# TOPOLOGICAL OPTIMIZATION OF STRUCTURES SUBJECT TO DRUCKER-PRAGER STRESS CONSTRAINTS

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***Abstract.*** *The topological asymptotic analysis provides the sensitivity of a given shape functional with respect to an infinitesimal domain perturbation. Therefore, this sensitivity can be naturally used as a descent direction in a structural topology design problem. According to the literature, the classical approach based on flexibility minimization for a given amount of material, without control on the stress level supported by the structural device, has been widely considered. On the other hand, one of the most important requirements in mechanical design is to find the lightest topology satisfying a material failure criterion. In this paper, therefore, we introduce a class of penalty functionals that mimic a pointwise constraint on the Drucker-Prager stress field. The associated topological derivative is obtained for plane stress linear elasticity. Then, a topology optimization algorithm based on these concepts is proposed, that allows for treating local stress criteria taking into account different behaviors of the structure under traction or compression loading. Finally, these features are shown through some numerical examples.*

***Keywords:*** *topological derivative, topology optimization, local stress criteria, Drucker-Prager stress field*

## 1. INTRODUCTION

Structural topology optimization is an expanding research field of computational mechanics which has been growing very rapidly in the last years. A relatively new approach for this kind of problem is based on the concept of topological derivative (4; 6). This derivative allows to quantify the sensitivity of a given shape functional with respect to an infinitesimal topological domain perturbation, like typically the nucleation of a hole. Thus, the topological derivative has been successfully applied in the context of topology optimization, inverse problems and image processing. However, in the context of structural topology design, the topological derivative has been used as a descent direction only for the classical approach based on minimizing flexibility for a given amount of material. Although widely adopted, through this formulation the stress level supported by the structural device cannot be controlled. This limitation is not admissible in several applications, because one of the most important requirements in mechanical design is to find the lightest topology satisfying a material failure criterion. Even the methods based on relaxed formulations have been traditionally applied to minimum compliance problems. In fact, only a few works dealing with local stress control can be found in the literature (see, for instance, (5)). This can be explained by the mathematical and numerical difficulties introduced by the large number of highly non-linear constraints associated to local stress criteria. Following the original ideas presented in (3) for the Laplace equation, in this paper we introduce a class of penalty functionals in order to approximate a pointwise constraint on the Drucker-Prager stress field. The associated topological derivative is then obtained for plane stress linear elasticity. We show that the obtained topological asymptotic expansion can be used within a topology optimization algorithm, which allows for treating local stress criteria. Finally, the efficiency of this algorithm is verified through some numerical examples. In particular, the obtained structures are free of geometrical singularity, unlike what occurs by the compliance minimization approach. We recall that such singularities lead to stress concentrations which are highly undesirable in structural design.

## 2. PROBLEM STATEMENT

In this Section we introduce a class of Drucker-Prager stress penalty functionals under plane stress linear elasticity assumptions.

### 2.1 The constrained topology optimization problem

Let $D$ be a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\Gamma$. We assume that $\Gamma$ is split into three disjoint parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_0$, where $\Gamma_D$ is of nonzero measure, and $\Gamma_N$ is of class $\mathcal{C}^1$. We consider the topology optimization problem:

$$\underset{\Omega\subset D}{\text{Minimize}}\; I_\Omega(u_\Omega) \tag{1}$$

subject to the state equations

$$\begin{cases} -\mathrm{div}(\gamma_\Omega \sigma(u_\Omega)) &= 0 \quad \text{in} \quad D, \\ u_\Omega &= 0 \quad \text{on} \quad \Gamma_D, \\ \gamma_\Omega \sigma(u_\Omega) n &= g \quad \text{on} \quad \Gamma_N, \\ \sigma(u_\Omega) n &= 0 \quad \text{on} \quad \Gamma_0, \end{cases} \tag{2}$$

and the constraint

$$\sigma_M(u_\Omega) + \eta \mathrm{tr}\sigma(u_\Omega) \leq \overline{\sigma} \qquad \text{a.e. in } \Omega \cap \tilde{D}, \tag{3}$$

The notations used above are the following. The system (2) is understood in the weak sense, as this will be the case throughout all the paper, and admits an unique solution $u_\Omega \in \mathcal{V} = \{u \in H^1(D)^2, u_{|\Gamma_D} = 0\}$. The material density $\gamma_\Omega$ is a piecewise constant function which takes two positive values: $\gamma_\Omega = \gamma_{in}$ in $\Omega$ and $\gamma_\Omega = \gamma_{out}$ in $D \setminus \overline{\Omega}$. In the applications, $D \setminus \overline{\Omega}$ is occupied by a weak phase that approximates an empty region, thus we assume that $\gamma_{out} \ll \gamma_{in}$. The stress tensor $\sigma(u_\Omega)$, normalized to a unitary Young modulus, is related to the displacement field $u_\Omega$ through the Hooke law: $\sigma(u) = Ce(u)$, where $e(u) = \nabla^s u$ is the strain tensor, and $C = 2\mu \mathbb{I} + \lambda(I \otimes I)$ is the elasticity tensor. Here, $I$ and $\mathbb{I}$ are the second and fourth order identity tensors, respectively, and the Lamé coefficients $\mu$ and $\lambda$ are given in plane stress by $\mu = 1/(2(1+\nu))$ and $\lambda = \nu/(1-\nu)^2$, where $\nu$ is the Poisson ratio. The Neumann data $g$ is assumed to belong to $L^2(\Gamma_N)^2$. The Drucker-Prager stress constraint (3) can be written in the following equivalent form

$$\left(\frac{\sigma_M(u)}{\overline{\sigma}}\right)^2 - \left(\frac{\eta}{\overline{\sigma}}\right)^2 \mathrm{tr}^2\sigma(u) + 2\frac{\eta}{\overline{\sigma}}\mathrm{tr}\sigma(u) \leq 1, \tag{4}$$

where $\overline{\sigma}$ and $\eta$ are prescribed positive numbers. The von Mises stress $\sigma_M(u)$ is given by

$$\sigma_M(u) = \sqrt{\frac{1}{2}(3\sigma(u).\sigma(u) - \mathrm{tr}^2\sigma(u))}. \tag{5}$$

