SOURCE RECONSTRUCTION FROM BOUNDAY DATA

Nilson Costa Roberty, nilson@con.ufrj.br
Nuclear Engineering Program, Coppe-Federal University of Rio de Janeiro

Antonio P. Carvalho, antoniorhoads@hotmail.com
Eletrical Engineering Department, EP-Federal University of Rio de Janeiro

Denis M. de Sousa, denis@ufrj.br
Mathematics Department - ICEx - PUVR, Fluminense Federal University, RJ, Brazil

Marcelo L.S. Rainha, marcelo.rainha@uniriotec.br
Mathematics Department, Unirio-Rio de Janeiro Federal University

Abstract. *We consider phenomenological, mathematical and computational aspects related with the problem of reconstruction of an unknown characteristic star shape thermal/acoustic source moving inside a domain. By introducing the definition of an Extended Dirichlet to Neumann map in the time space cylinder and the adoption of the anisotropic Sobolev-Hilbert spaces, we can treat the problem with methods similar to that used in the analysis of the stationary source reconstruction problem. Using Green formula we establish a reciprocity gap functional relating directly the boundary integral of Cauchy data with the domain integral of test functions in the unknown characteristic source support. An adjoint space of test functions is introduced in order to formulates a system of first kind Fredholm equations. Explicits Fourier series based solutions for the equations involved in the model are presented. Numerical experiment related with captured of a source inside a domain from boundary data are presented in one, two and three dimensional implementations. The problem of source centroid and shape determination also is addressed.*

source reconstruction, transient second order equation, Helmholtz equation, finite differences-scheme, shape capture.

1. INTRODUCTION

Inverse source transient heat problem has been studied by a huge number of authors. In relation to books with specific chapters in the subject, we can give special attention to Anger (1990) and Isakov (1990). This last gives specific results for the problem of source reconstruction in models with different operators and over specification of boundary conditions, and specifically demonstrates an uniqueness theorem related with the moving characteristic source studied in this work. Early works by Cannon and Perez-Estevas (1986) studied stationary support reconstruction under hypotheses of a known intensity function, $f(t,x) = f(t)\chi_\omega(x)$. Some years later they studied the same problem in a three-dimensional case, Cannon and Perez-Estevas (1991). More recently, Lefevre and Nilliot (2002) used the Boundary Elements Method for identification of static and moving point sources. Also, it is important to mention the authors Hettlich and Rundell (2001) who model the unmoving characteristic domain and Badia and Duong (1998), whose stationary source reconstruction and the transient point sources reconstructions have a fundamental influence in the present work. The present work has come from the investigation of the stationary source reconstruction by the fundamental solution method in Alves and all. (2008). The adoption of the reciprocity gap functional method to solve the stationary source in the Laplace Poisson equation, Roberty and Alves (2009), and the solution of the full identification of sources with the Helmholtz Poisson model, Alves *et al.* (2009), have developed to the modeling adopted to the transient heat transient characteristic source reconstruction in this work. The model is based on the modified Helmholtz Poisson equation that is obtained from the transient equation through the θ-scheme related to time finite differences discretization. Analysis of the related mathematical and computational work involved has been presented by the author in national conferences, Roberty and Alves (2007a), Roberty and Alves (2007b), Roberty and Sousa (2008).

2. DIRECT TRANSIENT HEAT WAVE EQUATION PROBLEM

By $\Omega \subset \Re^d$, $d = 1, 2, 3$ we denote a bounded space domain with smooth boundary $\Gamma = \partial\Omega$, which means that it will be locally parametrized with C^∞ functions and that Ω is locally on one side of its connected boundary. In the spatial surface Γ the normal ν is defined almost everywhere and the induced measure on the surface is denoted by $d\sigma$. In the time-space $\Re^{d+1}$, we consider the time interval $I := (0,T)$, $T > 0$ to form the bounded cylinder $Q := I \times \Omega$, whose lateral time-space surface is $\Sigma := I \times \Gamma$. A section in this cylinder is $\Omega_t := \{t\} \times \Omega$ and the complete cylinder boundary is

$$\partial Q = \overline{\Sigma} \cup \Omega_0 \cup \Omega_T,$$

where Ω_0 and Ω_T are, respectively, the cylinders' bottom and top sections. At cylinder top and bottom there exist the corners $\Gamma_0 = \overline{\Omega_0} \cap \overline{\Sigma} \subset \Re^{d-1}$ and $\Gamma_T = \overline{\Omega_T} \cap \overline{\Sigma} \subset \Re^{d-1}$, respectively.

The direct transient heat wave source initial boundary value problem consists in finding $u(t,x)$ with $(t,x) \in Q$ given a boundary input $g(t,x)$ with $(t,x) \in \overline{\Sigma}$, an initial input $u_0(x)$, $u_0'(x)$ with $(t,x) \in \Omega_0$ and a source distribution $f(t,x)$ with $(t,x) \in Q$ that verifies the problem :

$$(P_{u_0,u_0',g,f}) \begin{cases} \frac{1}{c^2}\partial_{tt}u + \alpha\partial_t u - \Delta u = f & \text{in } Q, \\ u = u_0 \,,\, \partial_t u = u_0' & \text{in } \Omega_0, \\ u = g & \text{on } \Sigma. \end{cases} \tag{2.1}$$

and Dirichlet data compatibility condition, $u_0 = g$ at the time-space cylinder corner Γ_0.

It is well known that this direct problem is well posed. For Hilbert space framework we need to introduce , following Lions and Magenes (1972), the anisotropic Sobolev spaces. For $r, s > 0$

$$H^{r,s}(Q) := L^2(I; H^r(\Omega)) \cap H^s(I; L^2(\Omega))$$

and the associated lateral boundary spaces

$$H^{r,s}(\Sigma) := L^2(I; H^r(\Gamma)) \cap H^s(I; L^2(\Gamma)).$$

Here $H^r(\Omega)$ and $H^r(\Gamma)$, $r \geq 0$ are the Hilbert family of Sobolev space in the L^2 theory, and the space $L^2(I;X)$ denotes the class of functions that are strongly measurable on $I = [0,T]$ with range in X with the following Hilbert norm:

$$||v||_{L^2(I;X)} = (\int_I ||v||_X^2 dt)^{\frac{1}{2}} < \infty.$$

The normal space with null lateral boundary trace will be

$$H^{r,s}_{0,\bullet}(Q) := L^2(I; H^r_0(\Omega) \cap H^s(I; L^2(\Omega)) \subset H^{r,s}(Q).$$

