# A DAMAGE MODEL FOR THE DUCTILE FAILURE ANALYSIS OF PLASTIC COMPONENTS

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***Abstract.*** *The objective of this work is to propose a mathematical model and a numerical scheme for the ductile failure analysis of polymeric materials. The proposed model is based in the works of Fremond and Lemaitre, makes use of the method of local state variables and is derived within the scope of the consistent thermodynamics of the continuum medium. The polymeric material is modeled as an elastoviscoplastic material coupled with a non local damage model. In order to model the cold drawing process that occurs during the ductile fracture process of polymers, one incorporates a damage locking condition. Since the problem is formulated within the scope of non smooth mechanics, a regularization process is also applied. In order to identify the material constants associated with the model one considers a uniaxial tensile test and other complementary tests, which were done at the Fiber and Polymer Technology Department of the Royal Institute of Technology (KTH), in Sweden. More general problems are then solved in order to attest the proposed model and to validate the employed numerical scheme.*

***Keywords****: viscoplasticity, non local damage, finite element.*

## 1. INTRODUCTION

Here, one proposes an elastic viscoplastic theory coupled with a non local damage model that may be applied in the analysis of the ductile failure analysis of plastic components. The proposed theory considers the problem to be subjected to small displacements and constrains and the deformation process to be isothermal. Due to the above simplifications, the proposed model is suitable for the local failure analysis in plastic components and takes into account the cold drawing phenomenon, which may occur at regions with high concentration of stresses. The theory considers some of the ideas presented in Lemaitre (1996) and Fremond and Nedjar (1996) and employs a gradient type of non local damage theory.

Let the scalar $\beta$ be defined as the cohesion variable with value 1 when the material is undamaged and value 0 when it is completely damaged. This variable is related with the links between material points and can be interpreted as a measure of the local cohesion state of the material. When $\beta = 1$, all the links are preserved. But, if $\beta = 0$, a local rupture is considered, since all the links between material points have been broken Fremond and Nedjar (1996).

## 2. THEORETICAL DEVELOPMENT

### 2.1. Principle of virtual power

Here, one consider the principle of virtual power to be given by

$$P_a(\vec{v},\gamma) = P_i(\vec{v},\gamma) + P_e(\vec{v},\gamma) \tag{1}$$

where the power of the internal, external and inertial forces are given by

$$P_i(\vec{v},\gamma) = -\int_\Omega \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}\; d\Omega - \int_\Omega \left(F\gamma + \vec{H}\cdot\vec{\nabla}\gamma\right) d\Omega \tag{2}$$

$$P_e(\vec{v},\gamma) = \int_\Omega \rho\vec{b}\cdot\vec{v}\; d\Omega + \int_{\partial d\Omega} \vec{t}\cdot\vec{v}\;\partial d\Omega + \int_\Omega A_v\;\gamma\; d\Omega + \int_{\partial d\Omega} A_s\;\gamma\;\partial d\Omega \tag{3}$$

and

$$P_a(\vec{v},\gamma) = \int_\Omega \rho\ddot{\vec{u}}\cdot\vec{v}\; d\Omega + \int_\Omega \rho\ddot{\beta}\gamma\; d\Omega\;. \tag{4}$$

Here, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $F$ is the internal work of damage and $\vec{H}$ is the flux vector of internal work of damage. Moreover, $\rho$ is the density, $\vec{b}$ represents the prescribed body force per unit mass, $\vec{t}$ is the prescribed external traction and $A_v$ and $A_s$ are, respectively, the volumetric and the surface external sources of damage work.
Now, assuming that $A_v = 0$ and $A_s = 0$, and considering the process to be quasi-static one derives differential equations:

$$div\left[\boldsymbol{\sigma}\left(\vec{x},t\right)\right] + \rho\vec{b}\left(\vec{x},t\right) = 0 \tag{5}$$

and

$$div(\vec{H}) - F = 0 \tag{6}$$

Subjected to the following boundary conditions

$$\begin{cases} \boldsymbol{\sigma}\left(\vec{x},t\right)\vec{n} = \vec{t}\left(\vec{x},t\right) & \text{on} \quad \left(\vec{x},t\right)\in\Gamma_t \\ \vec{u}\left(\vec{x},t\right) = \bar{u} & \text{on} \quad \left(\vec{x},t\right)\in\Gamma_u \end{cases} \tag{7}$$

and

$$\vec{\nabla}\beta\cdot\vec{n} = 0 \quad \text{on } \partial\Omega \tag{8}$$

### 2.2. Definition of the free energy potential $\Psi^{vp}$

Here, one assumes the free energy potential, $\Psi^{vp}(\boldsymbol{\varepsilon}^{e}, r, \beta, \vec{\nabla}\beta)$, to be given by

$$\rho\Psi^{vp}(\boldsymbol{\varepsilon}^{e}, r, \beta, \vec{\nabla}\beta) = \beta\frac{1}{2}\mathbb{D}\boldsymbol{\varepsilon}^{e}\cdot\boldsymbol{\varepsilon}^{e} + \frac{k}{2}\vec{\nabla}\beta\cdot\vec{\nabla}\beta + \int_0^r h(r)dr \tag{9}$$