Therefore, we can define the Drucker-Prager stress constraint as follows

$$\Upsilon(\sigma(u)) := \frac{1}{2}\tilde{B}\sigma(u).\sigma(u) + 2\eta\overline{\sigma}\mathrm{tr}\sigma(u) \leq \overline{\sigma}^2, \quad \text{with} \quad \tilde{B} = 3\mathbb{I} - (1+2\eta^2) I \otimes I, \tag{6}$$

or alternatively as

$$\frac{1}{2}B\sigma(u).e(u) + \xi \mathrm{tr}e(u) \leq \overline{\sigma}^2, \quad \text{with} \quad B = 6\mu\mathbb{I} + \lambda(1-4\eta^2)(I \otimes I) - 2\mu(1+2\eta^2)(I \otimes I) \tag{7}$$

and the constant $\xi$ defined as

$$\xi = 4(\mu + \lambda)\eta\overline{\sigma}. \tag{8}$$

The set $\tilde{D}$ is an open subset of $D$. Finally, the objective functional $I_\Omega : \mathcal{V} \to \mathbb{R}$ is assumed to admit a known topological derivative $D_T I_\Omega$ as defined in Section 2.3

### 2.2 Penalization of the constraint

Problem (1)-(3) is very difficult to address directly because of the pointwise constraint. Therefore we propose an approximation based on the introduction of a penalty functional. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function of class $\mathcal{C}^2$. To enable proper justifications of the subsequent analysis, we assume further that the derivatives $\Phi'$ and $\Phi''$ are bounded. We consider the penalty functional:

$$J_\Omega(u) = \int_{\tilde{D}} \gamma_\Omega \Phi(\Upsilon(\sigma(u)))dx. \tag{9}$$

Then, given a penalty coefficient $\alpha > 0$, we define the penalized objective functional: $I^\alpha_\Omega(u) = I_\Omega(u) + \alpha J_\Omega(u)$. Henceforth we shall solve the problem:

$$\underset{\Omega \subset D}{\mathrm{Minimize}}\, I^\alpha_\Omega(u_\Omega) \qquad \text{subject to (2).} \tag{10}$$

We will see that solving (10) instead of (1)-(3) leads to feasible domains provided that $\alpha$ and $\Phi$ are appropriately chosen, namely that the two following conditions are fulfilled:

- $\alpha$ is large enough,
- $\Phi'$ admits a sharp variation around $\bar{\sigma}$.

### 2.3 Topology perturbations

Given a point $x_0 \in D \setminus \partial\Omega$ and a radius $\varepsilon > 0$, we consider a circular inclusion $\omega_\varepsilon = B(x_0, \varepsilon)$, and we define the perturbed domain: $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$ if $x_0 \in \Omega$ and $\Omega_\varepsilon = (\Omega \cup \omega_\varepsilon) \cap D$ if $x_0 \in D \setminus \overline{\Omega}$. We denote for simplicity $(u_{\Omega_\varepsilon}, \gamma_{\Omega_\varepsilon})$ by $(u_\varepsilon, \gamma_\varepsilon)$ and $(u_\Omega, \gamma_\Omega)$ by $(u_0, \gamma_0)$. Then, for all $\varepsilon \in [0, 1]$, $\gamma_\varepsilon$ can be expressed as: $\gamma_\varepsilon = \gamma_0$ in $D \setminus \overline{\omega_\varepsilon}$ and $\gamma_\varepsilon = \gamma_1 \text{in} \omega_\varepsilon$. We note that $\gamma_0$ and $\gamma_1$ are two positive functions defined in $D$ and constant in a neighborhood of $x_0$. For all $\varepsilon \geq 0$, the state equations can be rewritten:

$$\begin{cases} -\text{div}\,(\gamma_\varepsilon \sigma(u_\varepsilon)) &= 0 \quad \text{in} \quad D, \\ u_\varepsilon &= 0 \quad \text{on} \quad \Gamma_D, \\ \gamma_\varepsilon \sigma(u_\varepsilon) n &= g \quad \text{on} \quad \Gamma_N, \\ \sigma(u_\varepsilon) n &= 0 \quad \text{on} \quad \Gamma_0. \end{cases} \tag{11}$$

In order to solve (10), we are looking for an asymptotic expansion, named as topological asymptotic expansion, of the form

$$I^\alpha_{\Omega_\varepsilon}(u_\varepsilon) - I^\alpha_\Omega(u_0) = f(\varepsilon) D_T I^\alpha_\Omega(x_0) + o(f(\varepsilon)), \tag{12}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a function that goes to zero with $\varepsilon$, and $D_T I^\alpha_\Omega : D \to \mathbb{R}$ is the so-called topological derivative of the functional $I^\alpha_\Omega$. Since such an expansion is assumed to be known for the objective functional $I_\Omega$, we subsequently focus on the penalty functional $J_\Omega$. We adopt the simplified notation:

$$J_\varepsilon(u) := J_{\Omega_\varepsilon}(u) = \int_{\tilde{D}} \gamma_\varepsilon \Phi(\Upsilon(\sigma(u))) dx. \tag{13}$$

## 3. TOPOLOGICAL SENSITIVITY ANALYSIS OF THE DRUCKER-PRAGER STRESS PENALTY FUNCTIONAL

In this section, the topological sensitivity analysis of the penalty functional $J_\Omega$ is carried out. We follow the approach described in (3) for the Laplace problem. Here, the calculations are more technical, but the estimates of the remainders detached from the topological asymptotic expansion are analogous. Hence we do not repeat these estimates. The reader interested in the complete proofs may refer to (3). Possibly shifting the origin of the coordinate system, we assume henceforth for simplicity that $x_0 = 0$.

### 3.1 A preliminary result

The reader interested in the proof of the proposition below may refer to (1).

**Proposition 31.** *Let $\mathcal{V}$ be a Hilbert space and $\varepsilon_0 > 0$. For all $\varepsilon \in [0, \varepsilon_0)$, consider a vector $u_\varepsilon \in \mathcal{V}$ solution of a variational problem of the form*

$$a_\varepsilon(u_\varepsilon, v) = \ell_\varepsilon(v) \quad \forall v \in \mathcal{V}, \tag{14}$$

*where $a_\varepsilon$ and $\ell_\varepsilon$ are a bilinear form on $\mathcal{V}$ and a linear form on $\mathcal{V}$, respectively. Consider also, for all $\varepsilon \in [0, \varepsilon_0)$, a functional $J_\varepsilon : \mathcal{V} \to \mathbb{R}$ and a linear form $L_\varepsilon(u_0) \in \mathcal{V}'$. Suppose that the following hypotheses hold.*

1. *There exist two numbers $\delta a$ and $\delta \ell$ and a function $\varepsilon \in \mathbb{R}_+ \mapsto f(\varepsilon) \in \mathbb{R}$ such that, when $\varepsilon$ goes to zero,*

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = f(\varepsilon)\delta a + o(f(\varepsilon)), \tag{15}$$

