RELIABILITY ANALYSIS FOR PROBABILISTIC FATIGUE CRACK GROWTH

Marcela R. Machado, marcelam@fem.unicamp.br
José M. C. Dos Santos, zema@fem.unicamp.br
Computational Mechanics Department, Faculty of Mechanical Engineeing, State University of Campinas - UNICAMP, C. P. 6122, 13.083-970,Campinas-SP, Brazil.

Abstract. *It is well known to engineers that the majority of structural failures come from fatigue. The rate of fatigue crack growth in a structure depends on naturally random parameters, such as, the tensile strength, fracture properties, stress history and stress intensity factor. In this work, fatigue crack growth problem is solved using three probabilistic reliability methods: First Order Reliability Method (FORM), Second Order Reliability Method (SORM), and Monte Carlo Simulation (MCS). The problem consists of a rectangular plate with an edge crack, where the crack growth is calculated using classical Paris's law equation. Simulated results with different number of cycles limit are used to compare the probability of failure results obtained by these three stochastic modeling methods.*

Keywords*: Crack Growth, Uncertainty, Fatigue*

1. INTRODUCTION

Probabilistic fracture mechanics is a field in modern engineering that is rapidly developing. For the specimen to be analyzed fracture mechanics requires a knowledge of the crack. By considering the randomness of crack size and location, material properties, external loads and geometry, deterministic analysis provides an incomplete evaluation of the safety of a specimen. To address these problem probabilistic fracture mechanics is a useful tool, since it combines fracture mechanics with stochastic methods. The basic of probabilistic fracture mechanics and its application to the analysis of uncertainties in structural systems can be founded in Liu and Belytschko (1989). By considering the complexity of the failure mechanism and the aim of this paper in to evaluate the reliability methods, a combination of Paris's equation (Paris and Erdogan, 1963) with the theory of statistics and structural reliability has become an interesting problem. Three probabilistic methods will be used: First Order Reliability Method (FORM), Second Order Reliability Method (SORM) and Monte Carlo Simulation (MCS). Simulated results with different number of cycles limit are used to compare the probability of failure results obtained by these three stochastic modeling methods.

2. CRACK GROWTH

2.1. Fatigue Crack Growth and Laws

Based on fracture mechanic theory, fatigue crack growth models have been developed to evaluate damage tolerance in structures. Several papers have been published on fatigue crack growth prediction during the last decades. A review of the literature, which includes many fatigue life prediction models, can be founded in Beden *at al*, 2009.

A better understanding of the fatigue life under cyclic loading can be obtained from the block diagram suggested by Beden *at al*, 2009 ("Fig 1"). It shows that the process consists of two phases, the crack initiation life (nucleation and micro crack growth) followed by a crack growth period (macro crack growth) until failure.

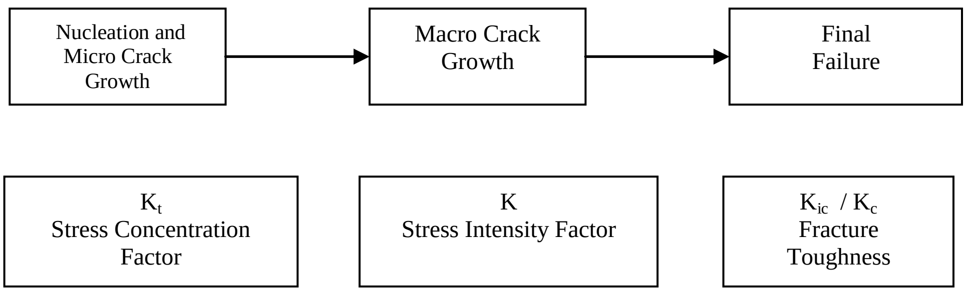

Figure 1. Block diagram of the fatigue life

This process can also be understood from the fatigue crack growth rate curve ("Fig 2"), also referred to as a da/dN versus ΔK curve, which is defined by Regions I, II and III. Region I includes the early stage of a fatigue crack and the crack growth rate. Also, defines the stress intensity factor threshold, ΔK_{th}, below which fatigue cracks should not propagate. Region II includes the intermediate crack propagation stage where the use of linear elastic fracture mechanics (LEFM) concepts is acceptable. Region III includes the fatigue crack growth at very high rates generated by the fast and unstable crack growth prior to final failure. The curve approaches to the fracture toughness, K_c.

