

SOLVING ROBUST TRUSS TOPOLOGY DESIGN PROBLEMS WITH TWO FEASIBLE DIRECTION INTERIOR POINT TECHNIQUES

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Abstract. *This paper gives two different approaches to solve a model of the Robust Truss Topology (RTT), where the robustness of the truss is obtained with respect both to given loading scenarios and small occasional loads. Both algorithms are based in the ideas of the Feasible Direction Interior Point Algorithm (FDIPA). The purpose of this paper is to describe those algorithms and to solve TTD problems with the semidefinite and nonsmooth formulations. A analyze of equivalent reformulations of the problem and illustrative numerical examples are presented.*

Keywords: *Robust truss topology; Nonsmooth optimization; Semidefinite programming.*

1. INTRODUCTION

Robust Truss Topology (RTT) deal with the selection of optimal configuration for structural systems. The model studied here was initially proposed by (Ben-Tal and Nemirovski, 1997), as well as a formulation based on semidefinite programming. A largely employed model for truss topology optimization considers structures submitted to a set of nodal loads, that we call “primary” loads, and looks for the volume of each of the bars that minimizes the structural compliance, see (Bendsøe, 1995). The structural topology changes if the volume of some of the bars are zero in the solution.

In the example considered we care about the issue of the robustness of the truss. Here we say that a truss is “robust" if it is reasonable rigid when any set of small possible uncertain loads act on it. In additional to the primary loads, we includes a set of “secondary” loads that are uncertain in size and direction and can act over the structure. The compliance to be minimized is the worst possible one, considering “primary” and “secondary” load cases.

In the following section, we describe the modeling approach in question. Section 3 are devoted to mathematical programmings where we present the semidefinite and nosmooth algorithms. The examples and numerical results are described in section 4.

2. OPTIMIZATION MODEL

Let us consider a two or a three-dimensional ground elastic truss with n nodes and m degrees of freedom, submitted to a finite set of loading conditions $P \equiv \{p^1, p^2, \dots, p^s\}$ such that $p^i \in \mathbb{R}^m$ for $i = 1, 2, \dots, s$, and let b be the number of initial bars. The design variables of the problem are the volumes of the bars, denoted x_j, $j = 1, 2, ..., b$. The reduced stiffness matrix is

$$K(x) = \sum_{j=1}^{b} x_j K_j, \tag{1}$$

where $K_j \in \mathbb{R}^{m \times m}$, $j = 1, 2, ..., b$, are the reduced stiffness matrices corresponding to bars of unitary volume. To obtain a well-posed problem, the matrix $\sum_{j=1}^{b} K_j$ must be positive definite (Ben-Tal and Nemirovski, 1997). The compliance related to the loading condition $p^i \in P$ can be defined as (Bendsøe, 1995):

$$\phi(x, p^i) = \sup\{2u^\top p^i - u^\top K(x)u : u \in \mathbb{R}^m\} \tag{2}$$

where u is the vector of nodal displacement. Let be $\hat{\phi}(x) = \sup\{\phi(x, p^i) : p^i \in P\}$ the worst possible compliance for the set P. A energy model for topology optimization with several loading conditions can be stated as follows:

$$\begin{cases} \min\limits_{x \in \mathbb{R}^b} \ \hat{\phi}(x) \\ \text{s.t.} \quad \sum\limits_{j=1}^{b} x_j \leq V \,, \\ \qquad\ x_j \geq 0 \,, \quad j = 1, \ldots, b \end{cases} \tag{3}$$

The value $V > 0$ is the maximum quantity of material to distribute in the truss.

Instead of maximizing ϕ on the finite domain P, we consider a model proposed by (Ben-Tal and Nemirovski, 1997) that maximizes ϕ on the ellipsoid M of loading conditions defined as follows:

$$M = \{Qe : e \in \mathbb{R}^q, e^\top e \leq 1\}, \tag{4}$$

where

$$[Q] = [p^1, \ldots, p^s, rf^1, \ldots, rf^{q-s}]. \tag{5}$$

The vectors $p^1, \ldots, p^s$ must be linearly independent and rf^i, represents the i-th secondary loading. The value r is the magnitude of the secondary loadings and the set $\{f^1, \ldots, f^{q-s}\}$ must be chosen as an orthonormal basis of a linear subspace orthogonal to the linear span of P. The procedure to chose a convenient basis $\{f^1, \ldots, f^{q-s}\}$ is explained later.