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = f(\varepsilon)\delta \ell + o(f(\varepsilon)), \tag{16}$$

$$\lim_{\varepsilon \to 0} f(\varepsilon) = 0, \tag{17}$$

*where $v_\varepsilon \in \mathcal{V}$ is an adjoint state satisfying*

$$a_\varepsilon(\varphi, v_\varepsilon) = -\langle L_\varepsilon(u_0), \varphi \rangle \quad \forall \varphi \in \mathcal{V}. \tag{18}$$

2. *There exist two numbers $\delta J_1$ and $\delta J_2$ such that*

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_0) + \langle L_\varepsilon(u_0), u_\varepsilon - u_0 \rangle + f(\varepsilon)\delta J_1 + o(f(\varepsilon)), \tag{19}$$

$$J_\varepsilon(u_0) = J_0(u_0) + f(\varepsilon)\delta J_2 + o(f(\varepsilon)). \tag{20}$$

*Then we have*

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = f(\varepsilon)(\delta a - \delta \ell + \delta J_1 + \delta J_2) + o(f(\varepsilon)). \tag{21}$$

### 3.2 Adjoint state

The bilinear and linear forms associated with Problem (11) are classically defined in the space $\mathcal{V}$ by:

$$a_\varepsilon(u,v) = \int_D \gamma_\varepsilon \sigma(u).e(v)\,dx \qquad \forall u,v\in\mathcal{V}, \tag{22}$$

$$\ell_\varepsilon(v) = \int_{\Gamma_N} g.v\,ds \qquad \forall v\in\mathcal{V}. \tag{23}$$

At the point $u_0$ (unperturbed solution), the penalty functional admits the tangent linear approximation $L_\varepsilon(u_0)$ given by:

$$\langle L_\varepsilon(u_0),\varphi\rangle = \int_{\tilde{D}} \gamma_\varepsilon k_1(B\sigma(u_0).e(\varphi)+\xi\mathrm{tr}e(\varphi))dx \qquad \forall\varphi\in\mathcal{V}. \tag{24}$$

We define the function

$$k_1 = \Phi'(\Upsilon(\sigma(u_0)))\chi_{\tilde{D}}, \tag{25}$$

where $\chi_{\tilde{D}}$ is the characteristic function of $\tilde{D}$. Then the adjoint state is (a weak) solution of the boundary value problem:

$$\left\{\begin{array}{rcll} -\mathrm{div}\,(\gamma_\varepsilon\sigma(v_\varepsilon)) &=& \mathrm{div}\,(\gamma_\varepsilon k_1(B\sigma(u_0)+\xi I)) & \text{in} \quad D,\\ v_\varepsilon &=& 0 & \text{on} \quad \Gamma_D,\\ \gamma_\varepsilon\sigma(v_\varepsilon)n &=& -\gamma_\varepsilon k_1(B\sigma(u_0)+\xi I)n & \text{on} \quad \Gamma_N\cup\Gamma_0.\\ [\![\gamma_\varepsilon\sigma(v_\varepsilon)]\!]n &=& -[\![\gamma_\varepsilon k_1(B\sigma(u_0)+\xi I)]\!]n & \text{on} \quad \partial\omega_\varepsilon \end{array}\right. \tag{26}$$

where $[\![\gamma_\varepsilon\sigma(v_\varepsilon)]\!]n\in H^{-1/2}(\partial\omega_\varepsilon)^2$ denotes the jump of the normal stress through the interface $\partial\omega_\varepsilon$.

### 3.3 Variation of the bilinear form

In order to apply Proposition 3.1, we need to obtain a closed form for the leading term of the quantity:

$$(a_\varepsilon-a_0)(u_0,v_\varepsilon) = \int_{\omega_\varepsilon}(\gamma_1-\gamma_0)\sigma(u_0).e(v_\varepsilon)dx. \tag{27}$$

In the course of the analysis, the remainders detached from this expression will be denoted by $\mathcal{E}_i(\varepsilon)$, $i=1,2,...$ By setting $\tilde{v}_\varepsilon = v_\varepsilon - v_0$ and assuming that $\varepsilon$ is sufficiently small so that $\gamma_\varepsilon$ is constant in $\omega_\varepsilon$, we obtain:

$$(a_\varepsilon-a_0)(u_0,v_\varepsilon) = (\gamma_1-\gamma_0)(x_0)\left(\int_{\omega_\varepsilon}\sigma(u_0).e(v_0)dx + \int_{\omega_\varepsilon}\sigma(u_0).e(\tilde{v}_\varepsilon)dx\right). \tag{28}$$

Since $u_0$ and $v_0$ are smooth in the vicinity of $x_0$, we approximate $\sigma(u_0)$ and $e(v_0)$ in the first integral by their values at the point $x_0$, and write:

$$(a_\varepsilon-a_0)(u_0,v_\varepsilon) = (\gamma_1-\gamma_0)(x_0)\left(\pi\varepsilon^2\sigma(u_0)(x_0).e(v_0)(x_0) + \int_{\omega_\varepsilon}\sigma(u_0).e(\tilde{v}_\varepsilon)dx + \mathcal{E}_1(\varepsilon)\right). \tag{29}$$

As $v_\varepsilon$ is solution of the adjoint equation (26), then the function $\tilde{v}_\varepsilon$ solves$\gamma_\varepsilon k_1(B\sigma(u_0)+$

$$\left\{\begin{array}{rcll} -\mathrm{div}(\gamma_\varepsilon\sigma(\tilde{v}_\varepsilon)) &=& 0 & \text{in} \quad \omega_\varepsilon\cup(D\setminus\overline{\omega_\varepsilon}),\\ [\gamma_\varepsilon\sigma(\tilde{v}_\varepsilon)n] &=& -(\gamma_1-\gamma_0)\left(k_1(B\sigma(u_0)+\xi I)+\sigma(v_0)\right)n & \text{on} \quad \partial\omega_\varepsilon,\\ \tilde{v}_\varepsilon &=& 0 & \text{on} \quad \Gamma_D,\\ \sigma(\tilde{v}_\varepsilon)n &=& 0 & \text{on} \quad \Gamma_N\cup\Gamma_0. \end{array}\right. \tag{30}$$