The adjoint transient heat wave problem has a straightforward definition

$$(P^*_{v_T,v_T',g,f}) \begin{cases} \frac{1}{c^2}\partial_{tt}u - \alpha\partial_t v - \Delta v = f & \text{in } Q, \\ v = v_T \,,\, \partial_t v = v_T' & \text{in } \Omega_T, \\ v = g & \text{on } \Sigma. \end{cases} \tag{2.2}$$

and Dirichlet data compatibility condition, $v_{0,T} = g|_{0,T}$, $v_{0,T}' = g'|_{0,T}$, at the time-space cylinder corner $\Gamma_{0,T}$. The time reversal operator

$$\kappa_T : H^{r,s}(Q) \to H^{r,s}(Q) \,;\, v(t,x) \mapsto \kappa[v](t,x) = v(T-t,x) \tag{2.3}$$

can be used to change changes of variables $u^*(t,x) = v(T-t,x)$ and convert the adjoint problem into an equivalent direct problem.

REMARK 2.1. *The data in the lateral and bottom d-dimensional surfaces of time-space cylinder can be considered as Dirichlet prescribed boundary data in the extended boundary $\overline{\Omega_0 \cup \Sigma}$ of the transient heat wave equation problem. This set is adjoint by the extended boundary $\overline{\Omega_T \cap \Sigma}$ for the adjoint problem. Since the transient heat wave problem has two derivatives in time and also two derivatives in space, the problem and its adjoint are posed with the following set of Cauchy data: prescribed both function (Dirichlet) and its time derivative (Neumann) on the bottom Ω_0 and on the top Ω_T of the cylinder and both Dirichlet and Neumann data at the lateral cylinder surface Σ. If there exists data compatibility at corners Γ_0 and Γ_T, it will give to the transient heat wave problem a character similar to the Poisson Laplace equation.*

LEMMA 2.2. *(SOLUTION OPERATOR) The solution operator is a continuous but not injective linear operator associated with the Direct problem 5$P_{u_0,u_0',g,f}$ defined by*

$$S(u_0, u_0', g, f) = u$$

when $u \in H^{2r+2,r+1}(Q)$ is solution of problem 5with initial data $(u_0, u_0', g) = (u|_{\Omega_0}, u'|_{\Omega_0}, u|_\Sigma)$.

DEFINITION 2.3. *(EXTENDED DIRICHLET TO NEUMANN MAP) We call The Extended Dirichlet to Neumann map for the problem 5the mapping defined by*

$$\Lambda^f_{\Omega,\Sigma}[(u_0, u_0', g)] = (u|_{\Omega_T}, u'|_{\Omega_T}, \partial_\nu u|_{\partial\Omega})$$

when $u \in H^{2r+2,r+1}(Q)$ is solution of problem 5with initial data $(u_0, u_0', g) = (u|_{\Omega_0}, u'|_{\Omega_0}, u|_\Sigma)$.

Note that this operator can be viewed as a combination of the standard Dirichlet to Neumann map in the spatial boundary with the Input to Output map in the time boundary, that is, in initial and final interval times, found in control theory.

DEFINITION 2.4. *(DIRICHLET GREEN FUNCTION) By the Dirichlet Green's function $G(t,x,\tau,\zeta)$ for the problem 5. we mean its adjoint problem solution 2.2 solution with source $\delta(x-\zeta, t-\tau)$, $(t,x,\tau,\zeta) \in Q \times Q$, and homogeneous Dirichlet data on the extended boundary $\Omega_T \times \Sigma$, i. e., $G(t,x,\tau,\zeta) = 0$ for (t,x) on $\Omega_T \times \Sigma$.*

REMARK 2.5. *For regular data the Green's function exist Cos, and we can show that*

$$u(t,x) = \int_{\Omega_0} [u_0(\zeta)(\alpha G(t,x,0,\zeta) - \frac{1}{c^2}\frac{\partial G(t,x,0,\zeta)}{\partial \tau}) + u_0'(\zeta) G(t,x,0,\zeta)] d\zeta \tag{2.4}$$

$$-\int_{\Sigma} g(\tau,\zeta)\frac{\partial G(t,x,\tau,\zeta)}{\partial \nu_{(\tau,\zeta)}} d\sigma_{(\tau,\zeta)} + \int_Q f(\tau,\zeta) G(t,x,\tau,\zeta) d\zeta d\tau \tag{2.5}$$

for $(t,x) \in \overline{Q}$ is an explicit solution S to problem 5. By using problem's 5linearity we formally decompose the solution in four parts

$$u = S[u_0, u_0', g, f] := S[u_0,0,0,0] + S[0,u_0',0,0] + S[0,0,g,0] + S[0,0,0,f] \tag{2.6}$$

where $S[u_0,0,0,0])$ is the homogeneous Dirichlet zero source zero time derivative initial value auxiliary problem solution and $S[0,u_0',0,0,0]$ is the homogeneous Dirichlet zero source zero initial value time derivative auxiliary problem solution and $S[0,0,g,0]$ is the zero source zero initial value zero time derivative auxiliary Dirichlet problem solution and $S[0,0,0,f]$ is the zero data auxiliary Dirichlet auxiliary problem.

LEMMA 2.6. *(EXTENDED DIRICHLET TO NEUMANN MAP IS COMPOSITION OF TRACE AND SOLUTION) The Extended Dirichlet to Neumann map is a composition of the final time trace, final time derivative trace and the lateral boundary normal trace with the Solution operator:*

$$\Lambda^f_{\Omega,\Omega,\Sigma}[u_0,u_0',g] = (\gamma_T, \gamma_{T,1}, \gamma_1) S[u_0,u_0',g,f] = (\gamma_T \circ S, \gamma_{T,1}\circ S, \gamma_1 \circ S)[u_0,u_0',g,f] = (u_T, u_T', g^\nu). \tag{2.7}$$

