VISCOELASTIC DAMPING IN SANDWICH PLATES IMPLEMENTED BY MEANS OF THE FINITE ELEMENT METHOD: USE OF FRACTIONAL CALCULUS AND STUDY OF THE THEORIES FSDT, HSDT AND LAYERWISE-FSDT

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Abstract. *This article concerns the characterization of dissipative effects in sandwich composite structures. The intrinsic damping of composite materials is increased by application of constrained viscoelastic treatments internal to the structure, resulting in a sandwich configuration. The theories employed in conjunction with the finite element modeling technique for sandwich plate structures are the First-order Shear Deformation Theory (FSDT), the Higher-order Shear Deformation Theory (HSDT) and the Layerwise-FSDT. In these formulations, each node of an arbitrary finite element has five, nine and eleven degrees of freedom, respectively. The chosen type of finite element is the quadrilateral Serendipity, which has three nodes at each interface and a total of eight nodes. The damping resulting from the viscoelastic behavior is included in the formulation by a constitutive equation established in the time domain by means of the Fractional Calculus. The corresponding fractional differential equation is discretized and allows the model incorporation to the Finite Element Model in a straightforward fashion. The dynamic responses are obtained as functions of time by using an adequate numerical integration scheme. Numerical simulations are performed aiming the validation of the modeling procedure for sandwich plate structures. Confrontation of the theories used for the displacements fields is carried out by comparing the corresponding time responses.*

Keywords*: Sandwich structures, Composites, Viscoelasticity, Fractional Calculus, Finite elements*

1. INTRODUCTION

Current engineering design involves minimization of production, operation and maintenance costs and must obey severe specific legislation requirements so as to ensure safety levels, low environmental impact, equipment accuracy levels, among others. For instance, the aviation industry demands very strict control in selection and application of materials that are employed in design and manufacturing stages of structural components. In general, structures are subjected to static (or quasi-static) and/or dynamical loads, as well as to environmental conditions, such as solar radiation, rain and wind, being such factors at times responsible for abrupt degradation-related failures due to long time periods of exposition. Many studies that associate metallic and/or composite materials for the purpose of passive or active vibration control techniques are reported in the literature, as vibration levels are commonly related to fatigue failure, damage and noise emission (Malekzadeh and Khalili, 2005; Correia *et al*., 2000). Finegam and Gibson (1999) state that the passive vibration control strategies, in contrast with the active ones, render improved reliability to machines and structures. In addition, they are inherently stable and do not depend on external energy sources to attain their goal.

Viscoelastic damping is one interesting passive approach that can be used in conjunction with composite and/or metallic materials to convert vibratory mechanical energy into heat. Examples of recent works involving viscoelasticity as a means for vibration control of composite plates are those of Meunier and Shenoi (2001), Makhecha *et al*. (2002), Malekzadeh and Khalili (2005) and Lima *et al*. (2010).

In addition, engineering design also involves the use of reliable analytical and numerical models so as to provide both qualitative and quantitative realistic insights on the structural behavior. Included in the numerical modeling techniques is the Finite Element Method (FEM), which has shown to be suitable for the characterization of the dynamical behavior of a broad category of systems.

In the context of the modeling of plate sandwich structures through finite elements, many theories are currently in use, being these distinguished mainly by the order of polynomial approximations adopted for displacement fields. Commonly used theories are the First-order Shear Deformation Theory (FSDT), the Higher-order Shear Deformation Theory (HSDT), and the Layerwise or Discrete Layer Theory. In the choice of a specific theory, one should consider the thickness of the laminated structure, as well as the trade-off between computational costs and accuracy. As reported by Faria (2010), for thick laminated structures, HSDT is a better alternative. In the cases of homogeneous thin plates, however, FSDT provides good results. Layerwise Theories, on the other hand, are better suited for assessing the dynamical behavior of heterogeneous sandwich systems.

This paper is devoted to the modeling of laminated sandwich composite plates with the inclusion of viscoelastic damping layers, by using the FEM. One highlights its main contribution being the use of different modeling approaches to describe the kinematics of laminated sandwich structures. One assesses, for instance, the influence of the modeling technique over time domain transient responses in a considered numerical example; in other words, one investigates which of the considered theories can give a satisfactory description of the viscoelastic damping behavior.

In the remainder, one firstly presents a brief introduction to the linear viscoelasticity modeling, as well as the fractional differential constitutive equation adopted in this work. The implementation of such model by means of the FEM is then highlighted. Later, attention is devoted to the modeling of sandwich composite laminated plate structures. Displacement fields as well as strain equations are given for the theories considered here, namely the FSDT, the HSDT and a Layerwise-FSDT. The finite element interpolation schemes are briefly presented in addition to the global system of equations of motion. Finally, a numerical example is presented to illustrate the modeling procedures and to compare the damping levels predicted by the different formulations considered, as mentioned earlier.

2. LINEAR VISCOELASTICITY AND FRACTIONAL DERIVATIVE MODEL

Viscoelastic materials are, in general, elastomeric materials that present long molecular chains. This molecular arrangement is responsible for transforming mechanical energy into heat when viscoelastic materials are cyclically loaded. Moreover, at low temperatures and high excitation frequencies, they present a low-stiffness rubber-like behavior, whilst at high temperatures and low excitation frequencies they exhibit stiffer, glassy-like behavior. In a transition zone between the former two limiting cases, it can be seen that their elastic properties are very sensitive to frequency and temperature, although it is in such situations that viscoelastic materials can more effectively dissipate vibratory energy (Nashif *et al.*, 1985). As being so, and also due to efficiency issues, when considering the design process of passive vibration control by means of viscoelastic damping augmentation, such environmental and operational dependencies should be accounted when modeling stress-strain relationships (Lima, 2003, 2007).

According to Nashif *et al.* (1985) and Lima (2003, 2007), basically two main classes of mathematical models for viscoelastic constitutive behavior can be found in the literature, namely classical and modern models. Classical approaches are based on the use of simple rheological models, such as those composed by the association of linear springs and viscous dampers. Among these one can mention the Generalized Kelvin-Voigt model, the Generalized Maxwell Model and the Zener model. However, to be able of representing the physical behavior of real viscoelastic materials used with the purpose of vibration mitigation over a wide frequency range, these models should encompass large number of parameters that must be identified from experimental data. Moreover, when associated with FE models, large computational costs need to be overcome in order to assess the system dynamical behavior.

Regarding the so-called modern viscoelastic models, one can mention the augmented thermodynamic field model (ATF, ADF) (Lesieutre, 1992; Lesieutre and Lee, 1996), the Golla-Hughes-McTavish model (GHM) (Golla and Hughes, 1985; McTavish and Hughes, 1993), and the fractional derivative models (Bagley and Torvik, 1983, 1985). In particular, finite element formulation implementation of fractional derivative models in the time domain has been investigated recently (Schmidt and Gaul, 2002; Deü and Matignon, 2010) and the technique proposed by Galucio *et al.* (2004) has proven to be one of the most effective in terms of computational burden. As a result, this model has been chosen for the developments presented in this paper.

