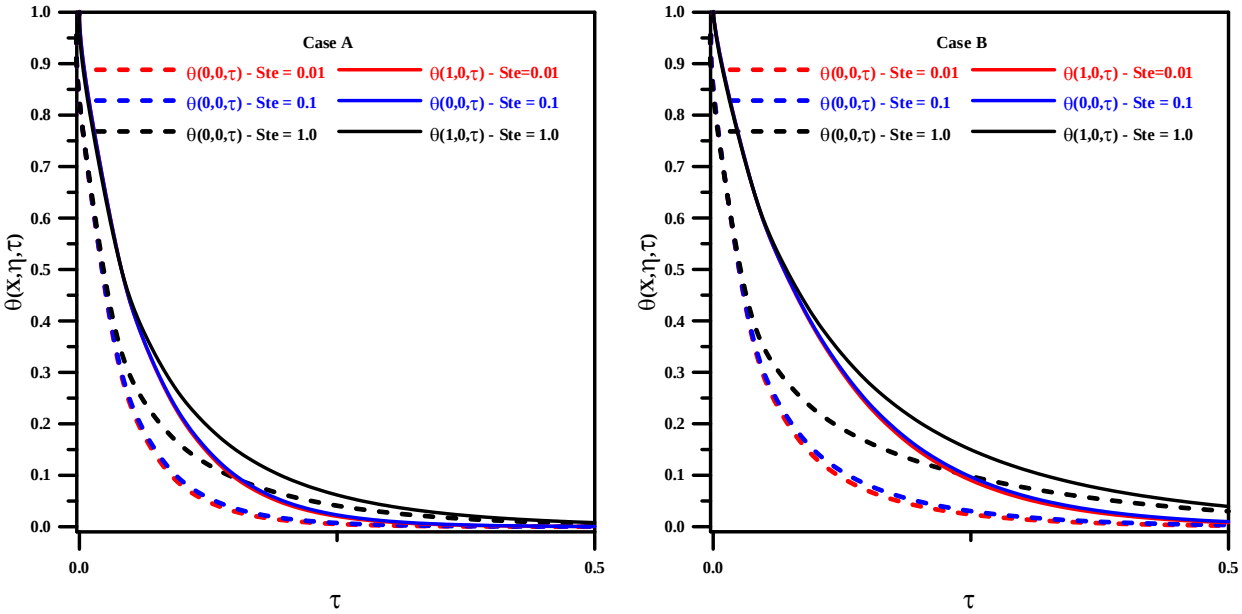

Figure 5. Temperature evolution for different Stefan numbers at X = 0 and X = 1 (Y=0): a) Case A. b) Case B.

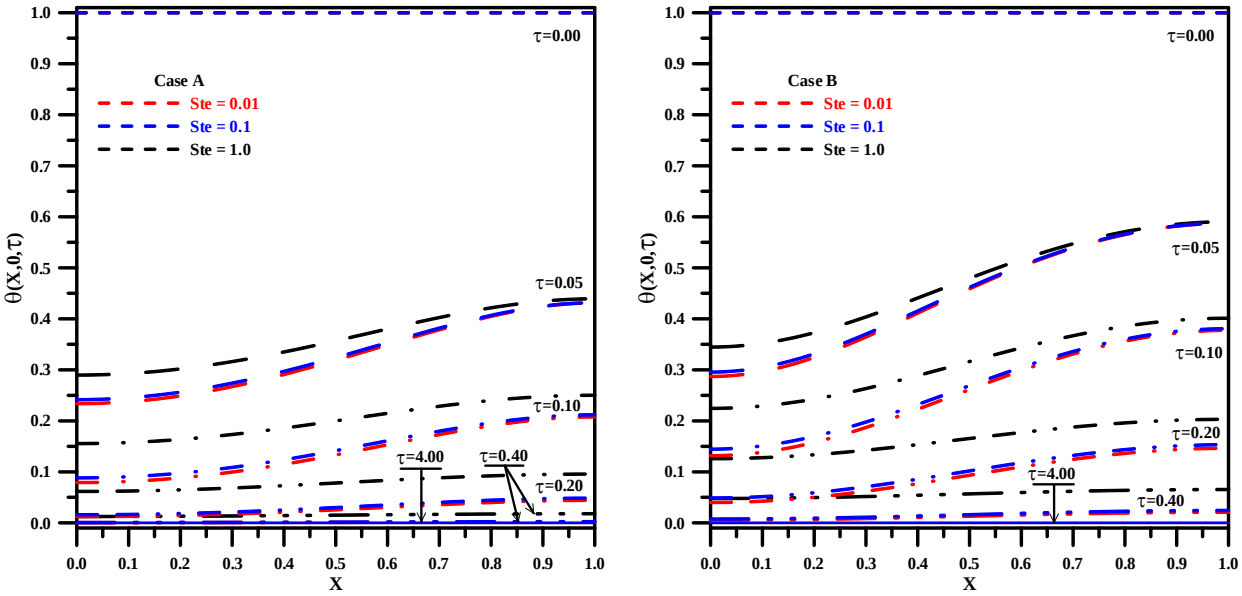

Figure 6. Temperature profiles at different times and position Y = 0, for different Stefan numbers: a) Case A. b) Case B.

4. CONCLUSIONS

A hybrid numerical-analytical treatment of the two-dimensional Stefan problem is advanced, by extending the ideas in the Generalized Integral Transform Technique (GITT). A double integral transformation of the temperature field is proposed based on an eigenvalue problem that incorporates the information on the moving boundary. A reordering scheme is organized to reduce the double summations into single ones, thus significantly reducing computational costs in the numerical solution of the transformed differential system. The moving boundary position itself is expanded in eigenfunctions, preceded by application of a filtering solution that extracts the information at the two borders, thus yielding a homogeneous eigenvalue problem for the boundary heat balance equation. A physical analysis is undertaken, for both the moving boundary position and the temperature distributions, demonstrating the adequacy of the proposed hybrid solution.

5. ACKNOWLEDGEMENTS

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