where $\boldsymbol{\varepsilon}^{e}$ is the elastic strain tensor, $h(r)$ represents a function that describes the isotropic hardening curve of the material, $k$ is a material parameter and $\mathbb{D}$ is the fourth order elasticity tensor, given by $\mathbb{D} = 2\bar{\mu}\mathbb{I} + \bar{\lambda}\left(\boldsymbol{I}\otimes\boldsymbol{I}\right)$, in which, $\bar{\mu}$ and $\bar{\lambda}$ are the Lamé constants, given respectively by $\bar{\mu} = \dfrac{E}{2\left(1+\nu\right)}$, $\bar{\lambda} = \dfrac{\nu E}{\left(1+\nu\right)\left(1-2\nu\right)}$.

The local state equations, defining the associated dual variables, are given by

$$\boldsymbol{\sigma} = \rho\frac{\partial\Psi^{vp}}{\partial\boldsymbol{\varepsilon}^{e}}, \qquad R = \rho\frac{\partial\Psi^{vp}}{\partial r}, \qquad F^{r} = \rho\frac{\partial\Psi^{vp}}{\partial\beta} \qquad \text{and} \qquad \vec{H} = \rho\frac{\partial\Psi^{vp}}{\partial\vec{\nabla}\beta} \tag{10}$$

where one considers

$$F = F^{r} + F^{reac} + F^{i} \tag{11}$$

in which $F^{reac}\in\rho\partial^{loc}I_{\mathbf{K}}\left(\beta\right)$, denoting the local sub differential of $I_K\left(\beta\right)$, $I_K\left(\beta\right) = \begin{cases} 0, & \text{if} \quad \beta\in\mathrm{K} \\ +\infty, & \text{if} \quad \beta\notin\mathrm{K} \end{cases}$, with $\mathrm{K} = \{\beta \mid 0\le\beta\le 1\}$.

### 2.3. Definition of the yield function

Different yield criteria have been proposed in the literature. Among them, one may highlight Quinson *et al.*(1997), Goldberg *et al.* (2003), Rottler and Robbins (2001) and Riande *et al.* (2000) propositions. Here, one has defined the following yield function:

$$f\left(\sigma, R; \circ\right) = \left(q + \mu\ p\right) - \left(1 + \frac{\mu}{3}\right)\left(\sigma_{yo} + R\right) \tag{12}$$

in which, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^D + \sigma_H \boldsymbol{I}$, $p = \tilde{\sigma}_H$, $q = \left( \frac{3}{2} \tilde{\boldsymbol{\sigma}}^D \cdot \tilde{\boldsymbol{\sigma}}^D \right)^{\frac{1}{2}}$, $\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{\beta}$, $\mu$ is the variable that incorporates the effect of the hydrostatic stress, $\sigma_{yo}$ is the initial yield stress and $R$ is the isotropic strain hardening variable.

### 2.4. Plastic flow rule and hardening law

In order to describe a dissipative process, one needs to introduce complementary kinetic laws. Therefore, to completely characterize the proposed viscoplastic with damage model, by defining the evolution laws for the internal variables, one assumes the existence of a pseudo-potential of dissipation, $\varphi^*(\sigma, R, F^i; \circ)$, that is a scalar continuous function, convex with respect to the dual/associate variables $(\sigma, R, F^i)$. One assume $\varphi^*(\sigma, R, F^i; \circ)$ to be decomposed as $\varphi^*(\circ) = \varphi^*_{vp}(\boldsymbol{\sigma}, R; \circ) + \varphi^*_D(F^i; \circ)$. By applying the normal dissipation criterion, yields.

$$\dot{\boldsymbol{\varepsilon}}^{vp} = \dot{\lambda} \frac{\partial \varphi^*_{vp}}{\partial \boldsymbol{\sigma}}, \; \dot{r} = -\dot{\lambda} \frac{\partial \varphi^*_{vp}}{\partial R}. \text{ and } \dot{\beta} \in \dot{\lambda} \partial \varphi^*_D(F^i; \circ) \tag{13}$$

in which, $\dot{\lambda}$, is given by

$$\dot{\lambda} = \beta \ln \left[ \left( 1 - \frac{f(\boldsymbol{\sigma}, R, \beta; \circ)}{K_\infty} \right)^{-M} \right] \tag{14}$$

where $M$ and $K_\infty$ are material parameters. Moreover, one assumes that

$$\varphi^*_{vp}(\boldsymbol{\sigma}, R, \beta; \circ) = f(\boldsymbol{\sigma}, R, \beta; \circ) = (q + \mu\ p) - \left(1 + \frac{\mu}{3}\right)(\sigma_{yo} + R) \tag{15}$$

in which $\mu$ is a material parameter. Considering $\sigma_{ef} = q + \mu\ p$ to denote the effective stress, one derives $\dot{e}^{vp}_{ef} = \frac{\dot{\lambda}}{\beta}$, defining the effective viscoplastic strain measure.

