MAXIMIZING THE FUNDAMENTAL FREQUENCY OF TRUSS STRUCTURES

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Abstract. *The objective of this work is to present a numerical algorithm to solve the structural optimization problem that consists to minimize the weight of a truss structure subject to a fundamental frequency constraint. The paper discusses nonlinear and semidefinite formulations for that problem. The numerical optimization model is solved with a new interior point technique for semidefinite programming. Finally, some numerical examples of truss topology optimization problems are solved with this approach.*

Keywords: *Truss Topology Design; Eigenvalue Optimization; Semidefinite Programming.*

1. INTRODUCTION

A truss is a mechanical system defined by thin elastic bars connected to each other at the end points. The points where the bars are linked are called nodes. The truss can be submitted to different external load cases. A load case is a set of forces acting at the nodes simultaneously. Under a load case, the structure deformes until the equilibrium of internal and external forces is obtained. The deformation of the truss is computed by the finite element method. The truss is defined when the set of nodes and set of bars are given.

The design of light-weight structures with bounded fundamental frequency is of supreme importance in real applications. In this work we consider the cross sectional areas of the bars as the desing variables of the optimization problem. In this case, the weight of the truss is a linear function. By limiting the fundamental frequency of a structure from below by a constant $\underline{\lambda}$, we preclude large deflections or failure of the system due to the resonance effect. A feasible design will be secure if the external excitation of the structure has a frequency far away and below $\underline{\lambda}$.

We use the following notation: $A \in \mathbb{S}^m$ means that A is a symmetric matrix of size $m \times m$. We denote by $A \in \mathbb{S}^m_+$ or $A \succcurlyeq 0$, a symmetric and positive semidefinite matrix of size $m \times m$. If $A \in \mathbb{S}^m_-$ or $A \preccurlyeq 0$, then A is a symmetric and negative semidefinite matrix of size $m \times m$. We write $A \in \mathbb{S}^m_{++}$ or $A \succ 0$ to denote a symmetric and positive definite matrix of size $m \times m$. If $A \in \mathbb{S}^m_{--}$ or $A \prec 0$, then A is a symmetric and negative semidefinite matrix of size $m \times m$. If $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, we write $x \geqslant a$ ($x > a$) to indicate that all components of x are greater than or equal to a (greater than a). For $x \in \mathbb{R}^n$, the symbol $Diag(x)$ is a diagonal matrix of size $n \times n$ with the main diagonal equal to x. The inverse operator is $diag$: $diag(Diag(x)) = x$. The set $\ker(A)$ is the null space of A.

In the next section we define basic concepts, important propositions and the optimization model we are going to solve. In section 3, we introduce a new algorithm for semidefinite programming. Finally, in section 4, we solve some numercial examples using the optimization model and algorithm defined in sections 3 and 4, respectively.

2. PROBLEM DEFINITION

2.1 BASIC CONCEPTS

Consider a general truss structure modeled by the finite element method. Let b be the number of bars where x_i is the cross sectional area of bar $i \in \{1, \ldots, b\}$. The vector of design variables is $x = [x_1 \ \ldots \ x_b]^\top$. Let m be the number of degrees of freedom of the structure and let s be the total number of load cases $\{p^1, \ldots, p^s\}$ where $p^i \in \mathbb{R}^m - \{0\}$.

We assume that the structure satisfy the following linear equilibrium equations:

$$K(x)U = P \tag{1}$$

where:

- $K(x) = \sum_{e=1}^{b} x_e K_e \in \mathbb{S}^m$ is the reduced stiffness matrix of the structure,

- $K_e \in \mathbb{S}^m$ is the reduced stiffness matrix of bar number e with unitary area,
- $U = [u_1 \ \dots \ u_s] \in \mathbb{R}^{m \times s}$ is the matrix with displacement vectors,
- $u_i \in \mathbb{R}^m$ is the displacement due to the load case p^i,
- $P = [p^1 \ \dots \ p^s] \in \mathbb{R}^{m \times s}$ is the matrix of the load cases and
- $p^i \in \mathbb{R}^m$ is the load case number i.