The 1-D viscoelastic constitutive equation that relates stress $\sigma(t)$ and strain $\varepsilon(t)$ adopted in this work was initially proposed by Bagley and Torvik (1985), and afterwards used by Galucio *et al.* (2004). The material behavior is modeled by the following fractional differential equation:

$$\sigma(t)+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\sigma(t)=E_0\,\varepsilon(t)+\tau^{\alpha}E_{\infty}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\varepsilon(t)\ , \tag{1}$$

where t denotes time, E_0 is the static modulus, E_∞ is the high frequency modulus and τ is the relaxation time. Furthermore, $\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}f(t)$ stands for the Riemman-Liouville fractional derivative of $f(t)$ with respect to time, mathematically stated as (Miller and Ross, 1993):

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\xi)^{-\alpha}f(\xi)\mathrm{d}\xi\ ,\quad 0<\alpha<1\ ,\quad \text{where }\Gamma(z):=\int_0^{\infty}u^{z-1}e^{-u}\mathrm{d}u\ ,\quad \mathrm{Re}(z)>0\ . \tag{2}$$

It is worthwhile introducing the Grünwald-Letnikov approximation for the fractional derivative operator. According to Miller and Ross (1993) and Galucio *et al.* (2004), one has:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} f(t) \approx (\Delta t)^{-\alpha} \sum_{j=0}^{N_P} A_{j+1}(\alpha)\, f(t - j\Delta t) , \tag{3}$$

where N_P is the number of points retained from the time discretization, which is assumed to be uniform, Δt is the fixed-size time increment and $A_{j+1}(\alpha)$ are the so-called Grünwald coefficients associated with an arbitrary α-order differentiation. These coefficients can be calculated either by means of the Gamma function or by the following recurrence formula:

$$A_{j+1}(\alpha) = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} = \frac{j-\alpha-1}{j} A_j(\alpha) , \tag{4}$$

where $A_1(\alpha) := 1$ for any order α. One may not need to consider all the history of the function $f(t)$ when computing the approximation given in Eq. (3) since Grünwald coefficients $A_{j+1}(\alpha)$ are strictly decreasing as the index *j* increases, as pointed out by Schmidt and Gaul (2002) and Galucio *et al.* (2004).

Galucio *et al.* (2004) introduce the variable change $\bar{\varepsilon}(t) = \varepsilon(t) - \sigma(t)/E_{\infty}$ into Eq. (1) to obtain a constitutive equation that contains only one fractional differentiation:

$$\bar{\varepsilon}(t) + \tau^{\alpha} \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} \bar{\varepsilon}(t) = \frac{E_{\infty} - E_0}{E_{\infty}} \varepsilon(t) . \tag{5}$$

This equation can be solved by means of the Grünwald-Letnikov approximation introduced above, to give:

$$\bar{\varepsilon}(t) = \left[\frac{(\Delta t)^{\alpha}}{(\Delta t)^{\alpha} + \tau^{\alpha}} \right] \frac{E_{\infty} - E_0}{E_{\infty}} \varepsilon(t) - \left[\frac{\tau^{\alpha}}{(\Delta t)^{\alpha} + \tau^{\alpha}} \right] \sum_{j=1}^{N_P} A_{j+1}(\alpha)\, \bar{\varepsilon}(t - j\Delta t) . \tag{6}$$

Such solution can be extended to more general 3-D cases by introducing first- and second-order tensors. By admitting that the viscoelastic material is isotropic and that its Poisson ratio ν is frequency-independent, one has:

$$\boldsymbol{\sigma}(t) = \mathbf{C} \frac{E_{\infty}}{E_0} \left[\boldsymbol{\varepsilon}(t) - \bar{\boldsymbol{\varepsilon}}(t) \right] = c_{\varepsilon} \mathbf{C} \boldsymbol{\varepsilon}(t) + c_{\bar{\varepsilon}} \mathbf{C} \sum_{j=1}^{N_P} A_{j+1}(\alpha)\, \bar{\boldsymbol{\varepsilon}}(t - j\Delta t) , \tag{7}$$

where $\mathbf{C}$ is the matrix of elastic low-frequency coefficients of the viscoelastic material and:

$$c_{\varepsilon} = 1 + \left[\frac{\tau^{\alpha}}{(\Delta t)^{\alpha} + \tau^{\alpha}} \right] \frac{E_{\infty} - E_0}{E_0} \;\; ; \;\; c_{\bar{\varepsilon}} = \left[\frac{\tau^{\alpha}}{(\Delta t)^{\alpha} + \tau^{\alpha}} \right] \frac{E_{\infty}}{E_0} . \tag{8}$$

The factor $\mathbf{C}/E_0$ is introduced in Eq. (7) to account for different behavior in the cases of axial and shear deformation. If the viscoelastic is either orthotropic or anisotropic, straightforward modifications must be introduced accordingly in the $\mathbf{C}$ elastic coefficients matrix.

2.1. Implementation of the Fractional Derivative Viscoelastic Model into Finite Element Models

One considers here a displacement-based finite element formulation for which the following interpolation is used:

$$\tilde{\mathbf{U}}(\mathbf{x}, t) = \tilde{\mathbf{N}}(\mathbf{x})\, \tilde{\mathbf{u}}(t), \tag{9}$$

where $\tilde{\mathbf{U}}(\mathbf{x},t)$ stands for the generalized displacement vector, $\tilde{\mathbf{N}}(\mathbf{x})$ is the interpolation matrix, $\tilde{\mathbf{u}}(t)$ is the nodal degrees of freedom vector and $\mathbf{x}$ the spatial coordinates vector. Following the procedure adopted by Galucio *et al.* (2004), one obtains the equations of motion at elementary level in the form:

$$\tilde{\mathbf{M}}\ddot{\tilde{\mathbf{u}}}(t)+c_{\varepsilon}\tilde{\mathbf{K}}\tilde{\mathbf{u}}(t)=\tilde{\mathbf{Q}}_e(t)-c_{\bar{\varepsilon}}\tilde{\mathbf{K}}\sum_{j=1}^{N_P}A_{j+1}(\alpha)\tilde{\bar{\mathbf{u}}}(t-j\Delta t)\,, \tag{10}$$

where $\tilde{\mathbf{M}}$ is the elementary mass matrix, $\tilde{\mathbf{K}}$ is the elementary stiffness matrix, $\tilde{\mathbf{Q}}_e(t)$ is the elementary externally applied generalized forces vector, and $\tilde{\bar{\mathbf{u}}}(t)$ is an internal variable vector that may be calculated by considering Eq. (6). As for the notation, quantities distinguished by an upper tilde ($\tilde{\bullet}$) stand for element-related ones.

3. FSDT, HSDT AND LAYERWISE-FSDT FINITE ELEMENT FORMULATIONS FOR SANDWICH COMPOSITE LAMINATED PLATES

In this section one presents, at first, the displacement fields related to the FSDT, HSDT and Layerwise-FSDT associated with composite laminated plate structures.