A robust design is then obtained by solving (3) with

$$\hat{\phi}(x) = \sup\{\phi(x, p) : p \in M\}. \tag{6}$$

For the rest of this paper, we use $\hat{\phi}$ as defined in (6). In order to introduce the semidefinite formulation, we show some equivalent formulations of problem (3). First, note that (3) is equivalente to

$$\begin{cases} \min\limits_{\tau, x \in \mathbb{R}^b} \ \tau \\ \text{s.t.} \quad \hat{\phi}(x) \leqslant \tau \,, \\ \qquad\ \sum\limits_{j=1}^{b} x_j \leq V \,, \\ \qquad\ x_j \geq 0 \,, \quad j = 1, \ldots, b \end{cases} \tag{7}$$

where we introduce an auxiliary variable $\tau \in \mathbb{R}$.

Sencondly, as proved in (Ben-Tal and Nemirovski, 1997), the following two expressions are equivalent:

$$\hat{\phi}(x) \leqslant \tau, \tag{8}$$

$$A(\tau, x) = \begin{pmatrix} \tau I^q & Q^\top \\ Q & K(x) \end{pmatrix} \succeq 0. \tag{9}$$

I^q is the identity matrix of size $q \times q$, $\tau \in \mathbb{R}$ and $A \succeq 0$ means that A is positive semidefinite.

Then, using the equivalence between (8) and (9), it follows that problem (3) is equivalent to:

$$\begin{cases} \min\limits_{\tau, x \in \mathbb{R}^b} \ \tau \\ \text{s.t.} \quad \begin{pmatrix} \tau I^q & Q^\top \\ Q & K(x) \end{pmatrix} \succeq 0 \,, \\ \qquad\ \sum\limits_{j=1}^{b} x_j \leq V \,, \\ \qquad\ x_j \geq 0 \,, \quad j = 1, \ldots, b \end{cases} , \tag{10}$$

Problem (10) is a semidefinite programming problem equivalent to the original problem (3). Note that the objective function and the constraints in problem (10) are differentiable functions. On the other hand, problem (7) is a nonsmooth optimization problem since $\hat{\phi}(x)$ is generally nondifferentiable. As proved in (Ben-Tal and Nemirovski, 1997), $\hat{\phi}(x)$ is the highest generalized eigenvalue of the system $(QQ^\top, K(x))$. If the highest eigenvalue is single, the function $\hat{\phi}$ is differentiable. If it is multiple, the function $\hat{\phi}$ is generally nondifferentiable. In both cases it is possible to compute the required subgradients, see (Seyranian *et al.*, 1994; Rodrigues *et al.*, 1995; Choi and Kim, 2004).

Recall that (10) is a convex optimization problem (Vandenberghe and Boyd, 1996). We can prove that using the following argument. In fact, the epigraph of $\hat{\phi}$ coincides with $\{(\tau, x) : A(\tau, x) \succeq 0\}$, and this last set is convex (Vandenberghe and Boyd, 1996) then $\hat{\phi}$ is a convex function.

Since the formulations (7) and (10) are equivalent, we can use a nonsmooth or a semidefinite technique to solve the RTT problem.

3. NUMERICAL ALGORITHMS

In this section we describe the two methods used to solve the Robust Truss Topology problem. Those methods are called Nonsmooth Feasible Direction Algorithm (NFDA) and Feasible Direction Algorithm for Semidefinite Programming (FDASDP), are general techniques for convex nonsmooth and semidefinite optimization problems, respectively.