### 2.5. Definition of the damage potential

Here one considers that

$$\varphi^*_D\left(F^i; \circ\right) = \begin{cases} \frac{1}{2} \beta S \left( \frac{F^i}{\beta S} \right)^2, & \text{if} \quad F^i \le 0 \\ 0, & \text{if} \quad F^i > 0 \end{cases} \tag{16}$$

which leads to

$$\dot{\beta} = \begin{cases} \dot{\lambda} \left( \frac{F^i}{\beta S} \right) \bar{H}\left(e^{vp}_{ef} - e^{vp}_{ef_{th}}\right), & \text{if} \quad F^i \le 0 \\ 0 & , \text{if} \quad F^i > 0 \end{cases} \tag{17}$$

where $\bar{H}\left(e^{vp}_{ef} - e^{vp}_{ef_{th}}\right) = \begin{cases} 1, & \text{if} \quad e^{vp}_{ef} \ge e^{vp}_{ef_{th}} \\ 0, & \text{if} \quad e^{vp}_{ef} < e^{vp}_{ef_{th}} \end{cases}$.

### 2.6. Regularization of the Free Energy Potential

Here, one considers a regularized free energy potential, given by

$$\Psi(\boldsymbol{\varepsilon}^e, r, \beta, \vec{\nabla}\beta) = \Psi^{vp}(\boldsymbol{\varepsilon}^e, r, \beta, \vec{\nabla}\beta) + \frac{\overline{\eta}_a}{\beta} + \frac{1}{2\overline{\eta}_b}\left(\langle \beta - 1 \rangle^+\right)^2 \tag{18}$$

where $\overline{\eta}_a$ and $\overline{\eta}_b$ are penalty parameters and $\langle \bullet \rangle^+$ is denotes the Macauley bracket.

Now, following the recommendation of Minak *et al.* (2007), it is more appropriate to use $D$ instead of $\beta$ as damage variable, since the definition of $D$ is closer to the one usually adopted in the traditional works of continuum damage mechanics. Thus, replacing $\beta = (1-D)$ in the above relations one derives the damage evolution equation, given by:

$$\dot{D} = \begin{cases} \dot{\lambda}\left[\dfrac{F_D^i}{(1-D)S}\right]\overline{H}\left(e_{ef}^{vp} - e_{ef_{th}}^{vp}\right)H\left(F_D^i\right), & \text{if } F_D^i \le 0 \\ 0 & \text{, if } F_D^i > 0 \end{cases} \tag{19}$$

in which

$$F_D^i = k\, div(\vec{\nabla}\beta) - (F_D^r + F_D^{reac}) \tag{20}$$

$$F_D^r = \frac{(\sigma_{eq}^{vm})^2 R_v}{2E(1-D)^2} \tag{21}$$

$$F_D^{reac} = -\frac{\overline{\eta}_a}{(1-D)^2} + \frac{1}{\overline{\eta}_b}\langle(-D)\rangle^+ \tag{22}$$

where $\langle(-D)\rangle^+ = \begin{cases} -D, & \text{if } D \le 0 \\ 0, & \text{if } D > 0 \end{cases}$, $\sigma_{eq}^{vm} = \left(\frac{3}{2}\boldsymbol{\sigma}^D \cdot \boldsymbol{\sigma}^D\right)^{\frac{1}{2}}$ and $R_v = \frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}^{vm}}\right)^2$.

**2.6. Analysis of the Cold Drawing Process**

When a ductile plastic material, like polypropylene or polyethylene, is submitted to a tensile load, its stress-strain diagram looks similar to that illustrated in Figure 1. The objective of this section is to improve the viscoplastic model in order to be capable of modeling the cold drawing process.

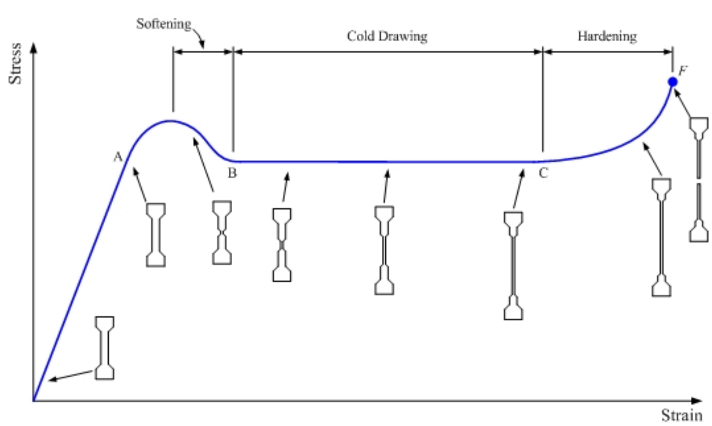


Fig. 1 – Description of the cold drawing phenomenon

In order to account for the cold drawing process one assume that

$$\dot{D} = \begin{cases} \dot{\lambda}\left[\dfrac{F_D^i}{(1-D)S_{\bar{\eta}_s}(e_{ef}^{vp})}\right]\bar{H}\left(e_{ef}^{vp} - e_{ef_{th}}^{vp}\right)H\left(F_D^i\right), \text{ if } F_D^i \le 0 \\ 0 \qquad , \text{ if } F_D^i > 0 \end{cases} \tag{23}$$

with

$$S_{\bar{\eta}_s}(e_{ef}^{vp}) = S_o + \frac{1}{2\bar{\eta}_s}\left[\left\langle\left(e_{ef}^{vp} - e_{ef_B}^{vp}\right)\right\rangle^+\right]^2. \tag{24}$$