For the FSDT, one adopts:

$$\begin{aligned}
u(x,y,z,t)&=u_0(x,y,t)+z\psi_x(x,y,t)\ ;\\
v(x,y,z,t)&=v_0(x,y,t)+z\psi_y(x,y,t)\ ;\\
w(x,y,z,t)&=w_0(x,y,t)\ ,
\end{aligned} \tag{11}$$

where (x,y,z) are the spatial cartesian coordinates, $(x,y,0)$ is the undeformed middle plane of the sandwich plate, $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ are the displacement fields in the directions of x, y and z, respectively, $u_0(x,y,t)$, $v_0(x,y,t)$ and $w_0(x,y,t)$ are the displacements of a material point that belongs to the reference plane $(x,y,0)$ in the directions of x, y and z, respectively, and $\psi_x(x,y,t)$ and $\psi_y(x,y,t)$ are y and x rotations, respectively (Reddy, 1997; Faria, 2010)

As for the HSDT, one has:

$$\begin{aligned}
u(x,y,z,t)&=u_0(x,y,t)+z\psi_x(x,y,t)+z^2\,\zeta_x(x,y,t)+z^3\,\phi_x(x,y,t)\ ;\\
v(x,y,z,t)&=v_0(x,y,t)+z\psi_y(x,y,t)+z^2\,\zeta_y(x,y,t)+z^3\,\phi_y(x,y,t)\ ;\\
w(x,y,z,t)&=w_0(x,y,t)+z\psi_z(x,y,t)+z^2\,\zeta_z(x,y,t)\ ,
\end{aligned} \tag{12}$$

where $\zeta_x(x,y,t)$, $\zeta_x(x,y,t)$, $\zeta_x(x,y,t)$ and $\zeta_x(x,y,t)$ are functions that do not have clear physical meaning, altough they can be viewed as higher order rotations (Faria, 2010).

Finally, considering the Layerwise-FSDT, one can establish:

$$\begin{aligned}
u^k(x,y,z,t)&=u_0^k(x,y,t)+\left(z-z^k\right)\psi_x^k(x,y,t)\ ;\\
v^k(x,y,z,t)&=v_0^k(x,y,t)+\left(z-z^k\right)\psi_y^k(x,y,t)\ ;\\
w^k(x,y,z,t)&=w_0(x,y,t)\ ,
\end{aligned} \tag{13}$$

where the superscript k is used to denote quantities associated with the k-th layer of the laminated plate, $k=1,\ \ldots,\ n_l$, in which n_l is the total number of layers. Furthermore, for Layerwise Theory formulations, such as the Layerwise-FSDT considered here, one needs also to consider displacement continuity between layers, expressed as follows:

$$\begin{bmatrix} u^k\left(x,y,z^k+\frac{1}{2}h^k,t\right)\\ v^k\left(x,y,z^k+\frac{1}{2}h^k,t\right)\end{bmatrix}\equiv\begin{bmatrix} u^{k+1}\left(x,y,z^{k+1}-\frac{1}{2}h^{k+1},t\right)\\ v^{k+1}\left(x,y,z^{k+1}-\frac{1}{2}h^{k+1},t\right)\end{bmatrix}, \tag{14}$$

from which $u_0^k(x,y,t)$ and $v_0^k(x,y,t)$ appearing in Eqs. (13) can be computed from the reference layer displacements $u_0(x,y,t)$ and $v_0(x,y,t)$.

Regarding strains, one can establish (space and time dependencies are omitted for the sake of simplicity):

$$\boldsymbol{\varepsilon}_{(6\times1)}^{\text{FSDT}} = \begin{bmatrix} \varepsilon_{xx}^{\text{FSDT}} \\ \varepsilon_{yy}^{\text{FSDT}} \\ \varepsilon_{zz}^{\text{FSDT}} \\ \varepsilon_{xy}^{\text{FSDT}} \\ \varepsilon_{yz}^{\text{FSDT}} \\ \varepsilon_{xz}^{\text{FSDT}} \end{bmatrix} = \begin{bmatrix} u_{0,x} + z\psi_{x,x} \\ v_{0,y} + z\psi_{y,y} \\ 0 \\ u_{0,y} + v_{0,x} + z\left(\psi_{x,y} + \psi_{y,x}\right) \\ \psi_y + w_{0,y} \\ \psi_x + w_{0,x} \end{bmatrix} ; \tag{15}$$

$$\boldsymbol{\varepsilon}_{(6\times1)}^{\text{HSDT}} = \begin{bmatrix} \varepsilon_{xx}^{\text{HSDT}} \\ \varepsilon_{yy}^{\text{HSDT}} \\ \varepsilon_{zz}^{\text{HSDT}} \\ \varepsilon_{xy}^{\text{HSDT}} \\ \varepsilon_{yz}^{\text{HSDT}} \\ \varepsilon_{xz}^{\text{HSDT}} \end{bmatrix} = \begin{bmatrix} u_{0,x} + z\psi_{x,x} + z^2\zeta_{x,x} + z^3\phi_{x,x} \\ v_{0,y} + z\psi_{y,y} + z^2\zeta_{y,y} + z^3\phi_{y,y} \\ \psi_z + 2z\zeta_z \\ u_{0,y} + v_{0,x} + z\left(\psi_{x,y} + \psi_{y,x}\right) + z^2\left(\zeta_{x,y} + \zeta_{y,x}\right) + z^3\left(\phi_{x,y} + \phi_{y,x}\right) \\ \left(\psi_y + w_{0,y}\right) + z\left(2\zeta_y + \psi_{z,y}\right) + z^2\left(3\phi_y + \zeta_{z,y}\right) \\ \left(\psi_x + w_{0,x}\right) + z\left(2\zeta_x + \psi_{z,x}\right) + z^2\left(3\phi_x + \zeta_{z,x}\right) \end{bmatrix} ; \tag{16}$$

$$\boldsymbol{\varepsilon}_{(6\times1)}^{k,\text{L-FSDT}} = \begin{bmatrix} \varepsilon_{xx}^{k,\text{L-FSDT}} \\ \varepsilon_{yy}^{k,\text{L-FSDT}} \\ \varepsilon_{zz}^{k,\text{L-FSDT}} \\ \varepsilon_{xy}^{k,\text{L-FSDT}} \\ \varepsilon_{yz}^{k,\text{L-FSDT}} \\ \varepsilon_{xz}^{k,\text{L-FSDT}} \end{bmatrix} = \begin{bmatrix} u_{0,x}^k + \left(z - z^k\right)\psi_{x,x}^k \\ v_{0,y}^k + \left(z - z^k\right)\psi_{y,y}^k \\ 0 \\ u_{0,y}^k + v_{0,x}^k + \left(z - z^k\right)\left(\psi_{x,y}^k + \psi_{y,x}^k\right) \\ \psi_y^k + w_{0,y} \\ \psi_x^k + w_{0,x} \end{bmatrix}, \quad k = 1,\ \dots,\ n_l\ , \tag{17}$$

where $(\bullet_{,i})$ stands for the partial derivative with respect to the parameter *i*, e. g., $u_{0,x} = \frac{\partial}{\partial x}u_0$, etc..