The methods we are going to describe has the same framework as the Feasible Direction Interior Point Algorithm (FDIPA). FDIPA is an interior point algorithm for smooth nonlinear optimization problems with equality and inequality constraints (Herskovits, 1998). In short, those techniques computes at each iteration k a search direction d^k based in a quasi-Newton iteration from the Karush-Kuhn-Tucker optimality condition of the original or auxiliary problems. The point x^{k+1} in the next iteration $k+1$ is computed from the previous one x^k by setting $x^{k+1} = x^k + t^k d^k$. The stepsize t^k is choosen with a line search criteria in order to hold the next point in the interior of a feasible region of the problem.

3.1 SEMIDEFINITE ALGORITHM

Consider the following semidefinite programming problem:

$$\begin{cases} \min\limits_{x \in \mathbb{R}^n} f(x) \\ \quad \text{s.t.} \quad G(x) \preccurlyeq 0, \end{cases} \tag{11}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{S}^m$ are smooth functions. The symbol $\mathbb{S}^m$ denotes the set of symmetric matrices of size $m \times m$ and the constraint $G(x) \preccurlyeq 0$ means that the matrix $G(x)$ must be negative semidefinite. The present algorithm generates a sequence of points in the interior of the feasible region $\Omega = \{x \in \mathbb{R}^n : G(x) \preccurlyeq 0\}$.

The first order Karush-Kunh-Tucker conditions (KKT) for the problem (11), proved in (Shapiro, 1994), are the following:

$$\begin{aligned} \nabla_x L(x, \Lambda) &= 0 \\ \Lambda G(x) &= 0 \\ G(x) &\preccurlyeq 0 \\ \Lambda &\succcurlyeq 0 \end{aligned} \tag{12}$$

where $\Lambda \in \mathbb{S}^m$ is a matrix of Lagrange multipliers and $L : \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R}$ is the lagrangian of problem (11) given by $L(x, \Lambda) = f(x) + tr\,(\Lambda G(x))$. Here, $tr(A)$ is the trace operator given by the sum of diagonal elements of A.

The Feasible Direction Algorithm for Semidefinite Programming (FDASDP) is an iterative method in the primal variables x and dual varibles Λ that converges assintotically to a pair (x^*, Λ^*) that verifies the KKT condition (12). In each iteration, FDASDP computes a search direction by solving linear systems of a quase-Newton iteration for the equalities of the KKT condition. The following quase-Newton matrix is defined for the equalities of KKT condition (12)

$$W = \begin{bmatrix} B & \nabla G(x) \\ (\Lambda \circledast I)\,\nabla G(x)^\top & I \circledast G(x) \end{bmatrix}, \tag{13}$$

where B could be any positive definite matrix, a quase-Newton aproximation of the lagrangean L or the identity matrix. The matrix $\nabla G(x)$ contains the components of partial derivatives of $G(x)$ (Shapiro, 1994). The symbol $\circledast$ is the symmetric Kronecker operator. Associated with this Kronecker operator is the symmetric vectorization of a matrix (de Klerk, 2002). Given $A \in \mathbb{S}^m$ with components a_{ij}, we define $\overline{m} = \frac{1}{2}m(m+1)$ as the number of upper diagonal components of A. The symmetric vectorization of A, denoted by $svec(A)$, is given in the following manner,

$$svec(A) = \begin{bmatrix} a_{11} & \sqrt{2}a_{12} & a_{22} & \sqrt{2}a_{13} & \sqrt{2}a_{23} & a_{33} & \ldots \end{bmatrix}^\top.$$

Each iteration of FDASDP solves two linear systems with matrix W. With the first linear system we obtain a descent direction d_0. Unfortunately when x is in the boudary of Ω, d_0 is a tangent direction to the feasible region. Then, we must solve a second linear system with matrix W to obtain a direction d_1 that points toward the interior of Ω. Once obtained d_0 and d_1, FDASDP computes a positive parameter ρ in such a way to make $d = d_0 + \rho d_1$ a feasible and descent direction.