We recall that, as before, the boundary value problem (30) is to be understood in the weak sense for $\tilde{v}_\varepsilon\in H^1(D)^2$. We set $S=S_1+S_2$, with $S_1 = k_1(x_0)(B\sigma(u_0)(x_0)+\xi I)$ and $S_2=\sigma(v_0)(x_0)$. We approximate $\sigma(\tilde{v}_\varepsilon)$ by $\sigma(h_\varepsilon^S)$ solution of the auxiliary problem:

$$\left\{\begin{array}{rcll} -\mathrm{div}(\sigma(h_\varepsilon^S)) &=& 0 & \text{in} \quad \omega_\varepsilon\cup(\mathbb{R}^2\setminus\overline{\omega_\varepsilon}),\\ [\![\gamma_\varepsilon\sigma(h_\varepsilon^S)]\!]n &=& -(\gamma_1-\gamma_0)(x_0)Sn & \text{on} \quad \partial\omega_\varepsilon,\\ \sigma(h_\varepsilon^S) &\to& 0 & \text{at} \quad \infty, \end{array}\right. \tag{31}$$

In the present case of a circular inclusion, the tensor $\sigma(h_\varepsilon^S)$ admits the following expression in a polar coordinate system $(r,\theta)$:

- for $r \geq \varepsilon$

$$\sigma_r(r,\theta) = -(\alpha_1+\alpha_2)\frac{1-\gamma}{1+a\gamma}\frac{\varepsilon^2}{r^2} - \frac{1-\gamma}{1+b\gamma}\left(4\frac{\varepsilon^2}{r^2} - 3\frac{\varepsilon^4}{r^4}\right)(\beta_1\cos 2\theta + \beta_2\cos 2(\theta+\phi)), \quad (32)$$

$$\sigma_\theta(r,\theta) = (\alpha_1+\alpha_2)\frac{1-\gamma}{1+a\gamma}\frac{\varepsilon^2}{r^2} - 3\frac{1-\gamma}{1+b\gamma}\frac{\varepsilon^4}{r^4}(\beta_1\cos 2\theta + \beta_2\cos 2(\theta+\phi)), \quad (33)$$

$$\sigma_{r\theta}(r,\theta) = -\frac{1-\gamma}{1+b\gamma}\left(2\frac{\varepsilon^2}{r^2} - 3\frac{\varepsilon^4}{r^4}\right)(\beta_1\sin 2\theta + \beta_2\sin 2(\theta+\phi)), \quad (34)$$

- for $0 < r < \varepsilon$

$$\sigma_r(r,\theta) = (\alpha_1+\alpha_2)\,a\frac{1-\gamma}{1+a\gamma} + b\frac{1-\gamma}{1+b\gamma}(\beta_1\cos 2\theta + \beta_2\cos 2(\theta+\phi)), \quad (35)$$

$$\sigma_\theta(r,\theta) = (\alpha_1+\alpha_2)\,a\frac{1-\gamma}{1+a\gamma} - b\frac{1-\gamma}{1+b\gamma}(\beta_1\cos 2\theta + \beta_2\cos 2(\theta+\phi)), \quad (36)$$

$$\sigma_{r\theta}(r,\theta) = -b\frac{1-\gamma}{1+b\gamma}(\beta_1\sin 2\theta + \beta_2\sin 2(\theta+\phi)), \quad (37)$$

Some terms in the above formulas require explanation. The parameter $\phi$ denotes the angle between the eigenvectors of tensors $S_1$ and $S_2$,

$$\alpha_i = \frac{1}{2}(s_I^i + s_{II}^i) \quad \text{and} \quad \beta_i = \frac{1}{2}(s_I^i - s_{II}^i), \quad i = 1,2, \quad (38)$$

where $s_I^i$ and $s_{II}^i$ are the eigenvalues of tensors $S_i$ for $i = 1,2$. In addition, the constants $a$ and $b$ are respectively given by

$$a = \frac{1+\nu}{1-\nu}, \qquad b = \frac{3-\nu}{1+\nu}, \quad (39)$$

and $\gamma$ is the contrast, that is, $\gamma = \gamma_1(x_0)/\gamma_0(x_0)$. From these elements, we obtain successively:

$$\int_{\omega_\varepsilon}\sigma(u_0).e(\tilde{v}_\varepsilon)dx = \int_{\omega_\varepsilon}\sigma(\tilde{v}_\varepsilon).e(u_0)dx = \int_{\omega_\varepsilon}\sigma(h_\varepsilon^S).e(u_0)dx + \mathcal{E}_2(\varepsilon). \quad (40)$$

Then approximating $e(u_0)$ in $\omega_\varepsilon$ by its value at $x_0$ and calculating the resulting integral with the help of the expressions (35)-(37) yields:

$$\begin{aligned}\int_{\omega_\varepsilon}\sigma(u_0).e(\tilde{v}_\varepsilon)dx &= \int_{\omega_\varepsilon}\sigma(h_\varepsilon^S).e(u_0)(x_0)dx + \mathcal{E}_2(\varepsilon) + \mathcal{E}_3(\varepsilon)\\ &= -\pi\varepsilon^2\rho\left(k_1T(B\sigma(u_0)+\xi I).e(u_0) + T\sigma(u_0).e(v_0)\right)(x_0) + \mathcal{E}_2(\varepsilon) + \mathcal{E}_3(\varepsilon),\end{aligned} \quad (41)$$

with

$$\rho = \frac{\gamma_1-\gamma_0}{b\gamma_1+\gamma_0}(x_0) \quad \text{and} \quad T = b\mathbb{I} + \frac{1}{2}\frac{a-b}{1+\gamma a}I\otimes I. \quad (42)$$

Finally, the variation of the bilinear form can be written in the form:

$$\begin{aligned}(a_\varepsilon - a_0)(u_0,v_\varepsilon) = -\pi\varepsilon^2(\gamma_1-\gamma_0)(x_0)\rho\bigg(&k_1b(B\sigma(u_0)+\xi I).e(u_0) + \frac{1}{2}k_1\frac{a-b}{1+\gamma a}\mathrm{tr}(B\sigma(u_0)+\xi I)\mathrm{tr}e(u_0)\\ &-\frac{b+1}{\gamma-1}\sigma(u_0).e(v_0) + \frac{1}{2}\frac{a-b}{1+\gamma a}\mathrm{tr}\sigma(u_0)\mathrm{tr}e(v_0)\bigg)(x_0) + (\gamma_1-\gamma_0)(x_0)\sum_{i=1}^{3}\mathcal{E}_i(\varepsilon).\end{aligned} \quad (43)$$

### 3.4 Variation of the linear form

Since here $\ell_\varepsilon$ is independent of $\varepsilon$, it follows trivially that