Introducing appropriate discretization for the variables appearing in Eqs. (11) – (13), finite elements models can be obtained for each one of the considered theories. For instance, here one uses the quadrilateral Serendipity element for multi-layered composite plates (Reddy, 1997; Faria, 2010), which presents eight nodes, three per interface. For FSDT, HSDT and Layerwise-FSDT formulations, one has 5, 11 and $2n_l + 3$ degrees of freedom per node, respectively. The linear transformation matrix that relates global (x, y) and local (ξ,η) coordinates, as well as Serendipity interpolation functions $N_i(\xi,\eta)$, $i = 1,\ \dots,\ 8$, can be encountered in Reddy (1997) and Faria (2010). Equations (11) – (13) can be rewritten by the use of finite elements discretization, respectively, as:

$$\tilde{\mathbf{U}}_{(3\times1)}(\xi,\eta,z,t) = \tilde{\mathbf{A}}_{(3\times5)}^{\text{FSDT}}(z)\,\tilde{\mathbf{N}}_{(5\times40)}^{\text{FSDT}}(\xi,\eta)\,\tilde{\mathbf{u}}_{(40\times1)}^{\text{FSDT}}(t) ; \tag{18}$$

$$\tilde{\mathbf{U}}_{(3\times1)}(\xi,\eta,z,t) = \tilde{\mathbf{A}}_{(3\times11)}^{\text{HSDT}}(z)\,\tilde{\mathbf{N}}_{(11\times88)}^{\text{HSDT}}(\xi,\eta)\,\tilde{\mathbf{u}}_{(88\times1)}^{\text{HSDT}}(t) ; \tag{19}$$

$$\tilde{\mathbf{U}}_{(3\times1)}^{k}(\xi,\eta,z,t) = \tilde{\mathbf{A}}_{(3\times(2\tilde{n}_l+3))}^{k,\text{L-FSDT}}(z)\,\tilde{\mathbf{N}}_{((2\tilde{n}_l+3)\times8(2\tilde{n}_l+3))}^{\text{L-FSDT}}(\xi,\eta)\,\tilde{\mathbf{u}}_{(8(2\tilde{n}_l+3)\times1)}^{\text{L-FSDT}}(t) , \quad k = 1,\ \dots,\ \tilde{n}_l\ , \tag{20}$$

where $\tilde{\mathbf{N}}^{\text{FSDT}}(\xi,\eta)$, $\tilde{\mathbf{N}}^{\text{HSDT}}(\xi,\eta)$ and $\tilde{\mathbf{N}}^{\text{L-FSDT}}(\xi,\eta)$ are the interpolation matrices associated with the FSDT, HSDT and Layerwise-FSDT formulations, respectively, and $\tilde{\mathbf{u}}^{\text{FSDT}}(t)$, $\tilde{\mathbf{u}}^{\text{HSDT}}(t)$ and $\tilde{\mathbf{u}}^{\text{L-FSDT}}(t)$ are the elementary d.o.f. vectors for the corresponding theories, which are given by:

$$\tilde{\mathbf{u}}_{(40\times1)}^{\text{FSDT}}(t)=\left[\left[\tilde{\mathbf{u}}_1^{\text{FSDT}}(t)\right]^{\text{T}}\quad\left[\tilde{\mathbf{u}}_2^{\text{FSDT}}(t)\right]^{\text{T}}\quad\cdots\quad\left[\tilde{\mathbf{u}}_8^{\text{FSDT}}(t)\right]^{\text{T}}\right]^{\text{T}}\ ;$$
$$\tilde{\mathbf{u}}_i^{\text{FSDT}}(t)=\left[\tilde{u}_0(\xi_i,\eta_i,t)\quad\tilde{v}_0(\xi_i,\eta_i,t)\quad\tilde{w}_0(\xi_i,\eta_i,t)\quad\tilde{\psi}_x(\xi_i,\eta_i,t)\quad\tilde{\psi}_y(\xi_i,\eta_i,t)\right]^{\text{T}}\ ,\quad i=1,\ \ldots,\ 8\ ;\tag{21}$$

$$\tilde{\mathbf{u}}_{(88\times1)}^{\text{HSDT}}(t)=\left[\left[\tilde{\mathbf{u}}_1^{\text{HSDT}}(t)\right]^{\text{T}}\quad\left[\tilde{\mathbf{u}}_2^{\text{HSDT}}(t)\right]^{\text{T}}\quad\cdots\quad\left[\tilde{\mathbf{u}}_8^{\text{HSDT}}(t)\right]^{\text{T}}\right]^{\text{T}}\ ;$$
$$\tilde{\mathbf{u}}_i^{\text{HSDT}}(t)=\left[\tilde{u}_0(\xi_i,\eta_i,t)\quad\tilde{v}_0(\xi_i,\eta_i,t)\quad\tilde{w}_0(\xi_i,\eta_i,t)\quad\tilde{\psi}_x(\xi_i,\eta_i,t)\quad\tilde{\psi}_y(\xi_i,\eta_i,t)\quad\tilde{\psi}_z(\xi_i,\eta_i,t)\quad\cdots\right.$$
$$\left.\cdots\quad\tilde{\zeta}_x(\xi_i,\eta_i,t)\quad\tilde{\zeta}_y(\xi_i,\eta_i,t)\quad\tilde{\zeta}_z(\xi_i,\eta_i,t)\quad\tilde{\phi}_x(\xi_i,\eta_i,t)\quad\tilde{\phi}_y(\xi_i,\eta_i,t)\right]^{\text{T}}\ ,\quad i=1,\ \ldots,\ 8\ ;\tag{22}$$

$$\tilde{\mathbf{u}}_{(8(2\tilde{n}_l+3)\times1)}^{\text{L-FSDT}}(t)=\left[\left[\tilde{\mathbf{u}}_1^{\text{L-FSDT}}(t)\right]^{\text{T}}\quad\left[\tilde{\mathbf{u}}_2^{\text{L-FSDT}}(t)\right]^{\text{T}}\quad\cdots\quad\left[\tilde{\mathbf{u}}_8^{\text{L-FSDT}}(t)\right]^{\text{T}}\right]\ ;$$
$$\tilde{\mathbf{u}}_i^{\text{L-FSDT}}(t)=\left[\tilde{u}_0(\xi_i,\eta_i,t)\quad\tilde{v}_0(\xi_i,\eta_i,t)\quad\tilde{w}_0(\xi_i,\eta_i,t)\quad\tilde{\psi}_x^1(\xi_i,\eta_i,t)\quad\tilde{\psi}_y^1(\xi_i,\eta_i,t)\quad\cdots\right.$$
$$\left.\cdots\quad\tilde{\psi}_x^k(\xi_i,\eta_i,t)\quad\tilde{\psi}_y^k(\xi_i,\eta_i,t)\quad\cdots\quad\tilde{\psi}_x^{\tilde{n}_l}(\xi_i,\eta_i,t)\quad\tilde{\psi}_y^{\tilde{n}_l}(\xi_i,\eta_i,t)\right]^{\text{T}}\ ,\quad i=1,\ \ldots,\ 8\ .\tag{23}$$