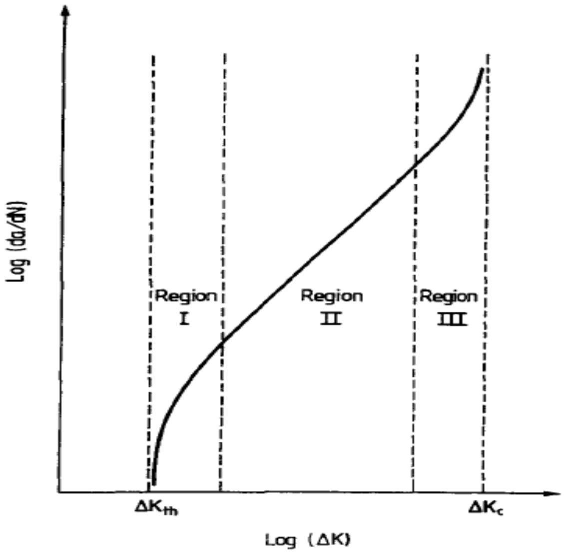

Figure 2. Fatigue crack growth rate curve

For this study we will be concerned about fatigue crack growth in the Region II, since it is where LEFM and Paris's Law can be applied. The steps of fatigue crack growth may be stated in this form: Determine the number of cycles N_f required for a crack grow from a certain initial crack size a_i to the final crack size a_f, the crack length a corresponds to N loading cycles (Gdouto, 1993).

Power law described by Paris and Erdogan (1963) is a simple method for predicting fatigue crack propagation. It describes the fatigue crack propagation behavior in Region II. The equation represents the first application of fracture mechanics to fatigue and is given by

$$\frac{da}{dN} = c\Delta K^m, \qquad (1)$$

where $\Delta K = K_{max} - K_{min}$, with K_{max} and K_{min} referring to the maximum and minimum values of stress intensity in the load cycles. The constants c and m are determined empirically from a log(ΔK) × log(da/dN) plot. The stress intensity factor is given by

$$K = \sigma\sqrt{\pi a}, \qquad (2)$$

where σ is stress at the crack. By calling the stress change as $\Delta\sigma$ we obtain,

$$\Delta K = \Delta\sigma\sqrt{\pi a} \qquad (3)$$

The number of cycles to failure, N_f , can be obtained by integrating "Eq. 1" with respect to a from the initial crack size, a_i , to the final crack size, a_f , to obtain

$$N_f = \int_{a_i}^{a_f} \frac{da}{c\left(\Delta\sigma\sqrt{\pi a}\right)^m} \qquad (4)$$

The final crack size is determined from

$$a_f = \frac{1}{\pi}\left(\frac{K_{IC}}{1.1215\Delta\sigma}\right)^2 \qquad (5)$$

where K_{IC} is the fracture toughness. By integrating and substituting final crack size “Eq. 5” into “Eq. 4”, the number of cycles to failure is obtained as

$$N_f = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{c\left(\Delta\sigma\sqrt{\pi a}\right)^m \pi^{m/2}\left(1-m/2\right)} \tag{6}$$

3. STOCHASTIC METHODS

This section presents the probabilistic analysis methods: first-order reliability method, second-order reliability method and Monte Carlo simulation. These methods evaluates the probability of failure by determining whether the limit-state functions are exceeded. Also, other statistical properties of structural response can be evaluated.

3.1. First-Order Reliability Method (FORM)

Knowing the system's limit-state equation or performance function, FORM method can be used for reliability analysis. The limit-state equation can be a linear function of correlated or uncorrelated normal variables or a non-linear function represented by its first-order approximation. If there are non-normal variables, the method requires users to work with equivalent normal variables (Haldar and Mahadevan, 2000).

The development of FORM method can be traced to second-moment methods: First-order second moment (FOSM) and Advanced First-order second-moment (AFOSM). They consider the first and the second moments of random variables, i.e. the mean and standard deviation.

In these methods, the performance function can be defined as $Z = R - S$. Assuming R and S statistically independent normal random variables, the failure is assumed when $R < S$ or $Z < 0$, so the probability of failure is obtained by

$$P_f = 1 - \Phi\left(\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 - \sigma_S^2}}\right) \tag{7}$$

where Φ is the Cumulative Density Function (CDF) of a standard normal distribution.

