Analyzing uncertainties in an energy pumping system

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Abstract. *The aim of this paper is to study the energy pumping (the irreversible energy transfer from one structure, linear, to another structure, nonlinear) robustness considering the uncertainties of the parameters of a two DOF mass-spring-damper, composed of two subsystems, coupled by a linear spring: one linear subsystem, the primary structure, and one nonlinear subsystem, the so-called NES (non-linear energy sink). Three parameters of the system will be considered as uncertain: the nonlinear stiffness, the damper from the nonlinear system and the damper from the linear subsystem. Random variables are associated to the uncertain parameters and probability density functions are constructed for the random variables applying the Maximum Entropy Principle. A sensitivity analysis is then performed, considering different levels of dispersion, and conclusions are obtained about the influence of the uncertain parameters in the robustness of the system.*
Keywords: *Energy pumping, uncertainties, modelling.*

1. INTRODUCTION

Energy pumping (EP) refers to a mechanism where energy is transferred in a one-way irreversible fashion from a source to a receiver. EP occurs in a wide range of both physical phenomena and engineering applications. In the context of passive vibration control of mechanical systems, it has been used to develop a new concept of nonlinear dynamic absorber. In this case, the energy pumping occurs from the main, or primary, structure, which needs to be protected, to the nonlinear absorber coupled with it. The nonlinear absorber, also named Nonlinear Energy Sink (NES), consists of a mass with a nonlinear spring. This concept involves nonlinear energy interactions which occur due to internal resonances making possible irreversible nonlinear energy transfers from the primary system to the attachment. The nonlinear energy pumping was first described in (Gendelman *et al.*, 2001; Vakakis and Gendelman, 2001). Comparing the NES with the corresponding linear dynamic absorber (also known as Frahm absorber or Helmholtz resonator), some interesting characteristics can be highlighted. The nonlinear absorbers operate in a large frequency band and not only with frequencies near the natural frequency of the primary system. Since the NES is nonlinear, these systems have no natural frequencies and they are effective for a large range of frequencies, while the linear absorbers attenuate well only one frequency. A complete description of the energy pumping phenomenon can be found in (Vakakis *et al.*, 2008).

Energy pumping phenomenon has been studied extensively in deterministic frameworks. However, very few studies have been devoted to analyse it in stochastic cases (Schmidt and Lamarque, 2009; Sapsis *et al.*, 2010). In this paper, uncertainties are taken into account to discuss the robustness of energy pumping. Some parameters of the model considered are taken as uncertain, random variables are associated to them and the corresponding probability density functions are constructed.

This paper is organized as follows: Section 2. presents the deterministic model which will be used to study the energy pumping phenomenon. Section 3. presents the parameters which will be considered uncertain and the methodology is applied to perform a stochastic analysis related to the robustness of the energy pumping. Finally, in Sec. 4. conclusions are outlined.

2. The deterministic model used

The system used to illustrate the energy pumping phenomenon and to discuss its robustness, taking into account uncertainties, has two-degrees-of-freedom and its sketch is shown in Fig. 1.

The system is composed of two subsystems (mass-spring-damper), coupled by a linear stiffness. The first subsystem, corresponding to the linear part (or primary system), is composed of the mass m_1, the linear spring k_1 and the linear damping c_1. The second subsystem, corresponding to the NES, is composed of the mass m_2, the cubic spring k_2 and the linear damping c_2. A linear spring γ couples the two subsystems. This configuration is referred as the grounded configuration because the NES is connected to the ground. The equations of motion are given by Eq.1:

$$\begin{cases} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + \gamma(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2^3 + \gamma(x_2 - x_1) = 0 \end{cases} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ denote the displacements of the primary system and the NES, respectively.

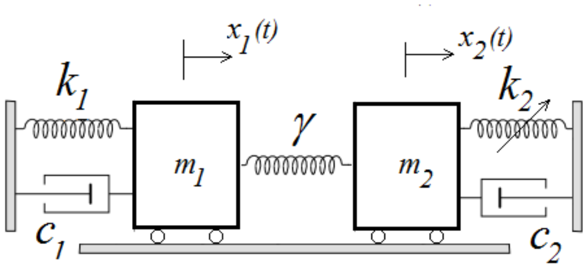

Figure 1. Energy pumping model.