In the former, $\tilde{\mathbf{u}}_i^{\text{FSDT}}(t)$, $\tilde{\mathbf{u}}_i^{\text{HSDT}}(t)$ and $\tilde{\mathbf{u}}_i^{\text{L-FSDT}}(t)$ are the d.o.f. vectors associated with the *i*-th node of the Serendipity element, whose coordinates are given by (ξ_i,η_i). Moreover, the matrices $\tilde{\mathbf{A}}^{\text{FSDT}}(z)$, $\tilde{\mathbf{A}}^{\text{HSDT}}(z)$ and $\tilde{\mathbf{A}}^{k,\text{L-FSDT}}(z)$ appearing in Eqs. (18) – (20) relate the mechanical displacement fields discretized by means of Eqs. (21) – (23) to the generalized displacements vectors:

$$\tilde{\mathbf{U}}_{(3\times1)}(\xi,\eta,z,t)=\left[\tilde{u}(\xi,\eta,z,t)\quad\tilde{v}(\xi,\eta,z,t)\quad\tilde{w}(\xi,\eta,z,t)\right]^{\text{T}}\ ;$$
$$\tilde{\mathbf{U}}_{(3\times1)}^{k}(\xi,\eta,z,t)=\left[\tilde{u}^k(\xi,\eta,z,t)\quad\tilde{v}^k(\xi,\eta,z,t)\quad\tilde{w}^k(\xi,\eta,z,t)\right]^{\text{T}}\ ,\quad k=1,\ \ldots,\ \tilde{n}_l\ .\tag{24}$$

Matrices $\tilde{\mathbf{A}}^{\mathcal{T}}(z)$ and $\tilde{\mathbf{N}}^{\mathcal{T}}(\xi,\eta)$, for $\mathcal{T}$ = FSDT, HSDT, L-FSDT , are not presented here, but can be encountered in the works of Reddy (1997) and Faria (2010).

The strain vectors given in Eqs. (15) – (17), can be computed according to:

$$\tilde{\boldsymbol{\varepsilon}}_{(6\times1)}^{\text{FSDT}}(\xi,\eta,z,t)=\tilde{\mathbf{D}}_{(6\times5)}^{\text{FSDT}}(z)\tilde{\mathbf{N}}_{(5\times40)}^{\text{FSDT}}(\xi,\eta)\tilde{\mathbf{u}}_{(40\times1)}^{\text{FSDT}}(t)\ ;\tag{25}$$

$$\tilde{\boldsymbol{\varepsilon}}_{(6\times1)}^{\text{HSDT}}(\xi,\eta,z,t)=\tilde{\mathbf{D}}_{(6\times11)}^{\text{HSDT}}(z)\tilde{\mathbf{N}}_{(11\times88)}^{\text{HSDT}}(\xi,\eta)\tilde{\mathbf{u}}_{(88\times1)}^{\text{HSDT}}(t)\ ;\tag{26}$$

$$\tilde{\boldsymbol{\varepsilon}}_{(6\times1)}^{k,\text{L-FSDT}}(\xi,\eta,z,t)=\tilde{\mathbf{D}}_{(6\times(2\tilde{n}_l+3))}^{k,\text{L-FSDT}}(z)\tilde{\mathbf{N}}_{((2\tilde{n}_l+3)\times8(2\tilde{n}_l+3))}^{\text{L-FSDT}}(\xi,\eta)\tilde{\mathbf{u}}_{(8(2\tilde{n}_l+3)\times1)}^{\text{L-FSDT}}(t)\ ,\quad k=1,\ \ldots,\ \tilde{n}_l\ ,\tag{27}$$

where $\tilde{\mathbf{D}}^{\text{FSDT}}(z)$, $\tilde{\mathbf{D}}^{\text{HSDT}}(z)$ and $\tilde{\mathbf{D}}^{k,\text{L-FSDT}}(z)$ are spatial differential operator matrices depending on *z* (Faria, 2010).

By considering the previous relations, as well as the constitutive equations for the base (metallic, composite, etc.) and viscoelastic materials, one can compute elementary kinetic and potential energies, this latter associated with the non-viscoelastic layers, only, and the elementary virtual work of the internal forces associated to the viscoelastic material. In addition, one may consider the Boolean transformation matrices $\tilde{\mathbf{L}}^{\text{FSDT}}$, $\tilde{\mathbf{L}}^{\text{HSDT}}$ and $\tilde{\mathbf{L}}^{\text{L-FSDT}}$ that relates elementary and global d.o.f. vectors:

$$\tilde{\mathbf{u}}_{(40\times1)}^{\text{FSDT}}(t)=\tilde{\mathbf{L}}_{(40\times n)}^{\text{FSDT}}\mathbf{u}_{(n\times1)}^{\text{FSDT}}(t)\ ;\quad\tilde{\mathbf{u}}_{(88\times1)}^{\text{HSDT}}(t)=\tilde{\mathbf{L}}_{(88\times n)}^{\text{HSDT}}\mathbf{u}_{(n\times1)}^{\text{HSDT}}(t)\ ;\quad\tilde{\mathbf{u}}_{(8(2\tilde{n}_l+3)\times1)}^{\text{L-FSDT}}(t)=\tilde{\mathbf{L}}_{(8(2\tilde{n}_l+3)\times n)}^{\text{L-FSDT}}\mathbf{u}_{(n\times1)}^{\text{L-FSDT}}(t)\ ,\tag{28}$$

where n is the total number of d.o.f. of the considered problem. In doing so, elementary energy contributions can be expressed in a global sense and, by means of Lagrange's equations, one can obtain the following global equations of motion for each of the theories considered:

$$\mathbf{M}_{(n\times n)}^{\text{FSDT}}\ddot{\mathbf{u}}_{(n\times1)}^{\text{FSDT}}(t)+\bar{\mathbf{K}}_{(n\times n)}^{\text{FSDT}}\mathbf{u}_{(n\times1)}^{\text{FSDT}}(t)=\mathbf{R}_{(n\times1)}^{\text{FSDT}}(t)-c_{\bar{\varepsilon}}\mathbf{K}_{(n\times n)}^{\text{FSDT},v}\sum_{j=1}^{N_P}A_{j+1}(\alpha)\bar{\mathbf{u}}_{(n\times1)}^{\text{FSDT}}(t-j\Delta t)\tag{29}$$