The eigenvalues of the truss are governed by the following problem:

$$K(x)v = \lambda M(x)v \tag{2}$$

where:

- $M(x) = \sum_{e=1}^{b} x_e M_e \in \mathbb{S}^m$ is the reduced mass matrix of the structure,
- $M_e \in \mathbb{R}^{m \times m}$ is the reduced mass matrix of bar number e with unitary area,
- $(\lambda, v) \in \mathbb{R} \times \mathbb{R}^m$ is an eigenvector-eigenvalue pair.

If the pair (λ, v) verifies equation (2), then the square root of λ is a natural frequency of the structure and v is the associated mode shape of vibration.

In order to define a well-posed problem, we assume that $\sum_{e=1}^{b} K_e \succ 0$ and $\sum_{e=1}^{b} M_e \succ 0$ (Ben-Tal and Nemirovski, 1997). These conditions mean that for each load case $p^i \in \mathbb{R}^m$ there exists a unique vector u_i verifying (1) for some design vector $x > 0$.

In general, if $x \geqslant 0$, then $K(x) \succcurlyeq 0$ and $M(x) \succcurlyeq 0$.

In the sequel, we use the following sets in $\mathbb{R}^b$:

- $X = \{x \in \mathbb{R}^b : x \geqslant 0, x \neq 0\}$,
- $X_\varepsilon = \{x \in \mathbb{R}^b : x \geqslant \varepsilon\}$, where $\varepsilon \in \mathbb{R}$.

A report from (Achtzinger and Kočvara, 2006) shows that there exist situations where the smallest eigenvalue of equation (2) is undefined. This situation happens when when (λ, v) verifies (2), with $v \neq 0$ and $v \in \ker(M(x))$. In that case, any pair (μ, v) is also a solution to (2), for any $\mu \in \mathbb{R}$. The well-defined smallest eigenvalue of problem (2), is in that case defined as (Achtzinger and Kočvara, 2006):

$$\begin{aligned} &\lambda_{min} : X \to \mathbb{R} \cup \{+\infty\} \\ &\lambda_{min}(x) = \min\{\lambda : \exists v \in \mathbb{R}^m, (\lambda, v) \ verify \ (2) \ and \ v \notin \ker(M(x))\} \end{aligned} \tag{3}$$

Proposition 1. *Some properties of the function λ_{min} are:*

(a) $\lambda_{min}(x) = \sup\{\lambda : K(x) - \lambda M(x) \succcurlyeq 0\}$ *for any* $x \in X$.

(b) λ_{min} *is quasiconcave in* X.

(c) λ_{min} *is upper semicontinuous in* X.

(d) λ_{min} *is Lipschitz continuous in* X_ε *for any* $\varepsilon > 0$.

(e) λ_{min} *is smooth in* $x \in X_\varepsilon$ *for any* $\varepsilon > 0$ *whenever the smallest eigenvalue of (2) is unique.*

The proof of (a)-(c) can be found in (Achtzinger and Kočvara, 2006). To prove (d) and (e), we can use the proposition 1.2.3 from (Šilhavý, 1949) and the fact that (2) can be expressed as $Aw = \lambda w$.

Finally, we want to emphasize that in this paper, the design variable x belong to X. Then we are dealing, additionally, with a problem of truss topology design (Bendsøe, 1995). As a consequence, we allow some components of x to be zero during the optimization process.

2.2 OPTIMIZATION MODEL

The classic formulation for the minimum weight problem with fundamental frequency constraint reads as:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^b, U\in\mathbb{R}^{m\times s}} & \sum_{e=1}^{b} x_e & & \\ subject\ to: & K(x)U & = & P \\ & diag(P^\top U) & \leqslant & \overline{\gamma} \\ & \lambda_{min}(x) & \geqslant & \underline{\lambda} \\ & x & \geqslant & 0 \end{cases} \tag{4}$$

where $\overline{\gamma}$ and $\underline{\lambda}$ are positive constants to bound the compliance and the smallest eigenvalue of the truss, respectively. The constraints

$$\begin{array}{rcl} K(x)U & = & P \\ diag(P^\top U) & \leqslant & \overline{\gamma} \end{array}$$

mean

$$\begin{array}{rcl} K(x)u_i & = & p^i \\ (p^i)^\top u_i & \leqslant & \overline{\gamma} \end{array}$$

for $i = 1, \ldots, s$.