3. INVERSE TRANSIENT HEAT WAVE EQUATION SOURCE PROBLEM

The inverse source problem that we address consists in the recovery of the source f, knowing the Extended Dirichlet to Neumann map $\Lambda^f_{\Omega,\Omega,\Sigma}$. When $r = 0$, the data are regular, the Green's function exists and $f \in L^2(Q)$. Let us investigate this situation. And then, we will show that the unique information available in this inverse problem is given only by one measurement, say, the bottom and top Dirichlet data and lateral cylinder Cauchy boundary data. The inverse problem $IP^f_{(u_0,u_0',g),(u_T,u_T',g^\nu)}$ is: To find $f \in L^2(Q)$ such that

$$(IP^f_{(u_0,u_0',g),(u_T,u_T',g^\nu)}) \left\{ \; (u_T, u_T', g^\nu) = \Lambda^f_{\Omega,\Omega,\Sigma}(u_0,u_0',g) \right. \tag{3.8}$$

for all given data pair $(u_0,u_0',g) \times (u_T,u_T',g^\nu)$ corresponding to different solutions to the direct problem. By taking the normal trace at lateral cylinder boundary of the solution 2.4, we obtain that

$$\gamma_1[u] = \frac{\partial u}{\partial \nu_{(t,x)}}|_\Sigma = \Lambda^f_{\bullet,\bullet,\Sigma}[(u_0,u_0',g)] =$$

$$\gamma_1[S[u_0,0,0,0]] + \gamma_1[S[0,u_0',0,0]] + \gamma_1[S[0,0,g,0]] + \gamma_1[S[0,0,0,f]]$$

or

$$\gamma_1[u](t,x) = \int_{\Omega_0} [u_0(\zeta)(\alpha\frac{\partial G(t,x,0,\zeta)}{\partial \nu_{(t,x)}}|_\Sigma - \frac{1}{c^2}\frac{\partial G(t,x,0,\zeta)}{\partial \nu_{(t,x)}\partial\tau}|_\Sigma) + u_0'(\zeta)\frac{\partial G(t,x,0,\zeta)}{\partial \nu_{(t,x)}}|_\Sigma] d\zeta$$

$$-\int_\Sigma g(\tau,\zeta)\frac{\partial^2 G(t,x,\tau,\zeta)}{\partial \nu_{(t,x)}\partial \nu_{(\tau,\zeta)}}|_\Sigma d\sigma_{(\tau,\zeta)} + \int_Q f(\tau,\zeta)\frac{\partial G(t,x,\tau,\zeta)}{\partial \nu_{(t,x)}}|_\Sigma d\zeta d\tau \tag{3.9}$$

is an explicit expression of the lateral part of the Extended Dirichlet to Neumann map $\Lambda^f_{\bullet,\bullet,\Sigma}$ mapping for $f \in L^2(Q)$. Note that it appears decomposed in its partial traces

$$\Lambda^f_{\bullet,\bullet,\Sigma}[(u_0,u_0',g)] := \Lambda^0_{\bullet,\bullet,\Sigma}[(u_0,0,0)] + \Lambda^0_{\bullet,\bullet,\Sigma}[(0,u_0',0)] + \Lambda^0_{\bullet,\bullet,\Sigma}[(0,0,g)] + \Lambda^f_{\bullet,\bullet,\Sigma}[(0,0,0)] \tag{3.10}$$

where $\Lambda^0_{\bullet,\bullet,\Sigma}[(.,0,0)]$ is the zero source auxiliary zero initial time derivative initial value zero Dirichlet problem solution lateral normal trace and $\Lambda^0_{\bullet,\bullet,\Sigma}[(0,.,0)]$ is the zero source auxiliary zero initial value zero Dirichlet problem solution lateral normal trace and $\Lambda^0_{\bullet,\bullet,\Sigma}[(0,0,.)]$ is the zero source non homogeneous Dirichlet zero initial auxiliary problem solution lateral normal trace and $\Lambda^f_{\bullet,\bullet,\Sigma}[(0,0,0)]$ is the homogeneous Dirichlet zero initial value zero initial time derivative source auxiliary problem solution lateral normal trace.

By taking the final time trace at cylinder top boundary of the solution 2.4 we obtain that

$$\gamma_T[u] = u(T,.) = \Lambda^f_{\Omega,\bullet,\bullet}[(u_0,u_0',g)] =$$

$$\gamma_T[S[u_0,0,0,0]] + \gamma_T[S[0,u_0',0,0]] + \gamma_T[S[0,0,g,0]] + \gamma_T[S[0,0,0,f]]$$

or

$$\gamma_T[u](t,x) = \int_{\Omega_0} u_0(\zeta)(\alpha G(T,x,0,\zeta) - \frac{1}{c^2}\frac{\partial G(T,x,0,\zeta)}{\partial \tau})d\zeta + \int_{\Omega_0} u_0'(\zeta)\frac{1}{c^2}G(T,x,0,\zeta)d\zeta$$

$$-\int_\Sigma g(\tau,\zeta)\frac{\partial G(T,x,\tau,\zeta)}{\partial \nu_{(\tau,\zeta)}}d\sigma_{(\tau,\zeta)} + \int_Q f(\tau,\zeta)G(T,x,\tau,\zeta)d\zeta d\tau \tag{3.11}$$

is an explicit expression of the final part of the input to output map $\Lambda^f_{\Omega,\bullet,\bullet}$ Note that its appears decomposed in its partial traces

$$\Lambda^f_{\Omega,\bullet,\bullet}[(u_0,g)] := \Lambda^0_{\Omega,\bullet,\bullet}[(u_0,0,0)] + \Lambda^0_{\Omega,\bullet,\bullet}[(0,u_0',0)] + \Lambda^0_{\Omega,\bullet,\bullet}[(0,0,g)] + \Lambda^f_{\Omega,\bullet,\bullet}[(0,0,0)] \tag{3.12}$$

where $\Lambda^0_{\Omega,\bullet,\bullet}[(.,0,0)]$ is the zero source zero initial derivative auxiliary initial value zero Dirichlet problem solution final trace, $\Lambda^0_{\Omega,\bullet,\bullet}[(0,.,0)]$ is the zero source auxiliary zero initial value zero Dirichlet problem solution final derivative trace, $\Lambda^0_{\bullet,\Sigma}[(0,0,.)]$ is the zero source non homogeneous Dirichlet zero initial auxiliary problem solution final trace and $\Lambda^f_{\Omega,\bullet}[(0,0,0)]$ is the homogeneous Dirichlet zero initial Cauchy data auxiliary source problem solution final trace.