The probability of failure depends on the ratio of the mean value of Z to its standard deviation. This ratio is known as the *safety index* or *reliability index* and can be written as

$$\beta = \frac{\mu_z}{\sigma_z} = \left(\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 - \sigma_S^2}}\right) \tag{8}$$

For the FORM method, the initial reliability index is of extreme importance, since the initial value may change completely the final results. Choi *at al* (2007), present a method to compute the initial reliability index using mean-value and standard deviation value of the approximation limit-state function $g(X)$ given by

$$\beta = \frac{\mu_g}{\sigma_g} \tag{9}$$

where the mean value and the standard deviation are given by

$$\mu_g = E\left[g\left(\mu_x\right)\right] = g\left(\mu_x\right) \tag{10}$$

$$\sigma_g = \sqrt{Var\left[g\left(\mu_x\right)\right]} = \sqrt{\left[\nabla g\left(\mu_x\right)^T\right]^2 Var\left(X\right)} = \left[\sum_{i=1}^{n}\left(\frac{\partial g\left(\mu_x\right)}{\partial x_i}\right)^2 \sigma_{x_i}^2\right]^{1/2} \tag{11}$$

Thus, one can rewrite the probability of failure as:

$$P_f = 1 - \Phi(\beta) \tag{12}$$

Having the safety index, it is possible to calculate the new design point that is the most probable failure point. These calculations are made in an iterative form based on the FORM algorithm (Machado *at al*, 2011).

3.2. Second-Order Reliability Method (SORM)

SORM is also employed to calculate the reliability of the system, but with a second order approximation. It includes additional information about the curvature of the limit-state function. Taylor series approximation used in the FORM method ignores the terms beyond the first-order, while the SORM method ignores the terms beyond the second order (Haldar and Mahadevan, 2000).

In FORM method first partial derivative of the limit-state equation were evaluated at the design point α, and can be written as

$$\alpha = \frac{\nabla g(U^*)}{\left|\nabla g(U^*)\right|} \tag{13}$$

while for the SORM method the second derivative of the limit-state equation will be calculated at B, and is given by

$$B = \frac{\nabla^2 g(U^*)}{\left|\nabla g(U^*)\right|} \tag{14}$$

Before compute the probability of failures it is necessary to calculate H matrix and k index. To generate the H matrix, an initial matrix is selected as

$$H_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{X1} & \alpha_{X2} & \cdots & \alpha_{Xn} \end{bmatrix} \tag{15}$$

Gram-Schmidt orthogonalization procedure is applied to "Eq.15", and the resulting matrix is H. Once the H matrix is obtained, a matrix A, whose elements are denoted as a_{ij} , is computed as

$$a_{ij} = \frac{(HBH^t)_{ij}}{\left|\nabla G(U^*)\right|} \tag{16}$$

In the rotated space, the last variable, Y_n, coincides with β vector computed by FORM. The limit-state can be rewritten in terms of a second-order approximation in this rotated standard normal space Y' expressed by

$$y_n = \beta + \frac{1}{2}\sum_{i=1}^{n-1} k_i y_i'^2 \tag{17}$$

Finally, for the calculated k_j used in Breitung approach, are computed as the eigenvalues of the matrix A. The limit-state equation is approximated by a second order function and in this method the probability of failure is calculated with

$$P_f \approx \Phi(-\beta)\prod_{j=1}^{n-1}(1 + k_j\beta)^{-1/2} \tag{18}$$

Researchers have developed different SORM methods, here will be using Breitung approach (Choi, Grandhi and Canfield, 2007).To calculate the probability of failure is necessary to make use of a Laplace method for the asymptotic approximation of multidimensional integrals

$$I(\beta) = \int_{g(Y)<0} \exp\left(\frac{-\beta^2 |Y|^2}{2}\right) dY \tag{19}$$

where $I(\beta)$ is a integral over a fixed domain whose integrand is an exponential function. Making use of given results, the asymptotic form of $I(\beta)$ is shown as

$$I(\beta) \sim (2\pi)^{(n-1)/2} \exp\left(\frac{-\beta^2}{2}\right) \beta^{-(n+1)} |J|^{-1/2}, \beta \to \infty \tag{20}$$