By summarizing the above equations, one may state the strong formulation of the problem, given by: Find $(\boldsymbol{u}, D)$ that solve:

**(i) Linear momentum equation**

$$div\left[\boldsymbol{\sigma}\left(\vec{x},t\right)\right] + \rho\vec{b}\left(\vec{x},t\right) = 0 \tag{25}$$

where

$$\boldsymbol{\sigma} = (1-D)\mathbb{D}\boldsymbol{\varepsilon}^e, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}, \quad \dot{\boldsymbol{\varepsilon}}^{vp} = \frac{\dot{\lambda}}{(1-D)}\left(\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}^D}{\sigma_{eq}^{vm}} + \frac{\mu}{3}\boldsymbol{I}\right), \quad R = h(r) \tag{26}$$

$$\dot{e}_{ef}^{vp} = -M\ln\left(1 - \frac{f(\boldsymbol{\sigma}, R, D;\circ)}{K_\infty}\right), \qquad \dot{r} = \left(1 + \frac{\mu}{3}\right)\dot{\lambda} \qquad \text{and} \qquad \dot{\lambda} = (1-D)\dot{e}_{ef}^{vp} \tag{27}$$

in which

$$f\left(\sigma, R, D;\circ\right) = \left(q + \mu\ p\right) - \left(1 + \frac{\mu}{3}\right)\left(\sigma_{yo} + R\right) \tag{28}$$

with $q = \left(\frac{3}{2}\tilde{\boldsymbol{\sigma}}^D \cdot \tilde{\boldsymbol{\sigma}}^D\right)^{\frac{1}{2}}$, $p = \tilde{\sigma}_H$ and $\tilde{\boldsymbol{\sigma}} = \dfrac{\boldsymbol{\sigma}}{(1-D)}$.

**(ii) Damage Evolution Law**

$$\dot{D} = \begin{cases} \dot{\lambda}\left[\dfrac{F_D^i}{(1-D)S_{\bar{\eta}_s}}\right]\bar{H}\left(e_{ef}^{vp} - e_{ef_D}^{vp}\right)H\left(F_D^i\right), \text{ if } F_D^i \le 0 \\ 0 \qquad , \text{ if } F_D^i > 0 \end{cases} \tag{29}$$

in which,

$$F_D^i = kdiv(\vec{\nabla}D) + (F_D^r + F_D^{reac}), \ F_D^r = \frac{(\sigma_{eq}^{vm})^2 R_v}{2E(1-D)^2} \quad F_D^{reac} = -\frac{\bar{\eta}_a}{(1-D)^2} + \frac{1}{\bar{\eta}_b}\left\langle(-D)\right\rangle^+, \tag{30}$$

and

$$S_{\bar{\eta}_s}(e_{ef}^{vp}) = S_o + \frac{1}{2\bar{\eta}_s}\left[\left\langle\left(e_{ef}^{vp} - e_{ef_B}^{vp}\right)\right\rangle^+\right]^2 \tag{31}$$

## 3. DISCRETIZATION OF THE PROBLEM

### 3.1. Operator Split Algorithm

Here, in order to solve the local viscoplastic with damage constitutive equations one applies the operator split strategy, described by:

(i) The trial elastic step problem, formulated as: Given the strain and damage histories, $\{\boldsymbol{\varepsilon}(t), D(t)\} \in [t_n, t_{n+1}]$, find $\boldsymbol{\varepsilon}_{n+1}^{e\ trial}$ and $\boldsymbol{\omega}_{n+1}^{trial} = (\boldsymbol{\varepsilon}_{n+1}^{vp\ trial}, r_{n+1}^{trial}, e_{ef_{n+1}}^{vp\ trial})$, so that $\dot{\varepsilon}^{e\ trial} = \dot{\varepsilon}$ and $\dot{\omega}^{trial} = 0$. As a result, one derives

$$\boldsymbol{\varepsilon}_{n+1}^{e^{trial}} = \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^c \quad \text{and} \quad \boldsymbol{\omega}_{n+1}^{trial} = \boldsymbol{\omega}_n \, . \tag{32}$$