Finally, by taking the final time derivative trace at cylinder top boundary of the solution 2.4 we obtain that

$$\gamma_{T,1}[u] = u'(T,.) = \Lambda^f_{\bullet,\Omega,\bullet}[(u_0,u_0',g)] =$$

$$\gamma_{T,1}[S[u_0,0,0,0]] + \gamma_{T,1}[S[0,u_0',0,0]] + \gamma_{T,1}[S[0,0,g,0]] + \gamma_{T,1}[S[0,0,0,f]]$$

or

$$\gamma_{T,1}[u](t,x) = \int_{\Omega_0} u_0(\zeta)(\alpha\frac{\partial G(T,x,0,\zeta)}{\partial t} - \frac{1}{c^2}\frac{\partial^2 G(T,x,0,\zeta)}{\partial t\partial \tau})d\zeta + \int_{\Omega_0} u_0'(\zeta)\frac{1}{c^2}\frac{\partial G(T,x,0,\zeta)}{\partial t}d\zeta$$

$$-\int_\Sigma g(\tau,\zeta)\frac{\partial^2 G(T,x,\tau,\zeta)}{\partial t\partial \nu_{(\tau,\zeta)}}d\sigma_{(\tau,\zeta)} + \int_Q f(\tau,\zeta)\frac{\partial G(T,x,\tau,\zeta)}{\partial t}d\zeta d\tau \tag{3.13}$$

is an explicit expression of the final part of the input to output map $\Lambda^f_{\bullet,\Omega,\bullet}$. Note that its appears decomposed in its partial traces

$$\Lambda^f_{\bullet,\Omega,\bullet}[(u_0,u_0',g)] := \Lambda^0_{\bullet,\Omega,\bullet}[(u_0,0,0)] + \Lambda^0_{\bullet,\Omega,\bullet}[(0,u_0',0)] + \Lambda^0_{\bullet,\Omega,\bullet}[(0,0,g)] + \Lambda^f_{\bullet,\Omega,\bullet}[(0,0,0)] \tag{3.14}$$

where $\Lambda^0_{\bullet,\Omega,\bullet}[(.,0,0)]$ is the zero source, zero initial derivative, zero Dirichlet problem solution final derivative trace, $\Lambda^0_{\bullet,\Omega,\bullet}[(0,.,0)]$ is the zero source, zero initial value, zero Dirichlet problem solution final derivative trace, $\Lambda^0_{\bullet,\Omega,\bullet}[(0,0,.)]$ is the zero source, zero initial value, zero derivative non homogeneous Dirichlet auxiliary problem solution final derivative trace and $\Lambda^f_{\bullet,\Omega,\bullet}[(0,0,0)]$ is the homogeneous Dirichlet zero initial Cauchy data auxiliary source problem solution final trace.

DEFINITION 3.1. *(RELATIVE EXTENDED DIRICHLET TO NEUMANN MAP) Consider two problems $P_{u_0,u_0',g,f}$ and $P_{u_0,u_0',g,0}$, one with source f and two other with zero source, but both with the same consistent initial time and Dirichlet data. By the Relative Extended Dirichlet to Neumann map for $f \in L^2(Q)$ we mean the application:*

$$\Lambda^f_{\Omega,\Omega,\Sigma} - \Lambda^0_{\Omega,\Omega,\Sigma}. \tag{3.15}$$

Note that the consistence of data (u_0,u_0',g) is necessary to existence of solution to the problems $P_{u_0,u_0',g,f}$ and $P_{u_0,u_0',g,0}$.

LEMMA 3.2. *Let u_j, $j = 1,2,3,...$ be different solutions of problem 5.with the same source $f \in L^2(Q)$ and different initial time value and derivative and Dirichlet data (u_{0_j}, u'_{0_j}, g_j), $j = 1,2,3,...$, respectively. Then*

- *(i) The Relative Extended Dirichlet to Newman operator* $\Lambda^f_{\Omega,\Omega,\Sigma} - \Lambda^0_{\Omega,\Omega,\Sigma}$ *is an operator whose functional value depends only on the source function* $f \in L^2(Q)$*, but is independent of the initial time value and derivative and Dirichlet data* (u_0, u'_0, g).
- *(ii) For all solution of consistent data problems* $P_{u_{0_j},u'_{0_j},g_j,f}$*,* $j = 1, 2, 3, ...$*, with the same source, the source satisfies the systems of integral equations*

$$\int_Q f(\tau,\zeta)\left(G(T,x,\tau,\zeta), \frac{\partial G(T,x,\tau,\zeta)}{\partial t}, \frac{\partial G(t,x,\tau,\zeta)}{\partial \nu_{(t,x)}}\right) d\zeta d\tau = \tag{3.16}$$

$$(\Lambda^f_{\Omega,\Omega,\Sigma} - \Lambda^0_{\Omega,\Omega,\Sigma})[u_{0_j}, u'_{0_j}, g_j] = \Lambda^f_{\Omega,\Omega,\Sigma}[0,0,0]. \tag{3.17}$$

which depend only on the Relative Extended Dirichlet to Neumann map.

REMARK 3.3. *Note that in this case the unique information available for source reconstruction is given by only one measurement, i.e., that final-Neumann boundary measurement*

$$(u_T, u'_T, \partial_{\nu_{(t,x)}} u) = \Lambda^f_{\Omega,\Omega,\Sigma}[u_0, u'_0, g] = \Lambda^f_{\Omega,\Omega,\Sigma}[0,0,0] \tag{3.18}$$

corresponding to some specific initial-Dirichlet data (u_0, u'_0, g)*, which may be assumed as zero without loss of generality.*

3.1 THE RECIPROCITY GAP FUNCTIONAL

THEOREM 3.4. *Let* v *a function in the following test function space*

$$H^{2,1}_{\frac{1}{c^2}\partial_{tt} - \alpha\partial_t - \Delta}(Q) = \{v \in H^{2,1}(Q) | \frac{1}{c^2}\partial_{tt} v - \alpha \partial_t v - \Delta v = 0\} \tag{3.19}$$

the the source f *in problem* $P_{u_0,u'_0,g,f}$ *() satisfies the Transient Wave Reciprocity Gap Equation*

$$\int_Q f v dx dt = -\alpha \int_{\Omega_T} \Lambda^f_{\Omega,\bullet,\bullet}[0,0,0]\gamma_T[v] dx - \int_\Sigma \Lambda^f_{\bullet,\bullet,\Sigma}[0,0,0]\gamma[v] d\sigma_{(t,x)} \tag{3.20}$$
$$-\frac{1}{c^2}\int_{\Omega_T} (\Lambda^f_{\Omega,\bullet,\bullet}[0,0,0]\gamma_{T,1}[v] - \Lambda^f_{\bullet,\Omega,\bullet}[0,0,0]\gamma_T[v]) dx$$