In the case of independent standard normal random variables, the joint probability density function is given by

$$P_f = (2\pi)^{(n-1)/2} \int_{g(Y)<0} \exp\left(-\frac{|U|^2}{2}\right) dU \tag{21}$$

Substituting $(x_1, x_2, ..., x_n) \to (y_1, y_2, ..., y_n)$ with $y_i = \beta^{-1} u_i$ and "Eq. 16", into this equation, we obtain:

$$P_f \sim (2\pi)^{(n-1)/2} \beta^{-1} \exp\left(\frac{-\beta^2}{2}\right) |J|^{-1/2}, \beta \to \infty \tag{22}$$

Figure 3 illustrates a flow-chart for SORM method.

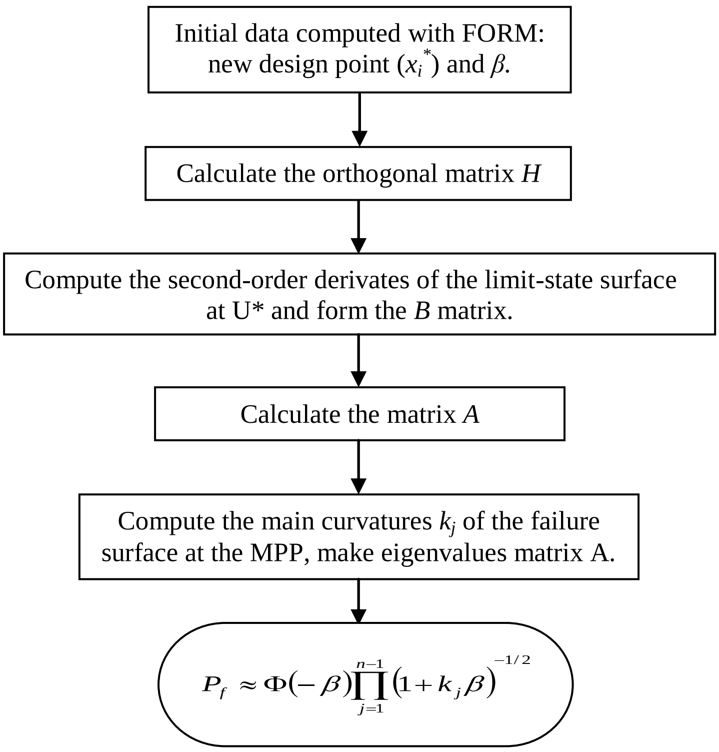

Figure 3. SORM method flow-chart

3.3. Monte Carlo Simulation (MCS)

Monte Carlo simulation is used as stochastic solver. It consists in solving the problem repeated times, each one of them with a new random input. The mean and the standard deviation of the result are calculated through the samples generated, and the computation procedure is quite simple. First, select a distribution type for the random variables; generate a sampling set from the distribution; and conduct simulations using the generated sampling set.

To generate random variables, the Matlab® functions *lognrdn* and *normrnd* were used. It is necessary to pay attention on these functions parameters for the purpose of generating the correct random values.

Haldar and Mahadevan (2000), suggest two different forms of applying MCS method, one more used form calculates the probability of failure as the ratio between the number of failures and the total number of events. It can be show in mathematical form as

$$P_f = \frac{N_f}{N} \tag{23}$$

where N is the total number of realizations and N_f is the number of failures. This first form will be referred as *MC1*. The second form is the same utilized in FORM method, the reliability index is calculated with "Eq. 8" and the probability of failure utilizes the "Eq. 9", this form will be called *MC2*.

4. NUMERICAL EXAMPLE AND DISCUSSIONS

For comparison between methods, an example of fatigue crack growth of a finite width rectangular plate with edge crack subject to constant amplitude is used (Figure 4).

The state-limit equation is given by $N_f - N_{allow} < 0$. The limit N_{allow} and the m index are deterministic variables, being m = 3.32 and limit N_{allow} will be changed from 3,000 up to 8,000 cycles. The random variables and their distributions were validated by Millwater *at* al, 1994, and are given in Table 1. Table 2 summarize the results of this crack growth problem and figure *5* to12 illustrate graphic results which presents a better visualization of methods comparison. Figure 13 show Cumulative Distribution Function (*CDF*) plots.