$$\mathbf{M}_{(n\times n)}^{\text{HSDT}}\ddot{\mathbf{u}}_{(n\times 1)}^{\text{HSDT}}\left(t\right)+\bar{\mathbf{K}}_{(n\times n)}^{\text{HSDT}}\ddot{\mathbf{u}}_{(n\times 1)}^{\text{HSDT}}\left(t\right)=\mathbf{R}_{(n\times 1)}^{\text{HSDT}}\left(t\right)-c_{\bar{\varepsilon}}\mathbf{K}_{(n\times n)}^{\text{HSDT},v}\sum_{j=1}^{N_P}A_{j+1}\left(\alpha\right)\bar{\mathbf{u}}_{(n\times 1)}^{\text{HSDT}}\left(t-j\Delta t\right) \tag{30}$$

$$\mathbf{M}_{(n\times n)}^{\text{L-FSDT}}\ddot{\mathbf{u}}_{(n\times 1)}^{\text{L-FSDT}}\left(t\right)+\bar{\mathbf{K}}_{(n\times n)}^{\text{L-FSDT}}\ddot{\mathbf{u}}_{(n\times 1)}^{\text{L-FSDT}}\left(t\right)=\mathbf{R}_{(n\times 1)}^{\text{L-FSDT}}\left(t\right)-c_{\bar{\varepsilon}}\mathbf{K}_{(n\times n)}^{\text{L-FSDT},v}\sum_{j=1}^{N_P}A_{j+1}\left(\alpha\right)\bar{\mathbf{u}}_{(n\times 1)}^{\text{L-FSDT}}\left(t-j\Delta t\right) \tag{31}$$

where:

- for the FSDT formulation:
  - $\mathbf{M}_{(n\times n)}^{\text{FSDT}}=\sum\left[\tilde{\mathbf{L}}_{(40\times n)}^{\text{FSDT}}\right]^{\text{T}}\tilde{\mathbf{M}}_{(40\times 40)}^{\text{FSDT}}\tilde{\mathbf{L}}_{(40\times n)}^{\text{FSDT}}$ and $\tilde{\mathbf{M}}_{(40\times 40)}^{\text{FSDT}}$ are the global and elementary mass matrices, respectively;
  - $\bar{\mathbf{K}}_{(n\times n)}^{\text{FSDT}}=\mathbf{K}_{(n\times n)}^{\text{FSDT},nv}+c_{\varepsilon}\mathbf{K}_{(n\times n)}^{\text{FSDT},v}$ is the modified global stiffness matrix;
  - $\mathbf{K}_{(n\times n)}^{\text{FSDT},nv}=\sum\left[\tilde{\mathbf{L}}_{(40\times n)}^{\text{FSDT}}\right]^{\text{T}}\tilde{\mathbf{K}}_{(40\times 40)}^{\text{FSDT},nv}\tilde{\mathbf{L}}_{(40\times n)}^{\text{FSDT}}$ and $\tilde{\mathbf{K}}_{(40\times 40)}^{\text{FSDT},nv}$ are the non-viscoelastic contributions to the global and elementary stiffness matrices, respectively;
  - $\mathbf{K}_{(n\times n)}^{\text{FSDT},v}=\sum\left[\tilde{\mathbf{L}}_{(40\times n)}^{\text{FSDT}}\right]^{\text{T}}\tilde{\mathbf{K}}_{(40\times 40)}^{\text{FSDT},v}\tilde{\mathbf{L}}_{(40\times n)}^{\text{FSDT}}$ and $\tilde{\mathbf{K}}_{(40\times 40)}^{\text{FSDT},v}$ are the viscoelastic contributions to the global and elementary stiffness matrices, respectively;
  - $\tilde{\mathbf{K}}_{(40\times 40)}^{\text{FSDT}}=\tilde{\mathbf{K}}_{(40\times 40)}^{\text{FSDT},nv}+\tilde{\mathbf{K}}_{(40\times 40)}^{\text{FSDT},v}$ is the elementary stiffness matrix;
  - $\mathbf{R}_{(n\times 1)}^{\text{FSDT}}\left(t\right)$ is the global externally applied generalized forces vector;
  - $\bar{\mathbf{u}}_{(n\times 1)}^{\text{FSDT}}\left(t\right)$ is an internal variable vector arising from the viscoelastic behavior of selected layers of the structure;
- for the HSDT formulation:
  - $\mathbf{M}_{(n\times n)}^{\text{HSDT}}=\sum\left[\tilde{\mathbf{L}}_{(88\times n)}^{\text{HSDT}}\right]^{\text{T}}\tilde{\mathbf{M}}_{(88\times 88)}^{\text{HSDT}}\tilde{\mathbf{L}}_{(88\times n)}^{\text{HSDT}}$ and $\tilde{\mathbf{M}}_{(88\times 88)}^{\text{HSDT}}$ are the global and elementary mass matrices, respectively;
  - $\bar{\mathbf{K}}_{(n\times n)}^{\text{HSDT}}=\mathbf{K}_{(n\times n)}^{\text{HSDT},nv}+c_{\varepsilon}\mathbf{K}_{(n\times n)}^{\text{HSDT},v}$ is the modified global stiffness matrix;
  - $\mathbf{K}_{(n\times n)}^{\text{HSDT},nv}=\sum\left[\tilde{\mathbf{L}}_{(88\times n)}^{\text{HSDT}}\right]^{\text{T}}\tilde{\mathbf{K}}_{(88\times 88)}^{\text{HSDT},nv}\tilde{\mathbf{L}}_{(88\times n)}^{\text{HSDT}}$ and $\tilde{\mathbf{K}}_{(88\times 88)}^{\text{HSDT},nv}$ are the non-viscoelastic contribution to the global and elementary stiffness matrices, respectively;
  - $\mathbf{K}_{(n\times n)}^{\text{HSDT},v}=\sum\left[\tilde{\mathbf{L}}_{(88\times n)}^{\text{HSDT}}\right]^{\text{T}}\tilde{\mathbf{K}}_{(88\times 88)}^{\text{HSDT},v}\tilde{\mathbf{L}}_{(88\times n)}^{\text{HSDT}}$ and $\tilde{\mathbf{K}}_{(88\times 88)}^{\text{HSDT},v}$ are the viscoelastic contribution to the global and elementary stiffness matrices, respectively;
  - $\tilde{\mathbf{K}}_{(88\times 88)}^{\text{HSDT}}=\tilde{\mathbf{K}}_{(88\times 88)}^{\text{HSDT},nv}+\tilde{\mathbf{K}}_{(88\times 88)}^{\text{HSDT},v}$ is the elementary stiffness matrix;
  - $\mathbf{R}_{(n\times 1)}^{\text{HSDT}}\left(t\right)$ is the global externally applied generalized forces vector;
  - $\bar{\mathbf{u}}_{(n\times 1)}^{\text{HSDT}}\left(t\right)$ is an internal variable vector arising from the viscoelastic behavior of selected layers of the structure;
- for the Layerwise-FSDT formulation:
  - $\mathbf{M}_{(n\times n)}^{\text{L-FSDT}}=\sum\left[\tilde{\mathbf{L}}_{(8(2\tilde{n}_l\times 3)\times n)}^{\text{L-FSDT}}\right]^{\text{T}}\tilde{\mathbf{M}}_{(8(2\tilde{n}_l\times 3)\times 8(2\tilde{n}_l\times 3))}^{\text{L-FSDT}}\tilde{\mathbf{L}}_{(8(2\tilde{n}_l\times 3)\times n)}^{\text{L-FSDT}}$ and $\tilde{\mathbf{M}}_{(8(2\tilde{n}_l\times 3)\times 8(2\tilde{n}_l\times 3))}^{\text{L-FSDT}}$ are the global and elementary mass matrices, respectively;
  - $\bar{\mathbf{K}}_{(n\times n)}^{\text{L-FSDT}}=\mathbf{K}_{(n\times n)}^{\text{L-FSDT},nv}+c_{\varepsilon}\mathbf{K}_{(n\times n)}^{\text{L-FSDT},v}$ is the modified global stiffness matrix;
  - $\mathbf{K}_{(n\times n)}^{\text{L-FSDT},nv}=\sum\left[\tilde{\mathbf{L}}_{(8(2\tilde{n}_l+3)\times n)}^{\text{L-FSDT}}\right]^{\text{T}}\tilde{\mathbf{K}}_{(8(2\tilde{n}_l+3)\times 8(2\tilde{n}_l+3))}^{\text{L-FSDT},nv}\tilde{\mathbf{L}}_{(8(2\tilde{n}_l+3)\times n)}^{\text{L-FSDT}}$ and $\tilde{\mathbf{K}}_{(8(2\tilde{n}_l+3)\times 8(2\tilde{n}_l+3))}^{\text{L-FSDT},nv}$ are the non-viscoelastic contribution to the global and elementary stiffness matrices, respectively;
  - $\mathbf{K}_{(n\times n)}^{\text{L-FSDT},v}=\sum\left[\tilde{\mathbf{L}}_{(8(2\tilde{n}_l+3)\times n)}^{\text{L-FSDT}}\right]^{\text{T}}\tilde{\mathbf{K}}_{(8(2\tilde{n}_l+3)\times 8(2\tilde{n}_l+3))}^{\text{L-FSDT},v}\tilde{\mathbf{L}}_{(8(2\tilde{n}_l+3)\times n)}^{\text{L-FSDT}}$ and $\tilde{\mathbf{K}}_{(8(2\tilde{n}_l+3)\times 8(2\tilde{n}_l+3))}^{\text{L-FSDT},v}$ are the viscoelastic contribution to the global and elementary stiffness matrices, respectively;
  - $\tilde{\mathbf{K}}_{(8(2\tilde{n}_l+3)\times 8(2\tilde{n}_l+3))}^{\text{L-FSDT}}=\tilde{\mathbf{K}}_{(8(2\tilde{n}_l+3)\times 8(2\tilde{n}_l+3))}^{\text{L-FSDT},nv}+\tilde{\mathbf{K}}_{(8(2\tilde{n}_l+3)\times 8(2\tilde{n}_l+3))}^{\text{L-FSDT},v}$ is the elementary stiffness matrix;
  - $\mathbf{R}_{(n\times 1)}^{\text{L-FSDT}}\left(t\right)$ is the global externally applied generalized forces vector;
  - $\bar{\mathbf{u}}_{(n\times 1)}^{\text{L-FSDT}}\left(t\right)$ is an internal variable vector arising from the viscoelastic behavior of selected layers of the structure.