Once $\boldsymbol{\varepsilon}_{n+1}^{e^{trial}}$ is determined, one computes $e_{H_{n+1}}^{e\ trial} = \frac{1}{3} tr\left[\boldsymbol{\varepsilon}_{n+1}^{e^{trial}}\right]$, $\boldsymbol{\varepsilon}_{n+1}^{e\ D\ trial} = \boldsymbol{\varepsilon}_{n+1}^{e\ trial} - e_{H_{n+1}}^{e\ trial}\mathbf{I}$, $p_{n+1}^{trial} = \frac{E}{(1-2\nu)} e_{H_{n+1}}^{e\ trial}$ and the trial elastic stresses $\sigma_{H_{n+1}}^{trial} = (1 - D_{n+1})\, p_{n+1}^{trial}$ and $\boldsymbol{\sigma}_{n+1}^{D\ trial} = 2G(1 - D_{n+1})\boldsymbol{\varepsilon}_{n+1}^{e\ D\ trial}$, where $\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^D + \sigma_{H_{n+1}}\boldsymbol{I}$.

(ii) The viscoplastic return mapping step problem
Here, by applying the fully implicit Euler method and performing some additional algebra, one derives the following coupled nonlinear problem, stated as: Given $(\varepsilon_{n+1}, D_{n+1})$, determine $(\Delta\lambda, p_{n+1}, q_{n+1})$, that solves

$$\begin{cases} R_1(\Delta\lambda, p_{n+1}, q_{n+1}) = \dfrac{\Delta\lambda\sigma_{yo}}{(1 - D_{n+1})} + \sigma_{yo} M \ln\left[1 - \dfrac{f(p_{n+1}, q_{n+1}, R_{n+1}, D_{n+1})}{K_\infty}\right]\Delta t = 0 \\ R_2(\Delta\lambda, p_{n+1}, q_{n+1}) = p_{n+1} + \dfrac{E}{3(1-2\nu)}\dfrac{\mu\Delta\lambda}{(1 - D_{n+1})} - p_{n+1}^{trial} = 0 \\ R_3(\Delta\lambda, p_{n+1}, q_{n+1}) = q_{n+1} + \dfrac{3G\Delta\lambda}{(1 - D_{n+1})} - q_{n+1}^{trial} = 0. \end{cases} \tag{33}$$

Once the set $(\Delta\lambda, p_{n+1}, q_{n+1})$ is computed, the Cauchy stress $\sigma$ may also de determined from

$$\boldsymbol{\sigma}_{n+1}^D = \boldsymbol{\sigma}_{n+1}^{D\ trial}\left[1 + \frac{3G\Delta\lambda}{(1 - D_{n+1})}\right]^{-1} \quad \text{and} \quad \sigma_{H_{n+1}} = (1 - D_{n+1})\, p_{n+1} \tag{34}$$

In addition, one can also determine

$$\boldsymbol{\varepsilon}_{n+1}^{vp} = \boldsymbol{\varepsilon}_n^{vp} + \frac{\Delta\lambda}{(1 - D_{n+1})}\left[\frac{3}{2}\frac{\boldsymbol{\sigma}_{n+1}^D}{(1 - D_{n+1})q_{n+1}} + \frac{\mu}{3}\boldsymbol{I}\right] \tag{35}$$

$$r_{n+1} = r_n + \left(1 + \frac{\mu}{3}\right)\Delta\lambda \quad \text{and} \quad e_{ef_{n+1}}^{vp} = e_{ef_n}^{vp} + \frac{\Delta\lambda}{(1 - D_{n+1})}. \tag{36}$$

in which, $r$ is the the accumulated viscoplastic strain and $e_{ef}^{vp}$ is the effective viscoplastic strain.

### 3.2. Incremental weak formulation of the problem

Once defined the constitutive model, one may solve the global equilibrium problem by employing an incremental procedure. The incremental formulation between $t_n$ and $t_{n+1}$ considers that

$$\vec{u}_{n+1} = \vec{u}_n + \Delta\vec{u}_n \quad \text{and} \quad D_{n+1} = D_n + \Delta D_n \tag{37}$$

so that, at time $t_{n+1}$, the weak formulation of the problem may be stated as: Determine $(\vec{u}_{n+1}, D_{n+1}) \in \overline{\mathrm{K}}$, which solves

$$F_1(\vec{u}_{n+1}, D_{n+1}; \vec{w}) = \int_\Omega \boldsymbol{\sigma}_{n+1} \cdot \varepsilon(\vec{w}) d\Omega - \int_\Omega \rho\vec{b}_{n+1} \cdot \vec{w} d\Omega - \int_{\Gamma_t} \vec{t}_{n+1} \cdot \vec{w} dA = 0, \quad \forall \vec{w} \in \mathrm{V}_u \tag{38}$$

and

$$F_2(\vec{u}_{n+1}, D_{n+1}; \gamma) = \int_\Omega S_{\overline{\eta}_{s_{n+1}}}(D_{n+1} - D_n)\gamma d\Omega + \int_\Omega k\, \phi_{n+1} \vec{\nabla} D_{n+1} \cdot \vec{\nabla}\gamma d\Omega - \int_\Omega (F^r_{D_{n+1}} + F^{reac}_{D_{n+1}})\phi_{n+1}\gamma d\Omega = 0, \quad \forall \gamma \in \mathrm{V}_D \tag{39}$$

in which

$$\phi_{n+1} = \frac{\Delta\lambda}{(1-D_{n+1})}\overline{H}(e^{vp}_{ef_{n+1}} - e^{vp}_{ef_{th}})H(F^i_{D_{n+1}}) \tag{40}$$

Since the above problem is non-linear, one applies Newton's method leading to the solution of a sequence of linearized problems.