In view of proposition 1, the function λ_{min} could be discontinuous. Additionally, the function λ_{min} could be nonsmooth if the smallest eigenvalue of problem (2) is multiple. Then it is inappropriate to solve (4) with smooth or even nonsmooth continuous techniques.

Fortunately, there is a pair of equivalences that transforms (4) into a semidefinite programming problem. The first equivalence is:

$$\begin{array}{rcl} K(x)U & = & P \\ diag(P^\top U) & \leqslant & \overline{\gamma} \end{array} \iff \begin{pmatrix} \overline{\gamma} & (p^i)^\top \\ p^i & K(x) \end{pmatrix} \succcurlyeq 0,\ i = 1, \ldots, s \tag{5}$$

The proof of (5) can be found in (Ben-Tal and Nemirovski, 1997).

The second equivalence is:

$$\lambda_{min}(x) \geqslant \underline{\lambda} \iff K(x) - \underline{\lambda} M(x) \succcurlyeq 0 \tag{6}$$

The proof of (6) can be found in (Achtzinger and Kočvara, 2006).

Using (5) and (6), problem (4) is equivalent to the following semidefinite programming problem:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^b} & \sum_{e=1}^{b} x_e & & \\ subject\ to: & & & \\ & \begin{pmatrix} \overline{\gamma} & (p^i)^\top \\ p^i & K(x) \end{pmatrix} & \succcurlyeq & 0, i = 1, \ldots, s \\ & K(x) - \underline{\lambda} M(x) & \succcurlyeq & 0 \\ & x & \geqslant & 0 \end{cases} \tag{7}$$

Problem (7) has nice properties. First, problem (7) is convex, then a local minimum is a global minimum. Second, the objective and constraint functions are smooth with respect to x. Third, we eliminated the design variable $U \in \mathbb{R}^{m\times s}$ from (4) reducing the number of variables from $b + m \times s$ to b.

In the next section we introduce a new semidefinite programming algorithm to solve problem (7).

3. SEMIDEFINITE TECHNIQUE

Consider the following semidefinite programming problem:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} f(x) \\ \quad \text{s.t.} \quad G(x) \preccurlyeq 0, \end{cases} \tag{8}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{S}^m$ are smooth functions. The present algorithm generates a sequence of points in the interior of the feasible region $\Omega = \{x \in \mathbb{R}^n : G(x) \preccurlyeq 0\}$.

The first order Karush-Kunh-Tucker conditions (KKT) for problem (8), proved in (Shapiro, 1994), are the following:

$$\begin{aligned} \nabla_x L(x,\Lambda) &= 0 \\ \Lambda G(x) &= 0 \\ G(x) &\preccurlyeq 0 \\ \Lambda &\succcurlyeq 0 \end{aligned} \tag{9}$$

where $\Lambda \in \mathbb{S}^m$ is a matrix of Lagrange multipliers and $L : \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R}$ is the Lagrangian of problem (8) given by $L(x, A) = f(x) + tr\,(AG(x))$. Here, $tr(A)$ is the trace operator given by the sum of diagonal elements of A.

The Feasible Direction Algorithm for Semidefinite Programming (FDASDP) is an iterative method in the primal variables x and dual variables Λ that converges asymptotically to a pair (x^*, Λ^*) that verifies the KKT condition (9). In each iteration, FDASDP computes a quasi-Newton search direction for the system of equalities of the KKT condition (9). The quasi-Newton matrix corresponding to the equalities of the KKT condition (9) is

$$W = \begin{bmatrix} B & \nabla G(x) \\ (\Lambda \circledast I)\,\nabla G(x)^\top & I \circledast G(x) \end{bmatrix}, \tag{10}$$