From results, in all situations the MC1 method converge better with FORM and SORM methods, while MC2 becomes more efficient when increase the probability of failure. Both, MC1 and MC2 methods start to converge with approximately 1,000 realizations.

As seen in Table 2 the FORM and SORM methods present minimal variance in this numerical example. A possible explanations for two methods have the same approximation, could be that only the first order polynomial is sufficient, the second order approach by SORM was not necessary. Therefore approximations about FORM that represent the linear system obtain satisfactory results.

To summarize, the three methods achieve similar reliability results and satisfactory approximations.

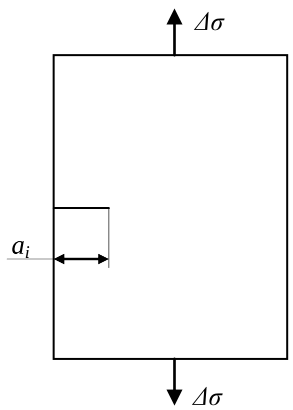

Figure 4. Rectangular plate with edge crack

Table 1. Random Variables Properties for Crack Growth Problem

Random Variables	Distribution	Mean	Standard Deviation
Load(ksi), *Δσ*	Lognormal	100	10
Initial Crack Size (in), *ai*	Lognormal	0.01	0.005
Paris constant, *c*	Lognormal	$1.2x10^{-10}$	$1.2x10^{-11}$
Fracture Toughness (ksi/√in), K_{IC}	Normal	60	6

Table 2. Probability of Failure for Crack Growth Problem

Limit (N_{allow})	$MC1_{(100,000)}$	$MC2_{(100,000)}$	FORM	SORM
1,000	0.0939	0.1540	0.083500935743235	0.083500935743190
3,000	0.4045	0.3420	0.391516055614330	0.391516055614317
3,704	0.5121	0.4224	0.500000044605869	0.500000044605869
4,000	0.5568	0.4633	0.541751124130008	0.541751124130003
5,000	0.6752	0.5829	0.662925705199224	0.662925705199210
6,000	0.7671	0.6939	0.753990830741950	0.753990830741932
7,000	0.8337	0.7925	0.819574591105803	0.819574591105786
8,000	0.8817	0.8682	0.865602475593301	0.865602475593286