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = 0. \quad (44)$$

### 3.5 Partial variation of the penalty functional with respect to the state

We now study the variation:

$$\begin{aligned} V_{J1}(\varepsilon) &= J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) - \langle L_\varepsilon(u_0), u_\varepsilon - u_0\rangle \\ &= \int_{\tilde D} \gamma_\varepsilon \left[\Phi(\Upsilon(\sigma(u_\varepsilon))) - \Phi(\Upsilon(\sigma(u_0))) - \Phi'(\Upsilon(\sigma(u_0)))(B\sigma(u_0).e(u_\varepsilon - u_0) + \xi \mathrm{tre}(u_\varepsilon - u_0))\right] dx. \end{aligned} \quad (45)$$

By setting $\tilde u_\varepsilon = u_\varepsilon - u_0$, we can write:

$$V_{J1}(\varepsilon) = \int_{\tilde D} \gamma_\varepsilon \Big[\Phi(\Upsilon(\sigma(u_0)) + B\sigma(u_0).e(\tilde u_\varepsilon) + \Upsilon(\sigma(\tilde u_\varepsilon))) - \Phi(\Upsilon(\sigma(u_0))) \\ - \Phi'(\Upsilon(\sigma(u_0)))(B\sigma(u_0).e(\tilde u_\varepsilon) + \xi \mathrm{tre}(\tilde u_\varepsilon))\Big] dx. \quad (46)$$

Since $u_\varepsilon$ is solution of the state equation (11), then by difference we find that $\tilde u_\varepsilon$ solves:

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\gamma_\varepsilon \sigma(\tilde u_\varepsilon)) &=& 0 & \text{in} \quad \omega_\varepsilon \cup (D \setminus \overline{\omega_\varepsilon}), \\ [\![\gamma_\varepsilon \sigma(\tilde u_\varepsilon)]\!] n &=& -(\gamma_1 - \gamma_0)\sigma(u_0) n & \text{on} \quad \partial\omega_\varepsilon, \\ \tilde u_\varepsilon &=& 0 & \text{on} \quad \Gamma_D, \\ \sigma(\tilde u_\varepsilon) n &=& 0 & \text{on} \quad \Gamma_N \cup \Gamma_0. \end{array} \right. \quad (47)$$

By setting now $S = \sigma(u_0(x_0))$, we approximate $\tilde u_\varepsilon$ by $h_\varepsilon^S$ solution of the auxiliary problem (31). It comes:

$$V_{J1}(\varepsilon) = \int_{\tilde D} \gamma_\varepsilon \Big[\Phi(\Upsilon(\sigma(u_0)) + B\sigma(u_0).e(h_\varepsilon^S) + \Upsilon(\sigma(h_\varepsilon^S))) - \Phi(\Upsilon(\sigma(u_0))) \\ - \Phi'(\Upsilon(\sigma(u_0)))(B\sigma(u_0).e(h_\varepsilon^S) + \xi \mathrm{tre}(h_\varepsilon^S))\Big] dx + \mathcal{E}_4(\varepsilon). \quad (48)$$

If $x_0 \in D \setminus \overline{\tilde D}$, we obtain easily, using a Taylor expansion of $\Phi$ and the estimate $|\sigma(h_\varepsilon^S)(x)| = O(\varepsilon^2)$ which holds uniformly with respect to $x$ a fixed distance away from $x_0$, that $V_{J1}(\varepsilon) = o(\varepsilon^2)$. Thus we assume that $x_0 \in \tilde D$ (the special case where $x_0 \in \partial \tilde D$ is not treated). In view of the decay of $\sigma(h_\varepsilon^S)$ at infinity and the regularity of $u_0$ near $x_0$, we write

$$V_{J1}(\varepsilon) = \int_{\mathbb{R}^2} \gamma_\varepsilon^* \Big[\Phi(\Upsilon(\sigma(u_0))(x_0) + B\sigma(u_0)(x_0).e(h_\varepsilon^S) + \Upsilon(\sigma(h_\varepsilon^S))) \\ - \Phi(\Upsilon(\sigma(u_0))(x_0)) - \Phi'(\Upsilon(\sigma(u_0))(x_0))(B\sigma(u_0)(x_0).e(h_\varepsilon^S) + \xi \mathrm{tre}(h_\varepsilon^S))\Big] dx + \mathcal{E}_4(\varepsilon) + \mathcal{E}_5(\varepsilon), \quad (49)$$

with $\gamma_\varepsilon^*(x) = \gamma_1(x_0)$ if $x \in \omega_\varepsilon$, $\gamma_\varepsilon^*(x) = \gamma_0(x_0)$ otherwise. The above expression can be rewritten as

$$V_{J1}(\varepsilon) = \int_{\mathbb{R}^2} \gamma_\varepsilon^* \Big[\Phi(\frac{1}{2}\tilde B S.S + 2\eta\overline{\sigma}\mathrm{tr} S + \tilde B S.\sigma(h_\varepsilon^S) + 2\eta\overline{\sigma}\mathrm{tr}\sigma(h_\varepsilon^S) + \frac{1}{2}\tilde B\sigma(h_\varepsilon^S).\sigma(h_\varepsilon^S)) \\ - \Phi(\frac{1}{2}\tilde B S.S + 2\eta\overline{\sigma}\mathrm{tr} S) - \Phi'(\frac{1}{2}\tilde B S.S + 2\eta\overline{\sigma}\mathrm{tr} S)(\tilde B S.\sigma(h_\varepsilon^S) + 2\eta\overline{\sigma}\mathrm{tr}\sigma(h_\varepsilon^S))\Big] dx + \mathcal{E}_4(\varepsilon) + \mathcal{E}_5(\varepsilon). \quad (50)$$

We denote by $V_{J11}(\varepsilon)$ and $V_{J12}(\varepsilon)$ the parts of the above integral computed over $\omega_\varepsilon$ and $\mathbb{R}^2 \setminus \overline{\omega_\varepsilon}$, respectively. Using the expressions (35)-(37), we find

$$V_{J11}(\varepsilon) = \pi\varepsilon^2\gamma_1(x_0)\Big[\Phi(\frac{1}{2}\tilde B S.S + 2\eta\overline{\sigma}\mathrm{tr} S - \rho(\tilde B S.TS + 2\eta\overline{\sigma}\mathrm{tr}(TS)) + \rho^2\frac{1}{2}\tilde B TS.TS) \\ - \Phi(\frac{1}{2}\tilde B S.S + 2\eta\overline{\sigma}\mathrm{tr} S) + \rho\Phi'(\frac{1}{2}\tilde B S.S + 2\eta\overline{\sigma}\mathrm{tr} S)(\tilde B S.TS + 2\eta\overline{\sigma}\mathrm{tr}(TS))\Big]. \quad (51)$$