where B could be any positive definite matrix, a quasi-Newton approximation of the Lagrangean L or the identity matrix. The matrix $\nabla G(x)$ contains the components of partial derivatives of $G(x)$ (Shapiro, 1994). The symbol $\circledast$ is the symmetric Kronecker operator. Associated with this Kronecker operator is the symmetric vectorization of a matrix (de Klerk, 2002). Given $A \in \mathbb{S}^m$ with components a_{ij}, we define $\overline{m} = \frac{1}{2}m(m+1)$ as the number of upper diagonal components of A. The symmetric vectorization of A, denoted by $svec(A)$, is given in the following manner,

$$svec(A) = \begin{bmatrix} a_{11} & \sqrt{2}a_{12} & a_{22} & \sqrt{2}a_{13} & \sqrt{2}a_{23} & a_{33} & \ldots & a_{mm} \end{bmatrix}^\top.$$

Each iteration of FDASDP solves two linear systems with the matrix W. The solution d_0 of the first linear system is a descent direction for the objective function (Aroztegui, 2010). Unfortunately, if x is in the boundary of Ω, d_0 can be infeasible. Then, we must solve a second linear system with the same matrix W to obtain a direction d_1 that points toward the interior of Ω. Once obtained d_0 and d_1, FDASDP computes a positive parameter ρ in such a way to make $d = d_0 + \rho d_1$ a feasible and descent direction. Finally, to obtain the next point x^{k+1} in the interior of Ω we compute a step size $t^k \in \mathbb{R}^n$ using an Armijo's line search along d. Those steps are repeated until the norm of the direction d_0 is less than a tolerance $TOL > 0$.

Under standard hypotheses we can prove the global convergence of FDASDP, i.e., the sequence generated by the algorithm converges to a stationary point of problem (8) (Aroztegui, 2010). FDASDP is a generalization of Feasible Direction Interior Point Algorithm (FDIPA) developed in (Herskovits, 1998). The statement of FDASDP is presented below.

Feasible Direction Algorithm for Semidefinite Programming- FDASDP

Parameters. $\xi \in (0,1)$, $\eta \in (0,1)$, $\varphi > 0$, $\nu \in (0,1)$, $TOL > 0$.
Data. $x \in int(\Omega_a)$, $B \in \mathbb{S}^n_{++}$ and $\Lambda \in \mathbb{S}^m_{++}$ such that Λ and $G(x)$ commute.
Step 1) Computation of the search direction d.
i) Solve the following linear system in $d_0 \in \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}^{\overline{m}}$

$$\begin{bmatrix} B & \nabla G(x) \\ (\Lambda \circledast I)\,\nabla G(x)^\top & I \circledast G(x) \end{bmatrix} \begin{bmatrix} d_0 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix} \tag{11}$$

If $|d_0| < TOL$, stop.
ii) Solve the following linear system in $d_1 \in \mathbb{R}^n$ and $\lambda_1 \in \mathbb{R}^{\overline{m}}$

$$\begin{bmatrix} B & \nabla G(x) \\ (\Lambda \circledast I)\,\nabla G(x)^\top & I \circledast G(x) \end{bmatrix} \begin{bmatrix} d_1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix} \tag{12}$$

where $\lambda = svec(\Lambda)$.
iii) Compute the parameter ρ such that

$$\rho \,=\, \min\left\{\varphi\|d_0\|^2, (\xi - 1)\frac{d_0^\top \nabla f(x)}{d_1^\top \nabla f(x)}\right\} \tag{13}$$

if $d_1^\top \nabla f(x) > 0$. Otherwise:

$$\rho \,=\, \varphi\|d_0\|^2. \tag{14}$$

iv) Compute the search direction d as

$$d = d_0 + \rho d_1. \tag{15}$$

Step 2) Line Search.
Find t, the first element of $\{1, v, v^2, v^3 \dots\}$ such that

$$f(x+td) \leqslant f(x) + t.\eta.d^\top \nabla f(x) \tag{16}$$

and

$$G(x+td) \prec 0. \tag{17}$$

Step 3) Updates.
i) Update the new point by $x = x + td$.
ii) Define a new value for $B \in \mathbb{S}^n_{++}$.
iii) Define a new value for $\Lambda \in \mathbb{S}^m_{++}$ such that commutes with $G(x)$.
iv) Go to Step 1).