Proof. *The second Green formula*

$$\int_Q ((\frac{1}{c^2}\partial_{tt} + \alpha\partial_t u - \Delta u) v - u(\frac{1}{c^2}\partial_{tt} - \alpha \partial_t v - \Delta v)) dx dt = \int_\Sigma (\gamma[u]\gamma_1[v] - \gamma_1[u]\gamma[v]) d\sigma_{(t,x)}$$
$$+\frac{1}{c^2}\int_{\Omega_0} (\gamma_0[u]\gamma_{0,1}[v] - \gamma_{0,1}[u]\gamma_0[v]) dx - \frac{1}{c^2}\int_{\Omega_T} (\gamma_T[u]\gamma_{T,1}[v] - \gamma_{T,1}[u]\gamma_T[v]) dx$$
$$-\alpha \int_{\Omega_T} \gamma_T[u]\gamma_T[v] dx + \alpha \int_{\Omega_0} \gamma_0[u]\gamma_0[v] dx \tag{3.21}$$

applied to problems $P_{u_0,u'_0,g,f}$ *with normal trace at the cylinder lateral boundary* Σ*,* $\gamma_1[u] = g^\nu$*, initial value at* Ω_0*,* $\gamma_0[u] = u_0$ *initial derivative at* Ω_0*,* $\gamma_{0,1}[u] = u'_0$ *and the adjoint problem* $P^*_{u_T,u'_T,\gamma[v],0}$ *with* u_T *time* T *value and zero source yield the following expression for reciprocity gap functional in the transient heat wave equation context:*

$$\int_Q f v dx dt = \int_\Sigma (g\gamma_1[v] - g^\nu\gamma[v]) d\sigma_{(t,x)} - \alpha \int_{\Omega_T} u_T \gamma_T[v] dx + \alpha \int_{\Omega_0} u_0 \gamma_0[v] dx \tag{3.22}$$
$$+\frac{1}{c^2}\int_{\Omega_0} (u_0\gamma_{0,1}[v] - u'_0\gamma_0[v]) dx - \frac{1}{c^2}\int_{\Omega_T} (u_T\gamma_{T,1}[v] - u'_T\gamma_T[v]) dx$$

or, by using the Extended Dirichlet to Neumann notation,

$$\int_Q f v dx dt = \int_\Sigma (g\gamma_1[v] - \Lambda^f_{\bullet,\bullet,\Sigma}[u_0, u'_0, g]\gamma[v]) d\sigma_{(t,x)} \tag{3.23}$$
$$-\alpha \int_{\Omega_T} \Lambda^f_{\Omega,\bullet,\bullet}[u_0, u'_0, g]\gamma_T[v] dx + \alpha \int_{\Omega_0} u_0\gamma_0[v] dx$$
$$+\frac{1}{c^2}\int_{\Omega_0} (u_0\gamma_{0,1}[v] - u'_0\gamma_0[v]) dx - \frac{1}{c^2}\int_{\Omega_T} (\Lambda^f_{\Omega,\bullet,\bullet}[u_0, u'_0, g]\gamma_{T,1}[v] - \Lambda^f_{\bullet,\Omega,\bullet}[u_0, u'_0, g]\gamma_T[v]) dx$$

By subtracting the Extended Dirichlet to Neumann map for the zero source problem $P_{u_0,u'_0,g,0}$ *with the same data, we obtain (3.20) as a weak variational form for the systems (3.16).*

4. EXPLICIT SOLUTION TO THERMAL WAVE PROBLEM WITH GREEN'S FUNCTION

Fourier's law has as consequence an infinity velocity propagation of the temperature field. In this way, if a perturbation as an external source located inside the domain introduces a modification in the temperature field, the diffusion based thermal equation will propagates this information to other regions inside the spatial domain, including the boundary with infinity velocity. Although this situation is clearly unrealistic, this classical theory works well in most situation , since the propagation velocity is usually 10 orders of magnitude greater thermal diffusivity. As a partial differential equation, the classical heat equation with can be obtaining by singular behavior o equation 5.with the coefficient in the second order time derivative $\frac{1}{c^2} \to 0$,i.e., $c \to \infty$. Its only characteristic surfaces are the time-space planes t =const. Unlike others usual equation in mechanics, as the wave equation, it is not preserved if we reverse the time t by $-t$. More can be observed in the model produced by the heat equation. In some discontinuous initial data is introduced in the problem at the instant t, after an infinitesimal lapse of time the heat equation model will produces a perfectly smooth solution. So, in this model a smoother future may be consequence to a very rougher initial condition. The good news for the heat equation model is that the information propagates with infinity velocity, and this will have an fundamental importance in the source reconstruction numerical implementation. The wave equation can also be obtaining from equation 5.when the coefficient in the first order time derivative α goes to 0. This model has the peculiarity that discontinuities in the initial data propagates along the time-space conical characteristic surfaces with finite velocity but without any smoothing. Finally, the model equation 5introduce smoothing to the finite velocity propagation.

We can give explicit formulation to the solution of the thermal wave direct problem using the variable separation method. The green's function satisfies equation

$$(\frac{1}{c^2}\partial_{tt} + \alpha\partial_t - \Delta)g(t,x|\tau,\zeta) = \delta(t-\tau)\delta(x-\zeta) \tag{4.24}$$

with homogeneous initial lateral traces. The auxiliary eigenvalue problem

$$\begin{cases} \Delta G_n = -\gamma_n \alpha G_n & \text{in } \Omega, \\ G_n = 0 & \text{on } \Gamma. \end{cases} \tag{4.25}$$

indicates the appropriated geometry dependent orthogonal spatial basis $\{G_n \in C_0^\infty(\overline{\Omega}), n = 1, ..., \infty\}$ when Ω has smooth boundary.

$$\int_\Omega G_n(x)G_m(x)dx = \begin{cases} 0 & \text{when } n \neq m, \\ \|G_n\|^2 & \text{when } n = m. \end{cases} \tag{4.26}$$