Next, we define the function independent of $\varepsilon$ such that $\Sigma_\rho^S(x) = \sigma(h_\varepsilon^S)(\varepsilon x)$. From the expressions (32)-(34) and after a change of variable we have

$$V_{J12}(\varepsilon) = \varepsilon^2\gamma_0(x_0)\left[\Psi_\rho(S) + \frac{1}{4}\pi\rho^2 k_1(x_0)\left((5 - 8\eta^2)(2S.S - \mathrm{tr}^2 S) + 3\left(\frac{1 + b\gamma}{1 + a\gamma}\right)^2 \mathrm{tr}^2 S\right)\right]. \quad (52)$$

where $\Psi_\rho(S)$ is given by

$$\Psi_\rho(S) = \int_0^1 \int_0^\pi \frac{1}{t^2}[\Phi(\Upsilon(S) + \Delta(t,\theta)) - \Phi(\Upsilon(S)) - \Phi'(\Upsilon(S))\Delta(t,\theta)]d\theta dt, \tag{53}$$

with

$$\Delta(t,\theta) = \rho\tfrac{t}{2}\left[(s_I - s_{II})[(s_I + s_{II})(2(1-4\eta^2) + 3\tfrac{1+b\gamma}{1+a\gamma}) + 8\eta\overline{\sigma}]\cos\theta + 3(s_I - s_{II})^2(2-3t)\cos 2\theta\right]$$
$$+\left(\rho\tfrac{t}{2}\right)^2\left[3(s_I + s_{II})^2(\tfrac{1+b\gamma}{1+a\gamma})^2 + (s_I - s_{II})^2(3(2-3t)^2 + 4(1-4\eta^2)\cos^2\theta) + 6\tfrac{1+b\gamma}{1+a\gamma}(s_I^2 - s_{II}^2)(2-3t)\cos\theta\right], \tag{54}$$

Finally we obtain:

$$VJ_1(\varepsilon) = \pi\gamma_1(x_0)\Big[\Phi(\frac{1}{2}\tilde{B}S.S + 2\eta\overline{\sigma}\mathrm{tr}S - \rho(\tilde{B}S.TS + 2\eta\overline{\sigma}\mathrm{tr}(TS)) + \rho^2\frac{1}{2}\tilde{B}TS.TS)$$
$$-\Phi(\frac{1}{2}\tilde{B}S.S + 2\eta\overline{\sigma}\mathrm{tr}S) + \rho\Phi'(\frac{1}{2}\tilde{B}S.S + 2\eta\overline{\sigma}\mathrm{tr}S)(\tilde{B}S.TS + 2\eta\overline{\sigma}\mathrm{tr}(TS))\Big]$$
$$+\gamma_0(x_0)\left[\Psi_\rho(S) + \frac{1}{4}\pi\rho^2 k_1(x_0)\left((5-8\eta^2)(2S.S - \mathrm{tr}^2 S) + 3\left(\frac{1+b\gamma}{1+a\gamma}\right)^2 \mathrm{tr}^2 S\right)\right] + \mathcal{E}_4(\varepsilon) + \mathcal{E}_5(\varepsilon). \tag{55}$$

### 3.6 Partial variation of the penalty functional with respect to the domain

The last term is treated as follows:

$$\begin{aligned} VJ_2(\varepsilon) &:= J_\varepsilon(u_0) - J_0(u_0) \\ &= \int_{\omega_\varepsilon\cap\tilde{D}} (\gamma_1 - \gamma_0)\Phi(\Upsilon(\sigma(u_0)))dx \\ &= \pi\varepsilon^2\chi_{\tilde{D}}(x_0)(\gamma_1 - \gamma_0)(x_0)\Phi(\frac{1}{2}\tilde{B}S.S + 2\eta\overline{\sigma}\mathrm{tr}S) + \mathcal{E}_6(\varepsilon). \end{aligned} \tag{56}$$

### 3.7 Topological derivative

Like in (3) for the Laplace equation, we can prove that the reminders $\mathcal{E}_i(\varepsilon)$, $i = 1, ..., 6$ behave like $o(\varepsilon^2)$. Therefore, after summation of the different terms according to Proposition 3.1 and a few simplifications, we arrive at the final formula for the topological asymptotic expansion of the penalty functional. It is given by

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \varepsilon^2 D_T J_\Omega(x_0) + o(\varepsilon^2) \tag{57}$$

with the topological derivative

$$D_T J_\Omega = -\pi(\gamma_1 - \gamma_0)\left[\rho k_1 T(BS + \xi I).E + (\rho T - II)S.E_a\right]$$
$$+\pi\gamma_1\chi_{\tilde{D}}\Big[\Phi(\frac{1}{2}\tilde{B}S.S + 2\eta\overline{\sigma}\mathrm{tr}S - \rho(\tilde{B}S.TS + 2\eta\overline{\sigma}\mathrm{tr}(TS)) + \rho^2\frac{1}{2}\tilde{B}TS.TS) + \rho k_1(\tilde{B}S.TS + 2\eta\overline{\sigma}\mathrm{tr}(TS))\Big]$$
$$+\gamma_0\chi_{\tilde{D}}\left[\Psi_\rho(S) + \frac{1}{4}\pi\rho^2 k_1\left((5-8\eta^2)(2S.S - \mathrm{tr}^2 S) + 3\left(\frac{1+b\gamma}{1+a\gamma}\right)^2\mathrm{tr}^2 S\right)\right]$$
$$-\pi\chi_{\tilde{D}}\gamma_0\Phi(\frac{1}{2}\tilde{B}S.S + 2\eta\overline{\sigma}\mathrm{tr}S). \tag{58}$$

Formula (58) is valid for all $x_0 \in D \setminus \partial\tilde{D} \setminus \partial\Omega$. We recall that $\rho$ and $T$ are given by (42); $\tilde{B}$, $B$, $k_1$ and $\Psi_\rho(S)$ are respectively given by (6), (7), (25), (53) and $S = \sigma(u_0)$, $E = e(u_0)$, $E_a = e(v_0)$. Moreover, $u_0 = u_\Omega$ is the solution of the state equation (2) and $v_0 = v_\Omega$ is the solution of the adjoint equation (26) for $\varepsilon = 0$, that is,

$$\left\{\begin{array}{rcll} -\mathrm{div}\,(\gamma_0\sigma(v_0)) &=& +\mathrm{div}\,(\gamma_0 k_1(B\sigma(u_0) + \xi I)) & \text{in} \quad D, \\ v_0 &=& 0 & \text{on} \quad \Gamma_D, \\ \gamma_0\sigma(v_0)n &=& -\gamma_0 k_1(B\sigma(u_0) + \xi I)n & \text{on} \quad \Gamma_N \cup \Gamma_0, \end{array}\right. \tag{59}$$

with $\xi$ given by (8).