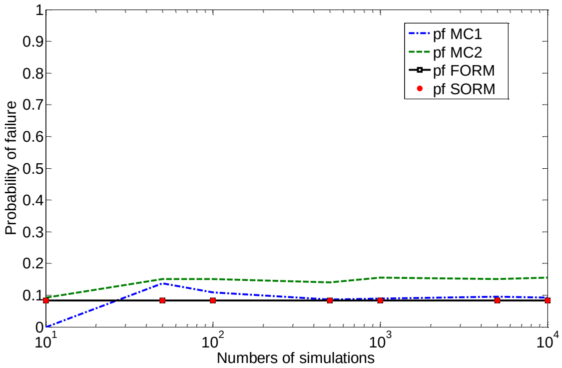

Figure 5. Probability of failure with N_{allow}= *1,000*

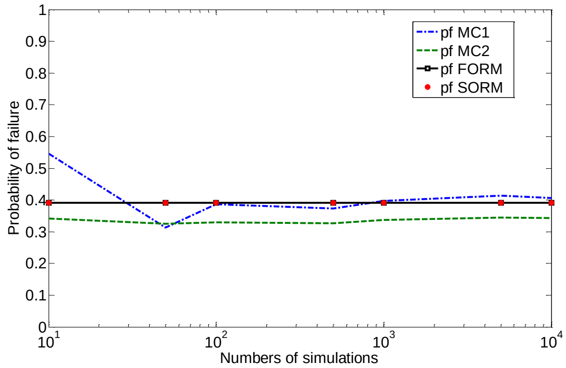

Figure 6. Probability of failure with N_{allow}= *3,000*

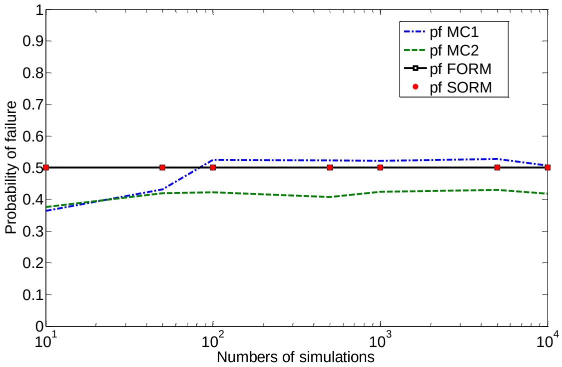

Figure 7. Probability of failure with N_{allow}= 3,704

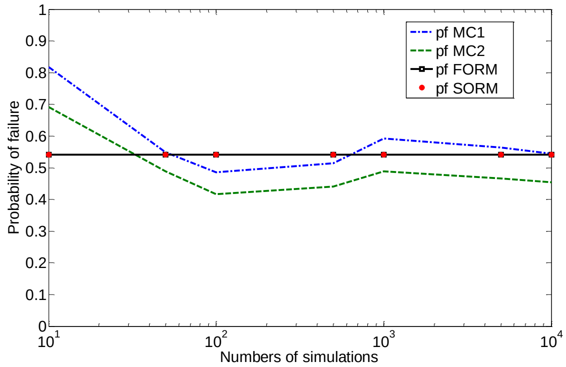

Figure 8. Probability of failure with N_{allow}= 4,000

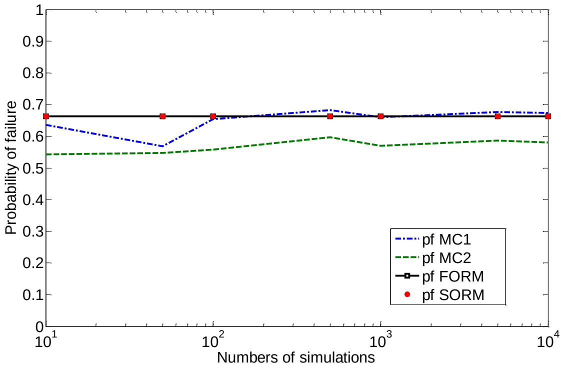

Figure 9. Probability of failure with N_{allow}= 5,*000*

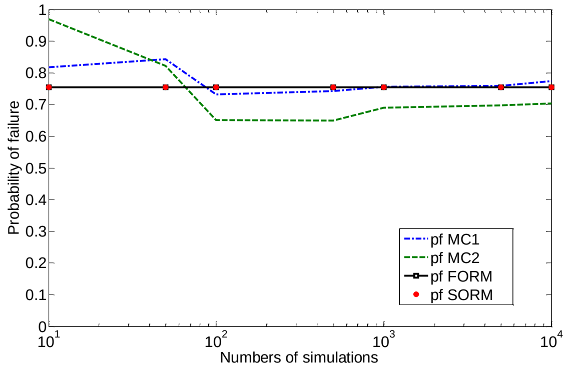

Figure 10. Probability of failure with N_{allow}=6,*000*

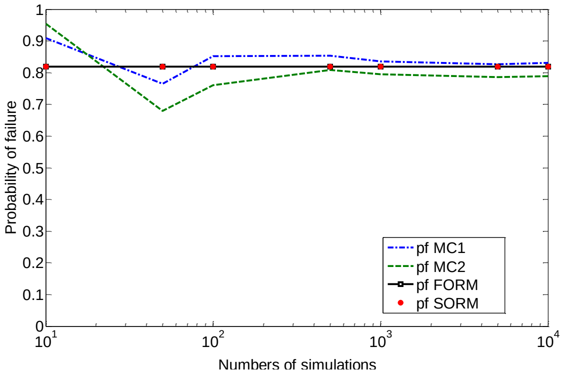

Figure 11. Probability of failure with N_{allow}=7,*000*

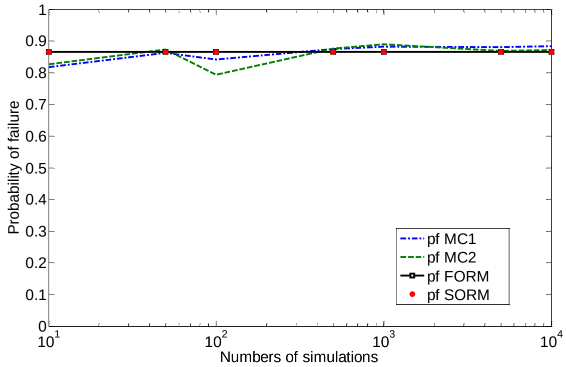

Figure 12. Probability of failure with N_{allow}=8,*000*

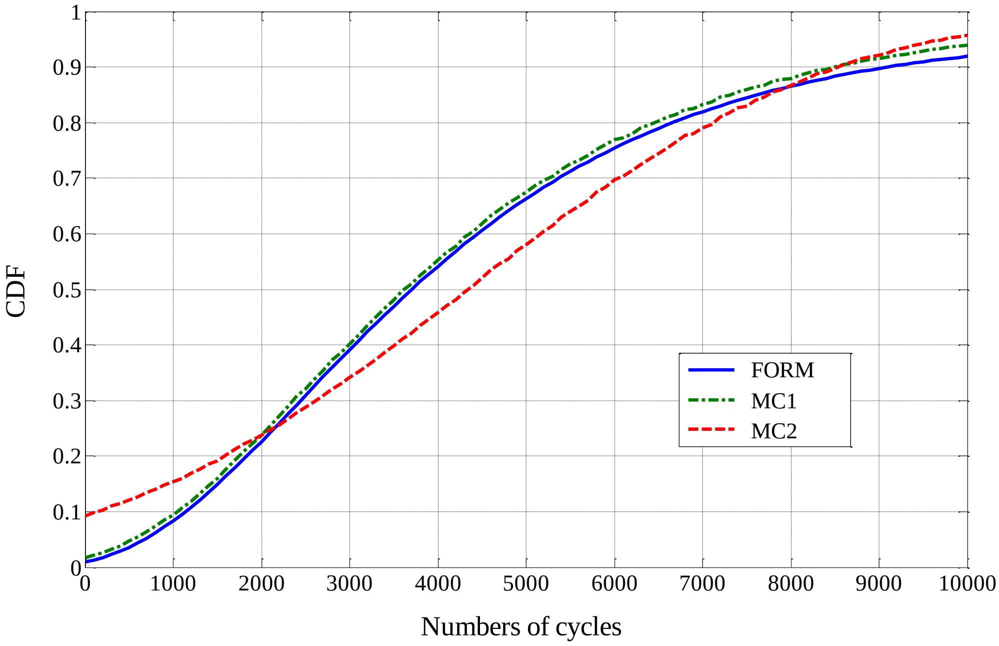

Figure 13. Cumulative Distribution Function of Cycles to Failure

5. ACKNOWLEDGEMENTS

The authors would like to thank Pro Engenharias, CNPq, Fapesp, and CAPES for supporting this work.

6. REFERENCES

Beden, S.M., Abdullah S., Ariffin, A.K., 2009. "Review of Crack Propagation Models for Metallic Components". European Journal pf Scientific Research, Vol. 28, No. 3 (2009), pp. 364-397.