Finally, to obtain the next point x^{k+1} in the interior of Ω we compute a step size $t^k \in \mathbb{R}^n$ using an Armijo's line search along d. Those steps are repeated until the norm of the direction d_0 is less than a tolerance $\varepsilon > 0$. The prove of global convergence of FDASDP can be found in (Aroztegui, 2010). The statement of FDASDP is presented below.

Feasible Direction Algorithm for Semidefinite Programming- FDASDP

Parameters. $\xi \in (0,1)$, $\eta \in (0,1)$, $\varphi > 0$, $\nu \in (0,1)$, $\varepsilon > 0$.
Data. $x \in int(\Omega_a)$, $B \in \mathbb{S}^n_{++}$ and $\Lambda \in \mathbb{S}^m_{++}$ such that Λ and $G(x)$ commute.
Step 1) Computation of the search direction d.
i) Solve the following linear system in $d_0 \in \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}^{\overline{m}}$

$$\begin{bmatrix} B & \nabla G(x) \\ (\Lambda \circledast I)\,\nabla G(x)^\top & I \circledast G(x) \end{bmatrix} \begin{bmatrix} d_0 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix} \tag{14}$$

If $|d_0| < \varepsilon$, stop.
ii) Solve the following linear system in $d_1 \in \mathbb{R}^n$ and $\lambda_1 \in \mathbb{R}^{\overline{m}}$

$$\begin{bmatrix} B & \nabla G(x) \\ (\Lambda \circledast I)\,\nabla G(x)^\top & I \circledast G(x) \end{bmatrix} \begin{bmatrix} d_1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix} \tag{15}$$

where $\lambda = svec(\Lambda)$.
iii) Compute the parameter ρ such that

$$\rho \;=\; \min\left\{ \varphi\|d_0\|^2, (\xi - 1)\frac{d_0^\top \nabla f(x)}{d_1^\top \nabla f(x)} \right\} \tag{16}$$

if $d_1^\top \nabla f(x) > 0$. Otherwise:

$$\rho \;=\; \varphi\|d_0\|^2. \tag{17}$$

iv) Compute the search direction d as

$$d = d_0 + \rho d_1. \tag{18}$$

Step 2) Line Search.
Find t, the first element of $\{1, v, v^2, v^3 \ldots\}$ such that

$$f(x + td) \leqslant f(x) + t.\eta.d^\top \nabla f(x) \tag{19}$$

and

$$G(x + td) \prec 0. \tag{20}$$

Step 3) Updates.
i) Take the new point as $x = x + td$.
ii) Define new value for $B \in \mathbb{S}^n_{++}$.
iii) Define new value for $\Lambda \in \mathbb{S}^m_{++}$ such that commute with $G(x)$.
iv) Go to Step 1).

3.2 NONSMOOTH ALGORITHM

We present a algorithm for solving the unconstrained optimization problem:

$$\left\{ \min_{x \in \mathbb{R}^n} f(x) \right. \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, not necessarily smooth. In fact, we reformulate the problem (P) as a equivalent constrained problem (EP)

$$\left\{ \begin{array}{ll} \displaystyle\min_{(x,z)\in\mathbb{R}^{n+1}} & z \\ \text{s.t.} & f(x) \leq z, \end{array} \right. \tag{EP}$$

where $z \in \mathbb{R}$ is an auxiliary variable. The algorithm which combines the feasible directions interior point algorithm (Herskovits, 1998) with some basic ideas of the classical cutting plane (Kelley, 1960) and bundle methods (Kiwiel, 1985) for evaluating candidate points. We have that $z^{k+1} < z^k$ and $z^k > f(x^k)$ for all k. At each iteration, an auxiliary linear program is defined using cutting planes. Let $g_i^k(x,z)$ be the current set of cutting planes such that

$$g_i^k(x,z) = f(y_i^k) + (s_i^k)^\top (x - y_i^k) - z, \quad i = 0,1,...,\ell$$

where $y_\ell^k \in \mathbb{R}^n$ are auxiliary points, $s_i^k \in \partial f(y_i^k)$ are subgradients at those points and ℓ represents the number of current cutting planes.