Expressing the Green's function as a linear combination of this basis

$$g(t,x|\tau,\zeta) = \sum_{n=1}^{\infty} C_n(t)G_n(x) \tag{4.27}$$

with $C_n(t)$ are time dependent unknown Fourier coefficients. Upon substituting 4.27 into 4.24 we obtain

$$\sum_{n=1}^{\infty} \frac{1}{c^2}\partial_{tt}C_n + \alpha\partial_t C_n + \lambda_n C_n)G_n = \delta(t-\tau)\delta(x-\zeta) \tag{4.28}$$

by using eigenfunction orthogonality 4.26 with obtain the following separated system of ordinary differential equations that must be solved with homogeneous initial conditions to explicitly gives the Fourier coefficients

$$\frac{1}{c^2}\partial_{tt}C_n + \alpha\partial_t C_n + \lambda_n C_n = \frac{G_n(\zeta)\delta(t-\tau)}{\|G_n\|^2}\ , n \in \mathbb{N}. \tag{4.29}$$

The most common method adopted in this solution is the Laplace transform method and it gives:

$$C_n(t|\tau,\zeta) = \frac{G_n(\zeta)}{\|G_n\|^2}c^2 \exp(-\frac{\alpha c^2}{2}(t-\tau))\frac{\sin((t-\tau)\sqrt{\lambda_n^2 - \frac{\alpha^2 c^4}{4}})}{\sqrt{\lambda_n^2 - \frac{\alpha^2 c^4}{4}}}H(t-\tau)\ , t,\tau \in \mathbb{R}\ , \zeta \in \Omega\ , n \in \mathbb{N}. \tag{4.30}$$

Substituting these coefficients in the Green's function expansion 4.27 we obtain

$$g(t,x|\tau,\zeta) = \sum_{n=1}^{\infty} \frac{G_n(x)G_n(\zeta)}{\|G_n\|^2}c^2 \exp(-\frac{\alpha c^2}{2}(t-\tau))\frac{\sin((t-\tau)\sqrt{\lambda_n^2 - \frac{\alpha^2 c^4}{4}})}{\sqrt{\lambda_n^2 - \frac{\alpha^2 c^4}{4}}}H(t-\tau)\ , t,\tau \in \mathbb{R}\ , x,\zeta \in \Omega$$

(4.31)

This deduced Green' function can be used to construct the homogeneous problem solution

$$u(t,x) = S[0,0,0,f](t,x) = \sum_{n=1}^{\infty} \int_0^t \int_\Omega \frac{G_n(x)G_n(\zeta)f(\tau,\zeta)}{\|G_n\|^2} c^2 \exp\left(-\frac{\alpha c^2}{2}(t-\tau)\right) \frac{\sin\left((t-\tau)\sqrt{\lambda_n^2 - \frac{\alpha^2 c^4}{4}}\right)}{\sqrt{\lambda_n^2 - \frac{\alpha^2 c^4}{4}}} d\tau d\zeta \tag{4.32}$$

From this formula an explicit expression to the Extended Dirichlet to Neumann map for the problem 5.can easily be formulated

$$\Lambda^f_{\Omega,\Sigma}[(0,0,0)] = (u|_{\Omega_T}, u'|_{\Omega_T}, \partial_\nu u|_{\partial\Omega})$$

and posing the integral equation system 3.16 or generate data for test the transient heat wave reciprocity gap equation 3.20 is straightforward.

REMARK 4.1. *(TRANSITION TO THE HEAT AND TO THE WAVE EQUATIONS)*
It is not difficult to verify the singular limits

- $\lim_{c\to\infty} g(t,x|\tau,\zeta)$ *to the heat equation Green's function and*
- *the limit* $\lim_{\alpha\to 0} g(t,x|\tau,\zeta)$ *to the wave equation.*

In the mix case in which we observe wave behavior with damping we can distinguish two class of eigenvalue:

- *Those for which* $\lambda_n \leq \frac{\alpha c^2}{2}$ *introduces a complex argument in the* $\sin$ *function, converting it in an hyperbolic* $\sinh$ *function with is typical of the heat equation and the damping process.*
- *Those for which* $\lambda_n > \frac{\alpha c^2}{2}$ *preserves the wave behavior of the mode and*

The parameter $\lambda_{\alpha,c} = \frac{\alpha c^2}{2}$ *is knower as the relaxation time and regulates the transition from over damping to under damping in the thermal wave model.*

5. THE HEAT EQUATION SINGULAR BEHAVIOR FOR $c >> \alpha$

In the heat equation limit, $c >> \alpha$ the test function space $H^{2,1}_{\frac{1}{c^2}\partial_{tt}-\alpha\partial_t-\Delta}(Q)$ can be approximated by the space

$$H^{2,1}_{-\alpha\partial_t-\Delta}(Q) = \{v \in H^{2,1}(Q) | -\alpha\partial_t v - \Delta v = 0\} \tag{5.33}$$

the the source f for the near singular problem given by () will satisfy approximately the the Transient Heat Reciprocity Gap Equation

$$\int_Q f v dx dt = -\alpha \int_{\Omega_T} \Lambda^f_{\Omega,\bullet,\bullet}[0,0,0]\gamma_T[v]dx - \int_\Sigma \Lambda^f_{\bullet,\bullet,\Sigma}[0,0,0]\gamma[v]d\sigma_{(t,x)} \tag{5.34}$$

We can further improve the approximation by considering a partition of the time interval $[0,T]$ into N subintervals of length $\tau > 0$. Let $\{t_0, t_1, t_2, ..., t_n, t_{n+1}, ...t_N\}$ be the knots of this partition, with $t_0 = 0$ and $t_N = T$. Let $H^2_\lambda(\Omega) := \{v \in H^2(\Omega) : -\Delta v + \lambda v = 0; \kappa^2 = \lambda = \frac{1}{\tau\theta}\}$ the space of Helmholtz functions. For $t_n < t < t_{n+1}, n = 0, 1, N-1$ the mean value theorem can be used to approximate the equation (5.34)

$$\int_\Omega f(t_{n+1},x)v(t_{n+1},x)dx \approx -\frac{\theta_2}{\theta_1}\int_\Gamma \Lambda^f_{\bullet,\Sigma}[0,0](t_{n+1},x)v(t_{n+1},x)d\sigma_x \tag{5.35}$$

where θ, θ_1 and θ_2 are same order numerical constants. This approximation to hight velocity of propagation $c >> \alpha$ combines hight velocity of propagation with high thermal dissipation. Information about the source almost instantaneously arrive to the spatial boundary and almost instantaneously is converted in temperature. So, a modified Helmholtz model will gives good source reconstruction.