### 3.3. Linearization procedure

Let $\vec{u}^k_{n+1}$ and $D_{n+1}$ be the estimate solution of (38-9) at the *k-th* iteration and consider that

$$\vec{u}^k_{n+1} = \vec{u}_n \text{ at } k=0 \text{ and } D^k_{n+1} = D_n \text{ at } k=0. \tag{41}$$

For the *k-th* iteration of the solution procedure, one has

$$\vec{u}^{k+1}_{n+1} = \vec{u}^k_{n+1} + \Delta\vec{u}^k_{n+1} \text{ and } D^{k+1}_{n+1} = D^k_{n+1} + \Delta D^k_{n+1}. \tag{42}$$

The determination of the increments $\left(\Delta\vec{u}^k_{n+1}, \Delta D^k_{n+1}\right)$ are obtained by imposing that

$$\begin{cases} F_1(\vec{u}^k_{n+1} + \Delta\vec{u}^k_{n+1}, D^k_{n+1} + \Delta D^k_{n+1}; \vec{w}) = 0, \quad \forall \vec{w} \in \mathrm{V}_u \\ F_2(\vec{u}^k_{n+1} + \Delta\vec{u}^k_{n+1}, D^k_{n+1} + \Delta D^k_{n+1}; \gamma) = 0, \quad \forall \gamma \in \mathrm{V}_D. \end{cases} \tag{43}$$

Considering $F_1(\circ)$ and $F_2(\circ)$ as being smooth and expanding them in a Taylor series, one derives, for a first order approximation,

$$\begin{bmatrix} \partial_{uu}F_1(\vec{u}^k_{n+1}, D^k_{n+1}; \vec{w}) & \partial_{ud}F_1(\vec{u}^k_{n+1}, D^k_{n+1}; \vec{w}) \\ \partial_{du}F_2(\vec{u}^k_{n+1}, D^k_{n+1}; \gamma) & \partial_{dd}F_2(\vec{u}^k_{n+1}, D^k_{n+1}; \gamma) \end{bmatrix} \begin{Bmatrix} \Delta\vec{u}^k_{n+1} \\ \Delta D^k_{n+1} \end{Bmatrix} = -\begin{Bmatrix} F_1(\vec{u}^k_{n+1}, D^k_{n+1}; \vec{w}) \\ F_2(\vec{u}^k_{n+1}, D^k_{n+1}; \gamma) \end{Bmatrix}. \tag{44}$$

where

$$\begin{aligned} \partial_{uu}F_1\left(\vec{u}^k_{n+1}, D^k_{n+1}; \vec{w}\right)\left[\Delta\vec{u}^k_{n+1}\right] &= \frac{d}{d\varepsilon}\left[F_1\left(\vec{u}^k_{n+1} + \varepsilon\Delta\vec{u}^k_{n+1}, D^k_{n+1}; \vec{w}\right)\right]_{\varepsilon=0} \\ \partial_{ud}F_1\left(\vec{u}^k_{n+1}, D^k_{n+1}; \vec{w}\right)\left[\Delta D^k_{n+1}\right] &= \frac{d}{d\varepsilon}\left[F_1\left(\vec{u}^k_{n+1}, D^k_{n+1} + \varepsilon\Delta D^k_{n+1}; \vec{w}\right)\right]_{\varepsilon=0} \\ \partial_{du}F_2\left(\vec{u}^k_{n+1}, D^k_{n+1}; \gamma\right)\left[\Delta\vec{u}^k_{n+1}\right] &= \frac{d}{d\varepsilon}\left[F_2\left(\vec{u}^k_{n+1} + \varepsilon\Delta\vec{u}^k_{n+1}, D^k_{n+1}; \gamma\right)\right]_{\varepsilon=0} \\ \partial_{dd}F_2\left(\vec{u}^k_{n+1}, D^k_{n+1}; \gamma\right)\left[\Delta D^k_{n+1}\right] &= \frac{d}{d\varepsilon}\left[F_2\left(\vec{u}^k_{n+1}, D^k_{n+1} + \varepsilon\Delta D^k_{n+1}; \gamma\right)\right]_{\varepsilon=0} \end{aligned} \tag{45}$$

## 4. NUMERICAL EXAMPLES

The discretization of the problem is obtained by the application of the Galerkin Finite Element method using a Tri 6 element. In order to verify the adequacy of the model and the proposed implicit algorithm, one solves a set of simple problems. The parameters used in these examples are given by: $M$= 2.0 s$^{-1}$, $K_\infty$=31.6 MPa, $\overline{\eta}_a = 1.0\times10^{-5}$,

$\overline{\eta}_b$ =$1.0\times10^{-9}$ , $S_o$ =0.15 MPa, $\overline{\eta}_s$ =$1.0\times10^{-9}$ , $\mathsf{k}$ =$2.0$ MPa $\cdot$ mm$^2$ , $e^{vp}_{ef_{th}}$ =0.75, E=1.54 GPa, $\sigma_{yo}$ =31.6MPa, $\nu$ =0.36 and $\rho=9.02\times10^2$ kg/m$^3$. The hardening parameters are given in "Table 1".