We call,

$$\tilde{g}_\ell^k(x,z) \equiv [g_0^k(x,z), ..., g_\ell^k(x,z)]^\top, \;\; \tilde{g}_\ell^k : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^{\ell+1}$$

and consider the current auxiliary problem

$$\begin{cases} \displaystyle\min_{(x,z)\in\mathbb{R}^{n+1}} & \psi(x,z) = z \\ \text{s.t.} & \tilde{g}_\ell^k(x,z) \le 0. \end{cases} \qquad (AP_\ell^k)$$

Instead of solving this problem, the present algorithm only computes with FDIPA a search direction d_ℓ^k of (AP_ℓ^k). We note that d_ℓ^k can be computed even if (AP_ℓ^k) has not a finite minimum.

The largest feasible step is

$$t = \max\{t \mid \tilde{g}_\ell^k((x^k, z^k) + t d_\ell^k) \le 0\}.$$

Since t is not always finite, it is taken

$$t_\ell^k := \min\{t_{max}/\mu, \;\; t\}$$

where $\mu \in (0,1)$. Then,

$$(x_{\ell+1}^k, z_{\ell+1}^k) = (x^k, z^k) + t_\ell^k d_\ell^k \qquad (10)$$

is feasible with respect to (AP_ℓ^k). Next we compute the following auxiliary point

$$(y_{\ell+1}^k, w_{\ell+1}^k) = (x^k, z^k) + \mu t_\ell^k d_\ell^k. \qquad (11)$$

If $(y_{\ell+1}^k, w_{\ell+1}^k)$ is strictly feasible with respect to (EP), that is, if $w_{\ell+1}^k > f(y_{\ell+1}^k)$ we consider that the current set of cutting planes is a good local approximation of $f(x)$ in a neighborhood of x^k. Then, we say that the "step is serious" and set the new iterate $(x^{k+1}, z^{k+1}) = (y_{\ell+1}^k, w_{\ell+1}^k)$. Otherwise, a new cutting plane $g_{\ell+1}^k(x,z)$ is added to the approximated problem and the procedure repeated until a serious step is obtained. We are now in position to state our algorithm.

Nonsmooth Feasible Direction Algorithm - NFDA

Parameters. $\xi, \mu \in (0,1), \;\; \varphi > 0, \;\; t_{max} > 0$.
Data. x^0, $f(a) > z^0 > f(x^0)$, $\lambda_0^0 \in \mathbb{R}^+$, $B^0 \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ symmetric and positive definite. Set $y_0^0 = x^0$, $k = 0$ and $\ell = 0$.
Step 1) Compute $s_\ell^k \in \partial f(y_\ell^k)$. A new cutting plane at the current iterate (x^k, z^k) is defined by

$$g_\ell^k(x,z) = f(y_\ell^k) + (s_\ell^k)^\top (x - y_\ell^k) - z.$$

Consider now

$$\nabla g_\ell^k(x,z) = \begin{bmatrix} s_\ell^k \\ \\ -1 \end{bmatrix} \in \mathbb{R}^{n+1},$$

define

$$\tilde{g}_\ell^k(x,z) = [g_0^k(x,z), ..., g_\ell^k(x,z)]^\top \in \mathbb{R}^{\ell+1},$$

and

$$\nabla \tilde{g}_\ell^k(x,z) = [\nabla g_0^k(x,z), ..., \nabla g_\ell^k(x,z)] \in \mathbb{R}^{(n+1)\times(\ell+1)}.$$

Step 2) Feasible Descent Direction d_ℓ^k for (AP_ℓ^k)

i) Compute $d^k_{\alpha\ell}$ and $\lambda^k_{\alpha\ell}$, solving

$$B^k d^k_{\alpha\ell} + \nabla\tilde{g}^k_\ell(x^k, z^k)\tilde{\lambda}^k_{\alpha\ell} = -\nabla\psi(x, z) \tag{12}$$

$$\tilde{\Lambda}^k_\ell[\nabla\tilde{g}^k_\ell(x^k, z^k)]^\top d^k_{\alpha\ell} + \tilde{G}^k_\ell(x^k, z^k)\tilde{\lambda}^k_{\alpha\ell} = 0. \tag{13}$$