4. NUMERICAL EXAMPLES

Here we show some numerical examples with one and two load cases, i.e. $s = 1$ and $s = 2$.
Example 1. The first example considers the ground structure of Figure 1 with a single load case.

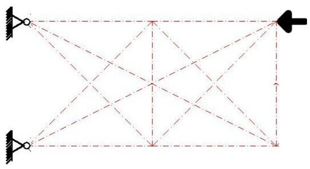

Figure 1. Truss of Example 1.

Example 2. The second example considers the ground structure of Figure 2 with a single load case.

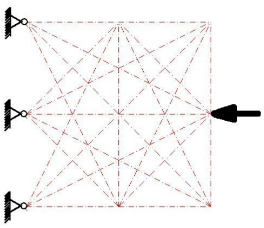

Figure 2. Truss of Example 2.

Example 3. The third example considers the ground structure of Figure 3 with two load cases.

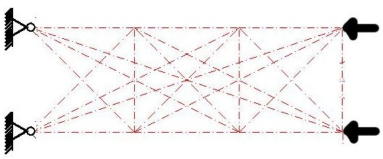

Figure 3. Truss of Example 3.

4.1 NUMERICAL RESULTS

In this section we solve problem (4) using the equivalent formulation (7) for each numerical example. In all examples we use the Feasible Direction Algorithm for Semidefinite Programming (FDASDP).

Figures 4, 5 and 6 show the final topology obtained with FDASDP for example 1, 2 and 3, respectively. In those figures we show the importance of the frequency constraint in truss topology optimization. To the left of each figure, we show the optimized structure without the frequency constraint, and to the right, the optimized structure with the frequency constraint.

The FDASDP algorithm stops when $|d_0| < 10^{-6}$. The compliance and frequency constraints are always active at the solutions obtained. In other words: $diag(P^\top U(x^*)) = \overline{\gamma}$ and $\lambda_{min}(x^*) = \underline{\lambda}$.

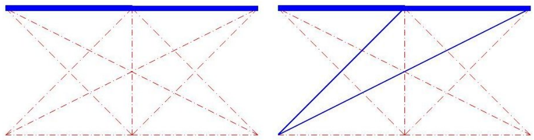

Figure 4. Optimum result of example 1: without (left) and with (right) frequency constraint.

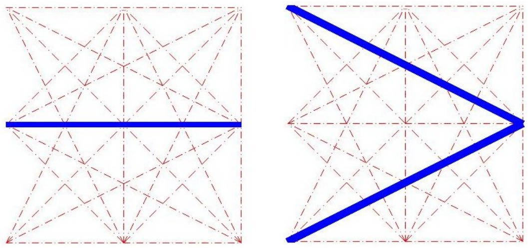

Figure 5. Optimum result of example 2: without (left) and with (right) frequency constraint.

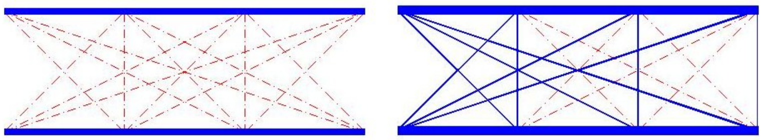

Figure 6. Optimum result of example 3: without (left) and with (right) frequency constraint.

5. CONCLUSIONS

This work shows a technique to solve structural optimization problem that minimizes the weight of a truss structure subject to a fundamental frequency constraint.

In our optimization model we allow the value of a design variable to be zero. The fundamental frequency, as a function of the design variables, could be discontinuous or non smooth when some variable design has zero value. Then, classical smooth optimization techniques could not be applied for that problem.

To overcome those difficulties we present an equivalent semidefinite formulation for the optimization problem. The equivalent formulation is solved with a general algorithm for semidefinite programming. We show some simple examples that confirms the effectiveness of the semidefinite approach. Those examples also shows the importance of fundamental frequency constraint in the design of robust structures.

6. ACKNOWLEDGEMENTS

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