In order to perform a transient dynamical analysis of the system, one must use a numerical integration scheme to solve either of Eqs. (29) – (31). Several of such numerical algorithms as applied to the solution of finite element models are provided by Bathe (1996). Galucio *et al.* (2004) use a Newmark explicit numerical integration procedure to solve equations in the form of Eqs. (10) and (29) – (31), which incorporate viscoelastic behavior by modifying the stiffness matrix of the model and by means of an internal variables vector. For further details about the solution of the dynamical differential equations associated with systems presenting viscoelasticity modeled by the fractional derivative constitutive equation herein considered, the reader should refer to Galucio *et al.* (2004), who firstly incorporated the procedure used herein.

4. NUMERICAL EXAMPLES

In this section, one presents a numerical example in order to validate the proposed procedures and to assess differences in the obtained time responses induced by the different formulations adopted.

As example, one considers a cantilever sandwich beam with length of 0.200 m and width of 0.010 m that presents three different layers and has been studied by Galucio *et al.* (2004). A schematic illustration of the addressed problem can be seen in Fig. 1. The bottom and upper layers have thickness of 0.001 m and their materials are an aluminum alloy, whose density, Poisson ratio and longitudinal Young modulus are $\rho_{\mathrm{Al}} = 2690\,\mathrm{kg/m^3}$, $\nu_{\mathrm{Al}} = 0.345$ and $E_{\mathrm{Al}} = 70.3\times 10^3$ MPa, respectively. As for the intermediate viscoelastic layer, its thickness is 0.0002 m and its material is 3M® ISD112™, for which, at 27 °C (≈ 300 K), $\rho_{\mathrm{ISD112}} = 1600\,\mathrm{kg/m^3}$, $\nu_{\mathrm{ISD112}} = 0.5$, $E_{0,\mathrm{ISD112}} = 1.5\,\mathrm{MPa}$, $E_{\infty,\mathrm{ISD112}} = 69.9495\,\mathrm{MPa}$, $\tau_{\mathrm{ISD112}} = 1.4052\times 10^{-5}$ s, and $\alpha_{\mathrm{ISD112}} = 0.7915$ are its density, Poisson ratio, static longitudinal Young modulus, high-frequency longitudinal Young modulus, relaxation time and fractional derivative order, respectively.

Spatial discretization was accomplished by means of a uniform mesh consisting of 20 elements along the length and one element along width of the beam.

In addition, one considers that a triangular impulsive load, given by:

$$f(t) = \begin{cases} 0\ \mathrm{N}, & \text{if } t \le 0 \text{ s or if } t > 2 \text{ ms}; \\ \dfrac{t}{0{,}002}\ \mathrm{N}, & \text{if } 0 < t \le 1 \text{ ms}; \\ 2 - \dfrac{t}{0{,}002}\ \mathrm{N}, & \text{if } 1 < t \le 2 \text{ ms}, \end{cases} \qquad (32)$$

for t expressed in seconds, is applied at the free-end of the beam as depicted in Fig.1.

The final observation time is chosen to be 250 ms. Time discretization was accomplished by dividing the time window [0–250] ms in 1000 equally spaced intervals, which implies in $\Delta t = 0.25$ ms. Numerical integration was performed by means of Newmark explicit integration algorithm, which can be found in (Bathe,1996) and in (Galucio *et al.*, 2004). Moreover, to describe the memory effect associated with the 3M® ISD112™ viscoelastic material one adopts $N_P = 50$, which is used in the computation of the weighted sums appearing in the right-hand side of Eqs. (29) – (31).