Table 1. Hardening curve data points.

| | | | | |
|---|---|---|---|---|
| $h(r)$ (MPa) | 0.0 | 1.0 | 2.0 | 2.5 |
| $r$ (mm/mm) | 0.0 | 0.2 | 0.4 | 2.0 |

**4.1. Uniaxial Tensile Test Simulation**

Here, one considers an uniaxial tension test, where the specimen is subjected to a prescribed displacement history, applied by a linear ramp function, ranging from zero to the maximum value of $\overline{u}_z = 60$ mm. The problem is considered to be axisymmetric and the specimen has a radius of 30 mm and a height of 100 mm, as depicted in Figure 2.

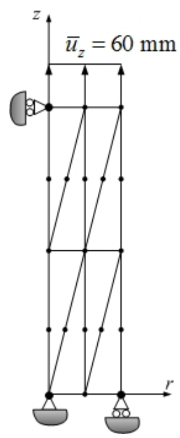


Fig. 2 – Definition of the problem

Figure 3 illustrates the comparison between both the experimental and numerical stress-strain diagrams for the tensile test. In order to reduce the processing time, one has considered the total time of the analysis equal to 60 seconds, which has been a sufficient time to reach the cold drawing region. In addition, one has also considered 1000 load steps and a global convergence tolerance of $10^{-5}$. The hardening curve, $h(r)$, has been obtained by interpolation, using spline functions and the given data in **Erro! Fonte de referência não encontrada.**.

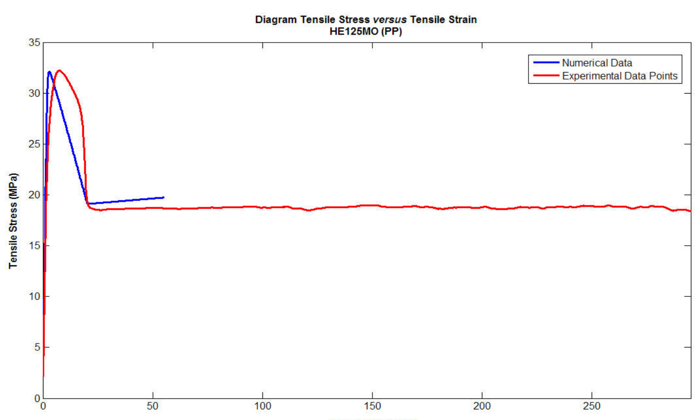


Fig. 3. - Stress-strain diagram for the tensile simulation.

Providing that several material parameters have been taken from literature, one considers that the error between experimental data and numerical prediction, observed in the softening region of Figure 3 is encouraging.

**4.2. Plastic pulley component**

Here, one considers a plastic pulley component, as depicted in Figure 4. The problem is considered to be subjected to aaxisymmetric condition. The component is subjected to a prescribed displacement applied by a linear ramp with a maximum displacement of $\bar{u} = -3.5mm$.

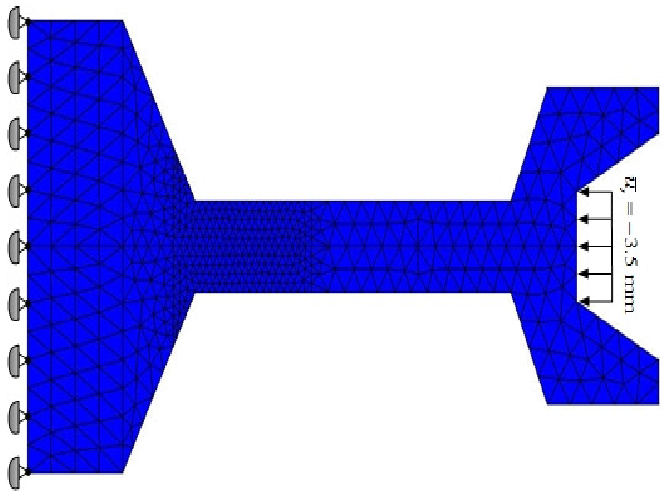


Fig. 4 – Plastic pulley mesh and boundary conditions

This analysis considered 10000 incremental displacement steps and global tolerance for convergenc of $10^{-5}$. Figure 5**Erro! Fonte de referência não encontrada.** illustrates the distribution of the norm of the displacement at the end of the analysis, in millimeters