Gdoutos, E.E., 1993, "Fracture Mechanics: an Introduction", Ed. Kluwer Academic Publishers, Netherlands, 307 p.

Haldar, A., Mahadevan, S., 2000, "Reliability Assessment Using Stochastic Finite Element Analysis", Ed. John Wiley & Sons,Inc, 328 p.

Liu, W. K. and Belytschko, T., 1989, "Computational Mechanics of Probabilistic and Reliability Analysis". Elme Press International.

Machado, M. R., Dos Santos, T.B., Dos Santos, J.M.C., and Arruda, J.R.F., 2011, "Reliability Analysis Using Simple Models Aiming at a Comparison Among Stochastic Methods", Proceedings of the 18th International Congress on Sound & Vibration, Rio de Janeiro, Brazil.

Millwater, H.R., Wu, Y-T, and Cardinal, J.W., 1994,"Probabilistic Structural Analysis of Fatigue and Fracture", AIAA-94-1507-CP, Proceedings of the 35th AIAA Structures, Structural Dynamics and Material Conference,, Hilton Head, SC,April.

Paris, P. and Erdogan, F., 1963, "A critical analysis of crack propagation laws". Trans. ASME, 85, 528-534.

Choi, Seung-Kyum, Ramana, V.G., Robert, A.C., 2007, "Reliability-based Structural Design", Ed. Springer, 306 p.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.