Compute $d^k_{\beta\ell}$ and $\lambda^k_{\beta\ell}$, solving

$$B^k d^k_{\beta\ell} + \nabla\tilde{g}^k_\ell(x^k, z^k)\tilde{\lambda}^k_{\beta\ell} = 0 \tag{14}$$

$$\tilde{\Lambda}^k_\ell[\nabla\tilde{g}^k_\ell(x^k, z^k)]^\top d^k_{\beta\ell} + \tilde{G}^k_l(x^k, z^k)\tilde{\lambda}^k_{\beta\ell} = -\tilde{\lambda}^k_\ell, \tag{15}$$

where

$$\tilde{\lambda}^k_{\alpha\ell} := (\lambda^k_{\alpha 0}, ..., \lambda^k_{\alpha\ell}), \quad \tilde{\lambda}^k_{\beta\ell} := (\lambda^k_{\beta 0}, ..., \lambda^k_{\beta\ell}),$$
$$\tilde{\lambda}^k_\ell := (\lambda^k_0, ..., \lambda^k_\ell), \quad \tilde{\Lambda}^k_\ell := \text{diag}(\lambda^k_0, ..., \lambda^k_\ell)$$
$$\text{and} \quad \tilde{G}^k_\ell(x, z) := \text{diag}(g^k_0(x, z), ..., g^k_\ell(x, z)).$$

ii) If $(d^k_{\beta\ell})^\top\nabla\psi(x, z) > 0$, set $\rho = \varphi\|d^k_{\alpha\ell}\|^2$.
Otherwise, set

$$\rho = \min\left\{\varphi\|d^k_{\alpha\ell}\|^2,\ (\xi - 1)\frac{(d^k_{\alpha\ell})^\top\nabla\psi(x, z)}{(d^k_{\beta\ell})^\top\nabla\psi(x, z)}\right\}.$$

iii) Compute the feasible descent direction

$$d^k_\ell = d^k_{\alpha\ell} + \rho d^k_{\beta\ell}.$$

Step 3) Compute the step length

$$t^k_\ell = \min\left\{t_{max}/\mu,\ \max\{t \mid \tilde{g}^k_\ell((x^k, z^k) + t d^k_\ell) \leq 0\}\right\}. \tag{16}$$

Step 4) Compute a new point
i) Set $(y^k_{\ell+1}, w^k_{\ell+1}) = (x^k, z^k) + \mu t^k_\ell d^k_\ell$.
ii) If $w^k_{\ell+1} \leq f(y^k_{\ell+1})$, we have a ***null step***. Then, define $\lambda^k_{\ell+1} > 0$ and set $\ell := \ell + 1$.
Otherwise, we have a ***serious step***. Then, call $d^k = d^k_\ell$, $d^k_\alpha = d^k_{\alpha\ell}$, $d^k_\beta = d^k_{\beta\ell}$, $\lambda^k_\alpha = \lambda^k_{\alpha\ell}$, $\lambda^k_\beta = \lambda^k_{\beta\ell}$ and $\ell^k = \ell$. Take $(x^{k+1}, z^{k+1}) = (y^k_{\ell+1}, w^k_{\ell+1})$, define $\lambda^{k+1}_0 > 0$, B^{k+1} symmetric and positive definite and set $k = k + 1$, $\ell = 0$, $y^k_0 = x^k$.
iii) Go to Step 1).

4. NUMERICAL EXAMPLES

We consider four test problems. In all of them, Young's modulus of the material is $E = 1.0$ and the maximum volume is $V = 1.0$.

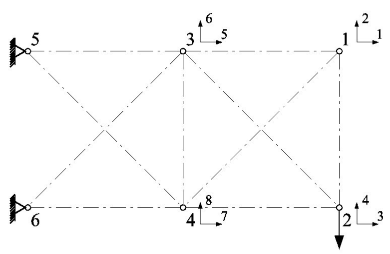


Figure 1. Truss of Example 1.