## 4. A TOPOLOGY DESIGN ALGORITHM

Given a real parameter $p \geq 1$, we consider the penalty function:

$$\Phi_p(t) = \Theta_p(\frac{t}{\bar{\sigma}^2}) \qquad \text{and} \qquad \begin{array}{rcl} \Theta_p : \mathbb{R}_+ & \rightarrow & \mathbb{R}_+, \\ t & \mapsto & (1+t^p)^{1/p} - 1. \end{array} \tag{60}$$

It is clear that this function satisfies the required assumptions. The penalized problem that we shall solve reads:

$$\underset{\Omega \subset D}{\text{Minimize}}\, I^\alpha_\Omega(u_\Omega) = I_\Omega(u_\Omega) + \alpha \int_{\tilde{D}} \gamma_\Omega \Phi_p(\Upsilon(\sigma(u_\Omega)))dx \qquad \text{subject to (2).} \tag{61}$$

In practice, $p$ must be chosen as large as possible, provided that the resolution of (61) can accommodate with the sharp variation of $\Theta'_p$ around 1. In all the numerical examples, we take the value $p = 32$ which, after several trials, proved to be a good compromise. In order to reduce the computer time, the function $\Psi_\rho$ has been tabulated. The unconstrained minimization problem (61) is solved by using the algorithm devised in (2). We briefly describe this algorithm here. It relies on a level-set domain representation and the approximation of topological optimality conditions by a fixed point method. Thus, the current domain $\Omega$ is characterized by a function $\psi \in L^2(D)$ such that $\Omega = \{x \in D, \psi(x) < 0\}$ and $D \setminus \bar{\Omega} = \{x \in D, \psi(x) > 0\}$. We compute the topological derivative $D_T I^\alpha(\Omega) = D_T I(\Omega) + \alpha D_T J(\Omega)$ where $D_T J(\Omega)$ is given by formula (58). Then we set $G(x) = D_T I^\alpha(\Omega)(x)$ if $x \in D \setminus \bar{\Omega}$ and $G(x) = -D_T I^\alpha(\Omega)(x)$ if $x \in \Omega$. We define the equivalence relation on $L^2(D)$: $\varphi \sim \psi \iff \exists \lambda > 0, \varphi = \lambda \psi$. Clearly, the relation $G \sim \psi$ is a sufficient optimality condition for the class of perturbations under consideration. We construct successive approximations of this condition by means of a sequence $(\psi_n)_{n \in \mathbb{N}}$ verifying

$$\psi_0 \in L^2(D) \quad \text{and} \quad \psi_{n+1} \in \text{co}(\psi_n, G_n)\ \forall n \in \mathbb{N}.$$

Above, the convex hull $\text{co}(\psi_n, G_n)$ applies to the equivalence classes, namely half-lines. Choosing representatives of unitary norm for $\psi_n$, $\psi_{n+1}$ and $G_n$, we obtain the algorithm:

$$\begin{aligned} &\psi_0 \in \mathcal{S}, \\ &\psi_{n+1} = \frac{1}{\sin\theta_n}\left[\sin((1-\kappa_n)\theta_n)\psi_n + \sin(\kappa_n\theta_n)G_n\right]\ \forall n \in \mathbb{N}. \end{aligned}$$

The notations are the following: $\mathcal{S}$ is the unit sphere of $L^2(D)$, $\theta_n = \arccos \frac{\langle G_n, \psi_n \rangle}{\|G_n\|\|\psi_n\|}$ is the angle between the vectors $G_n$ and $\psi_n$, and $\kappa_n \in [0,1]$ is a step which is determined by a line search in order to decrease the penalized objective functional. The iterations are stopped when this decrease becomes too small. At this stage, if the optimality condition is not approximated in a satisfactory manner (namely the angle $\theta_n$ is too large), an adaptive mesh refinement using a residual based a posteriori error estimate on the solution $u_{\Omega_n}$ is performed and the algorithm is continued.

## 5. NUMERICAL EXPERIMENTS

Given a fixed multiplier $\beta > 0$, we consider the objective functional

$$I_\Omega(u_\Omega) = \frac{K(u_\Omega)}{K_0} + \beta\frac{|\Omega|}{V_0}\ , \quad \text{with} \quad K(u_\Omega) = \int_{\Gamma_N} g \cdot u_\Omega\, ds\ , \tag{62}$$

where $|\Omega|$ is the Lebesgue measure of $\Omega$ and $K(u_\Omega)$ is the compliance. In addition, $V_0$ and $K_0$ are the area and the compliance associated to the hold-all domain $D$. The topological derivative of the area is obvious, namely,

$$D_T|\Omega| = -\beta\ , \tag{63}$$

and that of the compliance is known (1), which we repeat here for the sake of completeness

$$D_T K(u_\Omega) = -(\gamma_1 - \gamma_0)(\rho T - \mathbb{I})\sigma(u_\Omega) \cdot e(u_\Omega)\ . \tag{64}$$

In all the examples, the contrast $\gamma = 10^{-3}$ and the initial guess is the hold-all domain $D$.

### 5.1 Wall

Our first example is a wall under shear load (see Fig. 1). The computational domain is the rectangle of size $2 \times 1$ clamped on the bottom and the load $g = (1,0)$ is uniformly distributed along the line segment of length $0.2$. For comparison, we first address the von Mises stress constraint (*i.e.* $\eta = 0$). Then we compare with the Drucker-Prager stress

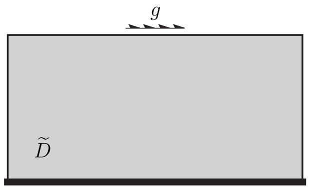



Figure 1. Wall: initial guess and boundary conditions.

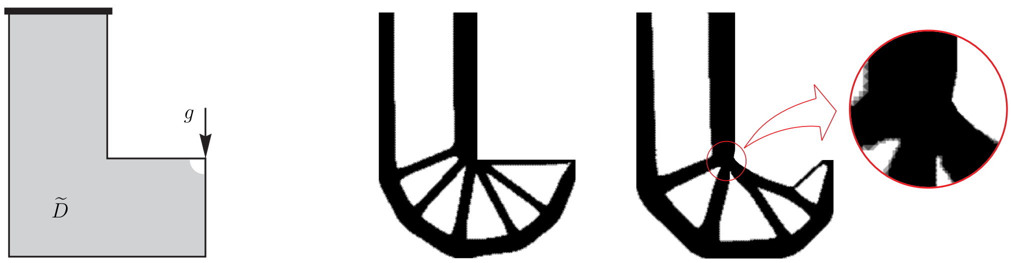
constraint for $\eta = 0.4$ and $\eta = -0.4$. In all cases we set $\overline{\sigma} = 1$, $\alpha = 100$ and $\beta = 4$. The material properties are $\gamma_0 = 1.0$ and $\nu = 0.3$. The initial mesh has 6400 elements. Then we perform one step of uniform mesh refinement, leading to 25600 elements.