Time response at the investigated d.o.f., namely the one associated with transverse motion of the beam free-end, is presented in Fig. 2. Results are given for the three different formulations considered. Also the displacement history obtained by Galucio *et al.* (2004) is superimposed to the results.

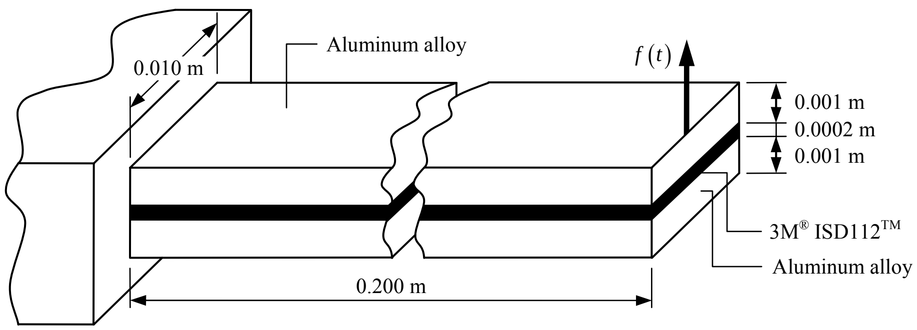

Figure 1. Schematic illustration of the sandwich beam considered (not in scale).

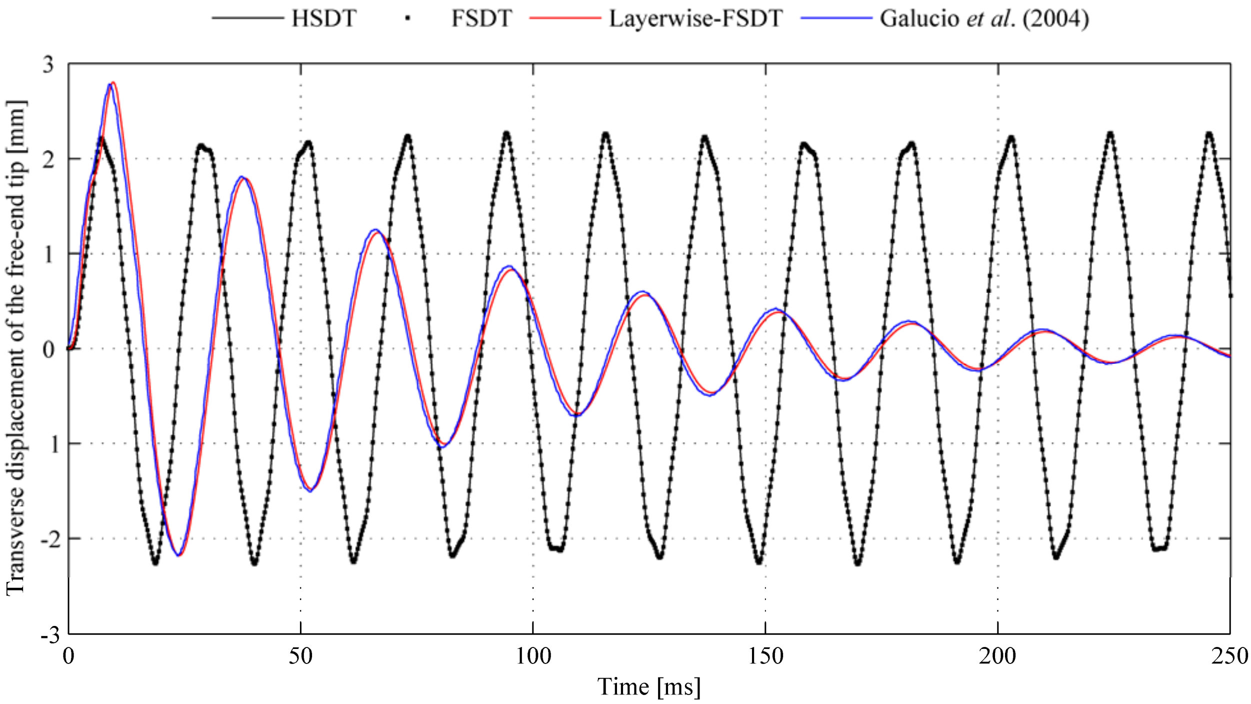

Figure 2. Time response of the d.o.f. associated with the transverse deflection of the free-end tip of the beam.

Responses shown in Fig. 2 allow the verification that the Layerwise-FSDT formulation lead to the same results included in the work of Galucio *et al*. (2004). Nevertheless, the FSDT and HSDT formulations cannot capture the damping effect provided by the viscoelastic layer. Such effect is observed inasmuch as two different materials compose the considered sandwich beam, namely an Aluminum alloy and the viscoelastic 3M® ISD112™ material. This leads to a heterogeneous configuration along the beam thickness, which cannot be properly modeled by the FSDT and HSDT formulations, as proven by the plots shown in Fig. 2. One can ascribe such modeling deficiency of the system dynamical behavior to the single displacement field admitted for all the layers of the sandwich structure. Therefore, although viscoelasticity is taken into account in the formulations, the strains related with the viscoelastic layer, in special shear strain, are not modeled correctly, leading to the inconsistencies shown in Fig. 2.

As to the results obtained by Galucio *et al.* (2004), it is important to mention that the authors use a three-layer sandwich beam theory as to assess the dynamic response included in Fig. 2. The formulation used by the previous authors in their work adopts Euler-Bernoulli assumptions for the external linear-elastic layers, as well as Timoshenko hypotheses for the viscoelastic core. In addition, plane stress state and perfect bonding between layers are assumed in their development. The response provided in Fig. 2 as being the one from Galucio *et al*. (2004) was achieved by digitizing the plot presented by the authors in their paper. Thus, minimal differences seen in Fig. 2 between such response and the one arising from the Layerwise-FSDT formulation are completely acceptable.

5. CONCLUSIONS

In this paper, general modeling of sandwich plate structures containing viscoelastic layers was carried out in the time domain by the use of the Finite Element Method and a fractional-derivative constitutive equation for the viscoelasticity phenomenon. Three different formulations of the problem were considered: the First-order Shear Deformation Theory (FSDT), the Higher-order Deformation Theory (HSDT), and a Layerwise-FSDT. The first two differ from each other in terms of interpolation order of the displacement fields, mainly. Layerwise Theories, on the other hand, account for different displacement fields for different layers, and the Layerwise-FSDT adopts a first order approximation, with respect to the transverse coordinate z , for such displacement fields.

Although the numerical example considered in this paper consists of a non-composite sandwich beam, the assessment of the correct modeling of plate structures containing viscoelastic damping treatments could be carried out, and indicated that FSDT and HSDT cannot correctly anticipate the dynamic response investigated. Still, it is important to stress that FSDT and HSDT may be useful for situations in which material homogeneity is encountered through thickness. Nevertheless, Layerwise-FSDT is better for situations alike the one analyzed in this text, including instances where composite materials are present.

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