Example 1. The first example considers the ground structure of Figure (1). The length of each of the horizontal and vertical bars is equal to 1.0 and the magnitude of the loads is 2.0. The secondary loadings have a magnitude $r = 0.3$ and define a basis of the orthogonal complement of the linear span of P, $L(P)$, in the linear space F of the degrees of freedoms of nodes 2 and 4. According to the numeration of the degrees of freedom of Figure (1), the primary loading and the matrix $A = [e^1, e^2, e^3, e^4]$ of the vectors of a basis of F are:

$$p^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have to find the orthonormal basis $\{f_1, f_2, f_3\}$ of the orthogonal complement of $L(P)$ in F. As each of the vectors f^i are in F, they satisfy $f^i = Av^i$ for some $v^i \in \mathbb{R}^4$. As they are normal to p^1, the vectors v^i satisfy $(p^1)^\top Av^i = 0$. Then, we can find $\{v^1, v^2, v^3\}$ as a orthonormal basis of the kernel of $(p^1)^\top A$. This basis can be found using the singular value decomposition of $(p^1)^\top A$ (Golub and Loan, 1996). The result obtained for v^i is

$$[v^1, v^2, v^3] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The final result is:

$$Q = [p^1, rf^1, rf^2, rf^3] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 \end{pmatrix}.$$

Example 2. This example considers the same ground structure of the Example 1, and a loading condition as shown in Figure (2). The secondary loadings have a magnitude $r = 0.4$ and define a basis of the orthogonal complement of $L(P)$ in the linear space F of all the degrees of freedoms of the structure.

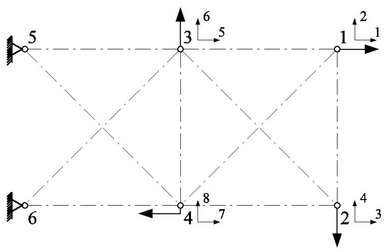


Figure 2. Truss of Example 2.

Example 3. This example consists of a three-dimensional truss with fixed nodes on the horizontal plane $z = 0$ and free nodes on the horizontal plane $z = 2$. The structure has 8 nodes of coordinates

$$\begin{array}{llll} x = \cos(2\pi i/N), & y = \sin(2\pi i/N), & z = 0, & i \in \{1, \ldots, N\}, \\ x = \frac{1}{2}\cos(2\pi i/N), & y = \frac{1}{2}\sin(2\pi i/N), & z = 2, & i \in \{N+1, \ldots, 2N\}, \end{array} \tag{17}$$

with $N = 4$. All the possible edges between free-free or free-fixed nodes are considered. The loading condition consists of four forces acting simultaneously and applied at the nodes on the plane $z = 2$. The nodal force at node i is

$$\begin{array}{ll} p_i & = \left(1/\sqrt{N(1+\rho^2)}\right) [\sin(2\pi i/N), -\cos(2\pi i/N), -\rho]^\top, \\ & i \in \{N+1, \ldots 2N\}, \end{array} \tag{18}$$

with $\rho = 0.001$. The secondary loadings have a magnitude $r = 0.3$ and define a basis of the orthogonal complement of $L(P)$ in the linear space F of all the degrees of freedoms of the structure.

Example 4. This example is similar to the previous one. The nodal coordinates and nodal forces are given by (17) and (18), respectively, but with $N = 5$ and $\rho = 0.01$. The secondary loadings have a magnitude $r = 0.3$ and define a basis of the orthogonal complement of $L(P)$ in the linear space F of all the degrees of freedoms of the structure.

4.1 NUMERICAL RESULTS

The solution obtained by both algorithm are very close. Figures (3) and (4) show the final topology obtained for both algorithms for the numerical examples presented in the later section.

We use the following notation in Table 1: NFDA: Nonsmooth algorithm, FDASDP: Semidefinite algorithm, NV: number of design variables, NI: number of iterations, F: optimal value of the objective function.