(a) $\eta = 0.0$ (b) $\eta = 0.4$ (c) $\eta = -0.4$

Figure 2. Wall: obtained design for different values of $\eta$.

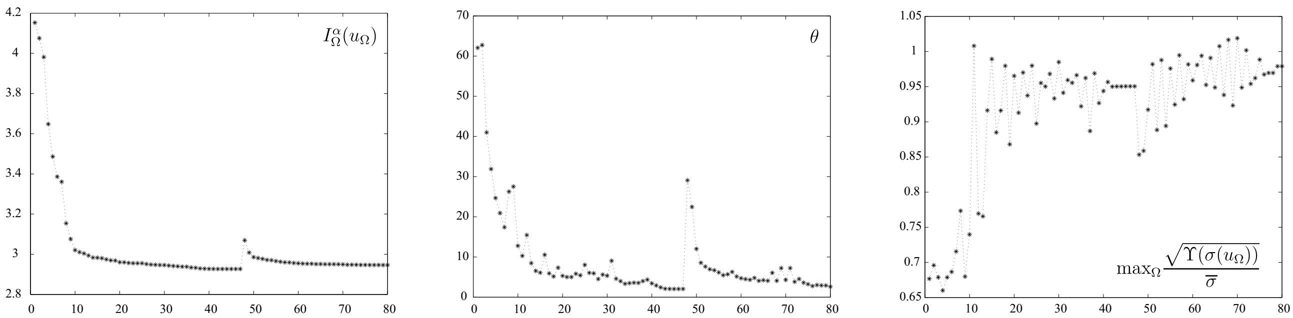
### 5.2 Hook

We now turn to a classical problem containing a geometrical singularity, namely the L-shaped beam (see Fig. 3a). The length of the two branches are 2.0 m and 2.5 m and their width are 1.0 m. The structure is clamped at the top, and a pointwise force $g = -(0, 40)$ KN/m is applied at the corner of the right tip. The material properties are $\gamma_0 = 12500$ MPa and $\nu = 0.2$. The stress constraint penalty function is not computed in the white region of radius 0.15 m. The initial mesh has 14236 elements, which is intensified at the reentrant corner. Then, we perform one step of uniform mesh refinement, leading to 58240 elements. We first show the result obtained for the unconstrained case ($\alpha = 0$). Then we take the parameters $\alpha = 10^5$, $\eta = -0.37027$ and admissible stress $\overline{\sigma} = 63.8546$ MPa. In both cases we set $\beta = 3$. We observe that, in the last case (Fig. 3c), the reentrant corner is rounded and the quantity $\max_\Omega \frac{\sqrt{\Upsilon(\sigma(u_\Omega))}}{\overline{\sigma}} = 0.98$. Yet, in the first case (Fig. 3b), the quantity $\max_\Omega \frac{\sqrt{\Upsilon(\sigma(u_\Omega))}}{\overline{\sigma}}$ blows-up when minimizing the compliance without stress constraint. The convergence history for the constrained case is presented in Fig. (4).




(a) initial guess (b) unconstrained (c) constrained

Figure 3. Hook: initial guess and boundary condition together with the obtained designs.




(a) shape function (b) angle $\theta$ (c) equivalent stress function

Figure 4. Hook: convergence history.

### 5.3 Bridge

In this last example the design of a bridge is considered, where the initial domain is given by a rectangle of $180\,\mathrm{m}$ of length and $60\,\mathrm{m}$ of high. The panel, clamped on the region $a = 9\,\mathrm{m}$, is submitted to a uniformly distributed traffic loading $g = -(0, 400)\,\mathrm{KN/m^2}$. This load is applied on the dark strip of height $b = 3\,\mathrm{m}$ positioned in a distance $c = 27\,\mathrm{m}$ from the top as shown in Fig. 5, which will not be optimized. The material properties are $\gamma_0 = 27500\,\mathrm{MPa}$ and $\nu = 0.2$. We first show the result obtained in the unconstrained case ($\alpha = 0$). Then we take the parameters $\alpha = 10^4$, $\eta = 0.417293$ and admissible stress $\overline{\sigma} = 5.04511\,\mathrm{MPa}$. In both cases we set $\beta = 10$. The stress constraint penalty function is not computed in the white region of size $15 \times 15\,\mathrm{m}^2$. Taking into account the symmetry od the problem, we discretize only a half part or the domain. The initial mesh has $4800$ elements. Then we perform two steps of uniform mesh refinement, leading to $76800$ elements. The result obtained for the unconstrained case is show in Fig. 6a, where we observe a well-known tie-arch bridge structure. However, the result obtained for the constrained case is quite different, namely, it eliminates the structural elements under uniaxial traction, which is reasonable from the practical point of view.

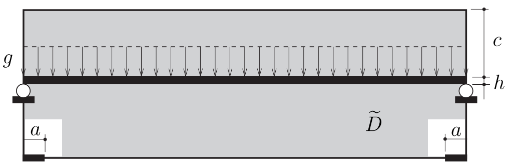
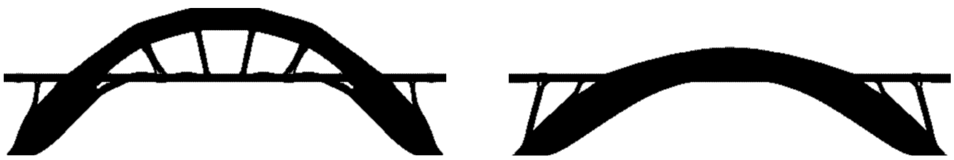



Figure 5. Bridge: initial guess and boundary conditions.



(a) unconstrained (b) constrained

Figure 6. Bridge: obtained design for the unconstrained and constrained cases.

## 6. CONCLUSIONS

In the paper we introduced a class of penalty functionals that mimic a pointwise constraint on the Drucker-Prager stress field. The associated topological derivative was obtained for plane stress linear elasticity. Then, a topology optimization algorithm was proposed allowing to treat local stress criteria taking into account different behaviors of the structure under traction or compression loading. These features were confirmed through the numerical experiments.

## 7. Responsibility notice

The authors are the only responsible for the printed material included in this paper.

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