5.1 The modified Helmholtz equation model

The averaged reciprocity gap equation 5.35 can be solve with parameters $\theta_1 = \theta_2$ to gives an approximates reconstruction of star shape characteristic source function. The uniqueness of the reconstruction of this kind of source is demonstrated in Roberty and Rainha (2011) and correspond to solves the following nonlinear integral system:

To find $R^{n+1}(\theta) \in C^2([0, 2\pi))$ such that

$$\int_0^{2\pi} \int_0^{R^{n+1}(\theta)} \rho \frac{\|m\| 2^{\|m\|} I_{\|m\|}(\kappa\rho)}{\kappa^{\|m\|}} \exp(im\theta) d\rho d\theta = \int_0^{2\pi} \partial_\nu u^{n+1} \frac{\|m\| 2^{\|m\|} I_{\|m\|}(\kappa\rho)}{\kappa^{\|m\|}} \exp(im\theta) d\sigma(\theta) \, , m \in \mathbb{N} \tag{5.36}$$

where we have used the following facts:

1. Ω is circle in $\mathbb{R}^{\not\vDash}$.
2. The set $\{ \frac{\|m\| 2^{\|m\|} I_{\|m\|}(\kappa\rho)}{\kappa^{\|m\|}} \exp(im\theta) \; ; m \in \mathbb{N}\}$ is dense in $H_{\kappa^2}(\Omega)$.
3. $\Lambda^f_{\bullet,\Sigma}[0,0](t_{n+1}, x) \approx \partial_\nu u^{n+1}$, that is, the data from the high velocity thermal wave problem approximates the spatial boundary trace to the transient heat model.

   This nonlinear problem is solved by Levemberg-Maquardt method or by Fourier series expansion of the boundary shape parametrization.

6. DETERMINING CENTROID AND SHAPE

An problem appears when we work with the modified Helmholtz approximation to thermal wave problem with high velocity, since functions $\{1, x_i, i = 1, ..., d\} \notin H_{\kappa^2}(\Omega) = \{v \in H^2(\Omega) : -\Delta v + \kappa^2 v = 0\}$. In this case we have

$$\{\exp(\kappa(\sum_{i=1}^{d} l_i x_i))\} \text{ for } (l_1, ...l_d) \in S^{d-1} \tag{6.37}$$

which are a special dense set in $H_{\kappa^2}(\Omega)$. We may construct an enumerable dense set by choosing some discrete set of directions $l^j \in S^{d-1}$ appropriately or by using the Jacobi-Anger expansion. An appropriate modification of this set will be obtained by substituting these exponentials with hyperbolic functions $\sinh$ and $\cosh$. These functions are respectively skew symmetric and symmetric with respect to origin of the coordinates system. If we know the star-shaped source centroid, it is best to choose the origin in the centroid and set the following basis

$$\{\frac{\sinh(\kappa(\sum_{i=1}^{d} l_i(x_i - \bar{x_i}))}{\kappa} \; ; \; \cosh(\kappa(\sum_{i=1}^{d} l_i(x_i - \bar{x_i}))\} \text{ for } (l_1, ...l_d) \in S^{d-1} \tag{6.38}$$

to have a more balanced system of test functions to use in the reciprocity gap functional. As already mentioned, contrary to the classical Novikov's star shape source reconstruction with boundary data problem for the Laplace operator ($\kappa = 0$), in which the centroid and the source volume may be obtained as zero and first order moments of the Neumann data at the boundary, the necessary functions for centroid calculations $\{1, (x_1, ..., x_d)\}$ are in the space $H^2_\lambda(\Omega)$. Fortunately, in this generic case of $\kappa \neq 0$ we may introduce a concept that we are naming meta centroid, κ-centroid or $\lambda-$space centroid. It may also be estimated from Neumann data in the boundary, and in the case in which the star shape source is a Cartesian domain interval, rectangle or parallelepiped rectangular voxel, this κ-centroid is equivalent to the $\kappa = 0$ centroid, that is, the harmonic centroid in the Novikov's problem, in the sense that if the source domain is star-shape with respect to one centroid, it also is star-shape with respect to the other meta centroid.

DEFINITION 6.1. *(META CENTROID) Let $\omega \subset \Omega \in R^d$. By meta centroid $\bar{x} = (\bar{x_1}, ..., \bar{x_d})$ of this sub domain we mean*

$$\bar{x_i} = \frac{\int_\omega x_i \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa(x_i - \bar{x_i})} dx}{\int_\omega \frac{\sinh(\kappa(x_i - \bar{x_i}))}{\kappa(x_i - \bar{x_i})} dx}, \text{ for } i = 1, ..., d. \tag{6.39}$$

LEMMA 6.2. *Suppose that the star shape source characteristic support border curve is symmetric with respect to the ordinates and the abscissa axis passing through the centroid. Then the meta centroid coincides with the harmonic centroid.*

Proof: *In fact, since the function* $\sinh$ *is skew symmetric, expression (6.39) will calculates zero in the coordinates system for with the harmonic source centroid is the origin.*

Since in the transient problem the source is moving inside the box, which means that its centroid position may vary with the time, the capacity of centroid position determination is fundamental for the solution of the source reconstruction problem.

6.1 Determining the meta centroid with the $\sinh$ function

Since the Neumann data are frequently noisy, the least square non linear method may be used to formulate an unconstrained minimizing problem for the determination of coordinates $\bar{x}_i$ of the centroid. If necessary, classical regularizations methods, such as the method of Tikhonov, may be adapted for the stabilization and improvement of the algorithm. Without any regularization other than truncation, the problem of centroid determination in the modified Helmholtz equation with boundary Dirichlet data zero and g^ν Neumann data on the boundary is

$$\bar{x}_i^\kappa = \arg\min\{\left|\int_\Gamma \frac{\sinh(\kappa(x_i - x_i^c))}{\kappa} g^\nu d\sigma(x)\right|^2 |x^c \in \Omega\} \text{ for } i = 1, ..., d. \tag{6.40}$$

6.2 Determining shape parameters with the $\cosh$ function

Once we have reconstructed the meta centroid, we may proceed with the shape parameters determination with the same modified Helmholtz data

$$\bar{\omega} = \arg\min\{\left|\int_\omega \cosh(\kappa(\sum_{i=1}^d l_i(x_i - \bar{x}_i^\kappa)))dx - I_\Gamma(\cosh, \kappa, l, \bar{x}_i^\kappa, g^\nu)\right|^2 : \omega \subset \Omega;\ l \in S^{d-1}\} \tag{6.41}$$

where the set of directions $l := (l_1, ...l_d) \in S^{d-1}$ is used to generate linearly independent functionals of the trial shape and

$$I_\Gamma(\cosh, \kappa, l, \bar{x}_i^\kappa, g^\nu) := \int_\Gamma \cosh(\kappa(\sum_{i=1}^d l_i(x_i - \bar{x}_i^\kappa)))g^\nu(x)d\sigma(x) \tag{6.42}$$

may be computed by using only the just calculated meta centroid coordinates and the already known Cauchy data on the boundary.