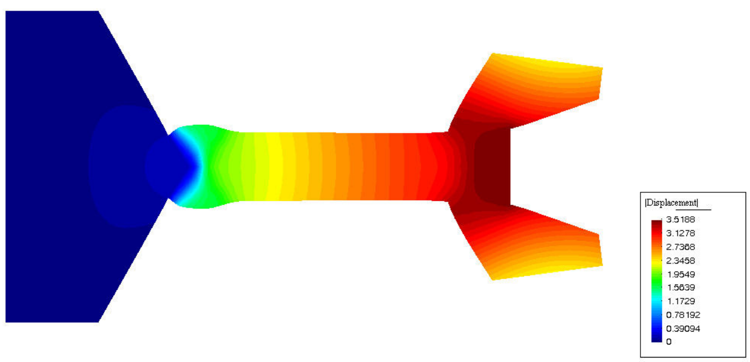


Fig. 5 - Displacement field at the end of the analysis, in millimiters

Figure 6 illustrates the distribution of equivalent visco plastic strain at the end of the analysis.

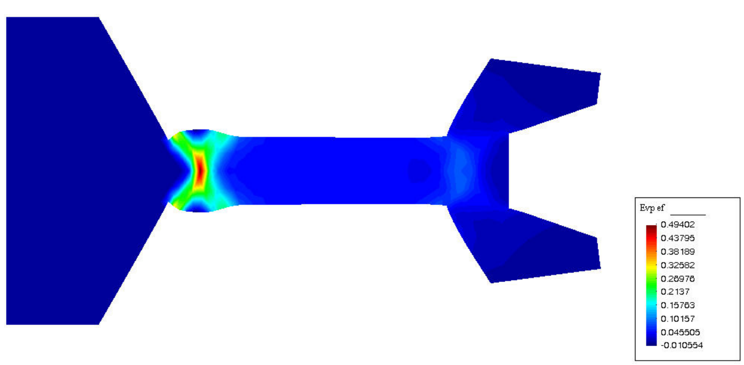


Fig. 6 – Distribution of equivalent viscoplastic strain, in mm/mm

The distribution of the damage variable at the end of the analysis, i.e., for a prescribed displacement of $\bar{u} = -3.5$ mm, is depicted in Figure 7.

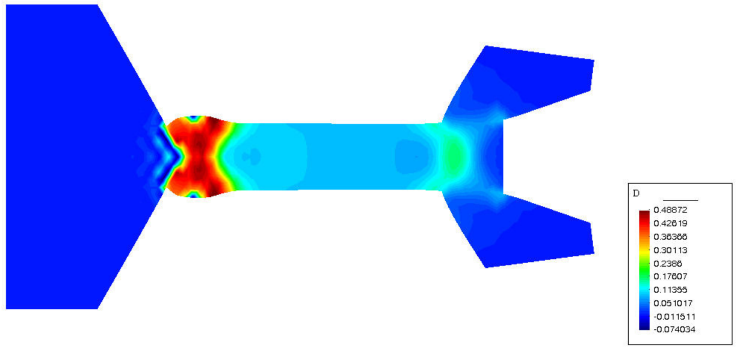


Fig. 7 - Distribution of damage

One can observe that the damaged region looks much like a "X" letter, in which each of its leg is inclined about 45° with respect to the radial direction. This peculiarity strongly suggests the formation of shear bands, which is a characteristic failure mode of plastic components under compressive loads. Additionally, one also observes that the maximum value for the effective viscoplastic strain occurs at the intersection of the shear bands.

## 5. ACKNOWLEDGEMENTS

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## 6. REFERENCES

Lemaitre, J., 1996. "A Course on Damage Mechanics", Springer, Berlin, Germany
Fremond M.; Nedjar B., 1996. "Damage, Gradient of Damage and Principle of Virtual Power". International Journal of Solids Structures, Vol. 33, n°. 8, pp. 1083 – 1103.
Quinson, R.; Perez, J.; Rink, M.; Pavan A., 1997. "Yield Criteria for Amorphous Glassy Polymers", Journal of Materials Science, Vol. 32, n°. 5, pp. 1371 – 1379.
Goldberg, R. K.; Roberts, G. D.; Gilat, A., 2003. "Implementation of an Associative Flow Rule Including Hydrostatic Stress Effect Into the High Strain Rate Deformation Analysis of Polymer Matrix Composites", NASA Center for Aerospace Information. Report Number 2003-212382.
Rottler, J.; Robbins, M. O., 2001. "Yield Conditions for Deformation of Amorphous Polymer Glasses", Physical Review E, Vol. 64, n°. 5, pp. 1 – 9.
Riande, E.; Diaz-Calleja, R.; Prolongo, M. G.; Masegosa, R. M.; Salom, C., 2000, "Polymer Viscoelasticity: Stress and Strain in Practice", Marcel Dekker.
Minak, G.; Chimisso, F. E. G.; Costa Mattos, H. S. da., 2007. "Cyclic Plasticity and Damage of Metals by a Gradient-Enhanced CDM Model", 1st International Symposium on Solid Mechanics, USP, São Paulo - Brazil.

## 7. RESPONSIBILITY NOTICE

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