The FDASDP algorithm stops when $|d_0| < 10^{-6}$. The stopping criterium for the NFDA algorithm is verified when $|d_\ell^k| < 10^{-4}$.

Table 1. Optimal results.

		NFDA		FDASDP	
Example	NV	NI	F	NI	F
1	10	42	258.24	80	257.26
2	10	37	278.40	238	256.78
3	22	25	110.56	686	110.55
4	35	35	135.27	200	136.26

Table 2. Bar volumes of the optimal structure. Volume of bars less than 10^{-10} are not shown.

Example 1			Example 2			Example 3			Example 4		
bar	NFDA	FDASDP	bar	NFDA	FDASDP	bar	NFDA	FDASDP	bar	NFDA	FDASDP
5–3	2.478e-1	2.487e-1	5–3	2.448e-1	2.498e-1	1–6	1.247e-1	1.247e-1	1–7	1.000e-1	9.908e-2
6–4	1.276e-1	1.264e-1	3–1	1.195e-1	1.247e-1	1–8	1.246e-1	1.245e-1	1–10	9.926e-2	9.899e-2
4–2	1.251e-1	1.250e-1	6–4	2.448e-1	2.498e-1	2–5	1.246e-1	1.245e-1	2–6	9.947e-2	9.899e-2
5–4	3.715e-3	2.098e-3	4–2	1.195e-1	1.247e-1	2–7	1.247e-1	1.247e-1	2–8	1.003e-1	9.908e-2
6–3	2.478e-1	2.487e-1	5–4	1.265e-2	4.818e-4	3–6	1.246e-1	1.245e-1	3–7	9.922e-2	9.899e-2
3–2	2.478e-1	2.487e-1	6–3	1.265e-2	4.819e-4	3–8	1.247e-1	1.247e-1	3–9	1.002e-1	9.908e-2
			3–2	2.368e-1	2.494e-1	4–5	1.247e-1	1.247e-1	4–8	9.943e-2	9.899e-2
			4–1	9.195e-3	3.504e-4	4–7	1.246e-1	1.245e-1	4–10	1.001e-1	9.908e-2
						5–6	4.842e-4	4.839e-4	5–6	1.003e-1	9.908e-2
						5–7	4.343e-4	4.330e-4	5–9	9.935e-2	9.899e-2
						5–8	4.847e-4	4.839e-4	6–7	2.573e-4	1.280e-4
						6–7	4.847e-4	4.839e-4	6–8	2.169e-4	1.792e-3
						6–8	4.343e-4	4.330e-4	6–9	2.366e-4	1.792e-3
						7–8	4.842e-4	4.839e-4	6–10	2.530e-4	1.280e-4
									7–8	2.548e-4	1.280e-4
									7–9	2.345e-4	1.792e-3
									7–10	2.162e-4	1.792e-3
									8–9	2.728e-4	1.280e-4
									8–10	2.137e-4	1.792e-3
									9–10	2.669e-4	1.280e-4

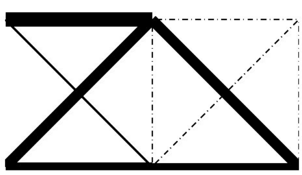
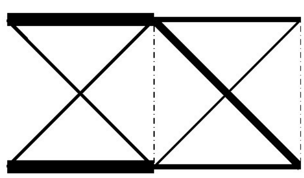


Figure 3. Truss of Example 1



Result of Example 2.

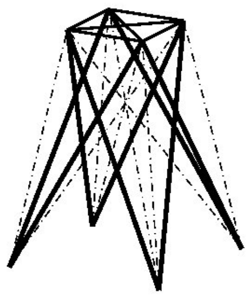
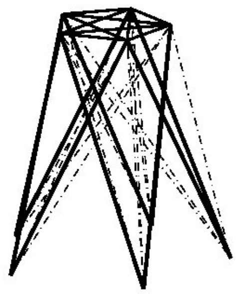


Figure 4. Result of Example 3



Result of Example 4.

5. ACKNOWLEDGEMENTS

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