6.3 Three dimensional parallelepiped block (voxel) movement inside the unitary cube numerically captured

The example presented here is for the heat equation case only. The model studied in this case is an source inside the domain $\Omega = (0, 1)^d \in \mathcal{R}^d$ with a parallelepiped block (voxel) shape. It is supported with an harmonic centroid evolution following the parametric curve

$$(.5(1 + .6\sin(2\pi t)), .5(1 + .6\cos(2\pi t)), .25 + .25t)$$

and deforming equally in all directions by the following time rule

$$h_x = h_y = h_z = h = .15(1 + .25\left|\cos(\frac{\pi t}{2})\right|),$$

where block edge is $2h$. The number of harmonics in the Fourier sine series is 20, the Δx for spacial collocation is .01 and θ is choose as .8. The evolution is calculated for various time step $\tau = [.1, .01]$ in the interval $t \in [0, 2]$. For these values of time inclement the modified Helmholtz equation parameter varies as $\kappa = 3, ..., 12$. As the κ value approaches to 6 the reconstruction start to become worst, so we may here observe that for this special set of heat equation coefficients, which means a thermal inertia equal one, the minimum time increment for the present methodology without any kind of regularization procedure is approximately $\tau = 0.05$. Others experiments enforcing the potentialities of the present methodology wil be presented during the conference.

7. CONCLUSIONS

We have presented a methodology for star shape source reconstruction in the transient second order problem by using one set of Cauchy data history. The method is based on a modified Helmholtz system based algorithm derived with the aid of finite differences time schemes. A methodology for centroid and shape capture is introduced. Numerical experiments in Cartesian geometry are investigated to stresses associates difficulties.

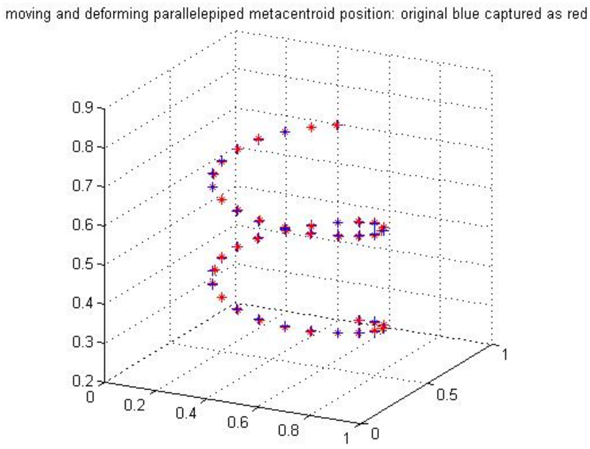

Figure 1. **3D meta centroid movement with time increment** .05

8. ACKNOWLEDGEMENTS

N.C. Roberty work is partially supported by the Brazilian agencies CNPq and Coopetec Foundation.
Denis M. de Sousa and Marcelo L. S. Rainha are supported by CNPq.

REFERENCES

Alves, C., Martins, N. and Roberty, N., 2009. "Full identification of acoustic sources with multiples frequencies and boundary measurements". *Inverse Problems and Imaging*, Vol. 3, No. 3, pp. 275–294.

Alves, C.J.S. and all., 2008. "Recovering the source term in a linear diffusion problem by the method of fundamental solutions." *Inverse Problems in Science and Engineering*, Vol. 16, No. 8, pp. 1005–1021.

Anger, G., 1990. *Inverse Problems in Differential Equations*. Plenum Press, New York, USA.

Badia, A.E. and Duong, T.H., 1998. "Some remarks on the problem of source identification from boundary measurements." *Inverse Problems*, Vol. 16, pp. 883–891.

Cannon, J.R. and Perez-Estevas, S., 1986. "An inverse problem for the heat equation". *Inverse Problems*, Vol. 2, pp. 395–403.

Cannon, J.R. and Perez-Estevas, S., 1991. "Uniqueness and stability of 3-d heat sources". *Inverse Problems*, Vol. 7, pp. 57–62.

Hettlich, F. and Rundell, W., 2001. "Identification of a discontinuous source in the heat equation". *Inverse Problems*, Vol. 17, p. 1465.

Isakov, V., 1990. *Inverse Sources Problems*. American mathematical Society, Providence, Rhode Island, USA.

Lefevre, F. and Nilliot, C.L., 2002. "The bem for point heat source estimation: application to multiple static sources and moving sources." *International Journal of Thermal Science*, Vol. 41, pp. 536–545.

Lions, J.L. and Magenes, E., 1972. "Non-homogeneous boundary value problems and applications- vol. 2".

Roberty, N.C. and Alves, C.J., 2007a. "On the uniqueness of helmholtz equation star shape sources reconstruction from boundary data." In . Fapemig-São João del Rei, ed., *65 Semiário Brasileiro de Análise-SBA 2007*. pp. 141–150.

Roberty, N.C. and Alves, C.J.S., 2007b. "Star-shape source reconstruction in the helmholtz equations-frequency parameter limit". In U. de São Paulo, ed., *66 Seminário Brasileiro de Análise-SBA 2007*. pp. 1–12.

Roberty, N.C. and Sousa, D.M., 2008. "Source reconstruction for the helmholtz equation." In U. de São Paulo, ed., *68 Seminário Brasileiro de Análise-SBA 2008*. pp. 1–10.

Roberty, N. and Alves, C.J.S., 2009. "On the identification of star-shape sources from boundary measurements using a reciprocity functional." *Inverse Problems in Science and Engineering*, Vol. 17, No. 1, pp. 187–202.

Roberty, N.C. and Rainha, M.L.S., 2011. "Reconstruction of star shape sources in the modified helmholtz equation model". *Submited to Inverse problems in Science and Engineering*.