The basic values of the parameters which will be used in simulations are: $m_1 = 1000\,kg$, $m_2 = 1000\,kg$, $k_1 = 1000\,N/m$, $k_2 = 150\,N/m^3$, $c_1 = 25\,Ns/m$, $c_2 = 75\,Ns/m$ and $\gamma = 250\,N/m$. Only free responses associated to impulsive excitation of the primary system will be analyzed, which corresponds to the following initial conditions: $x_1 = 0$, $x_2 = 0$, $\dot{x}_1 = \sqrt{2h/m_1}$ and $\dot{x}_2 = 0$, where h corresponds to the initial energy given to the system.

Energy pumping occurs when the value of the initial injected energy is above a specific value. Energy pumping is noted, especially at early times of the response, when the energy of the system is relatively high and the non-linear effects are profound. Figure 2 shows the displacement of the masses m_1 and m_2 considering two simulations, one of them considering low level of initial energy ($h = 5$) (Fig. 2 (a) and (b)), no energy pumping occurs, and the other considering high level of initial energy ($h = 14$) (Fig. 2 (c) and (d)), energy pumping occurs. The plots corresponding to the case when there is no coupling are also shown with dotted lines.

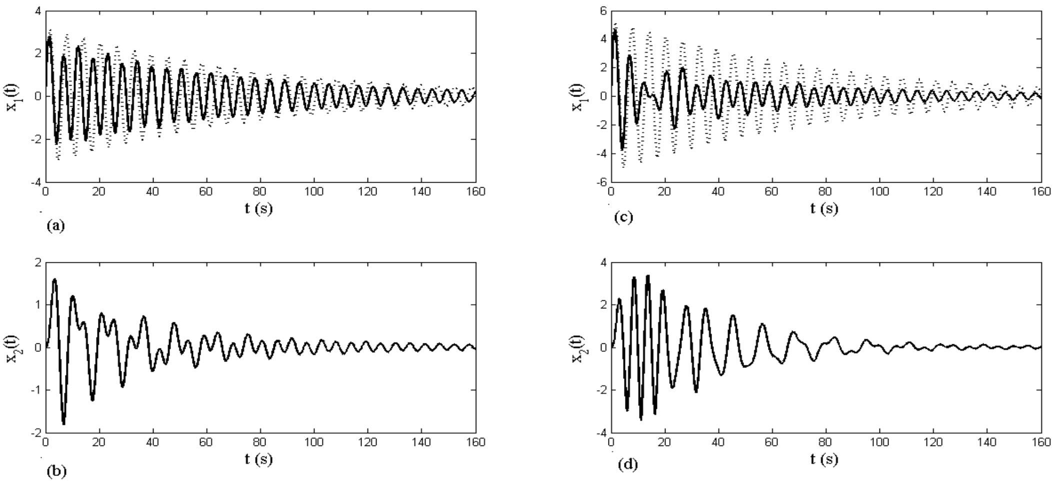

Figure 2. (i) $h = 5$ (no energy pumping occurs) (a) Displacement $x_1(t)$ of the linear oscillator: dotted line denotes the case without coupling ($\gamma = 0$) and the solid line denotes the case with coupling ($\gamma = 250$) and (b) Displacement $x_2(t)$ of the nonlinear oscillator. (ii) $h = 14$ (energy pumping occurs) (c) Displacement $x_1(t)$ of the linear oscillator and (d) Displacement $x_2(t)$ of the nonlinear oscillator.

Now, the initial energy is considered too high. Two cases are discussed: $h = 5$ and $h = 100$. In both cases the energy pumping do not occurs. The corresponding plots are shown in Fig. 3.

In this case, the energy pumping is not observed anymore. Then, one can say that the level cannot be increased indiscriminately in order to obtain better energy pumping performance, there is an optimum value to it. However, the energy pumping phenomenon has been presented here by using plots.

A natural question which appears is how the energy pumping can be measured. An idea is to measure the energy (stored and dissipated) related to each one of the oscillators.

A measure considered here will be called *Total Energy* and it is composed of the energy stored and the energy dissipated in each mass.

The total energy related to the linear oscillator will be denoted by ET_1 and it will be given by Eq. 2:

$$ET_1 = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}k_1x_1^2 + c_1\int_0^t \dot{x}_1^2(t)dt\,. \qquad (2)$$

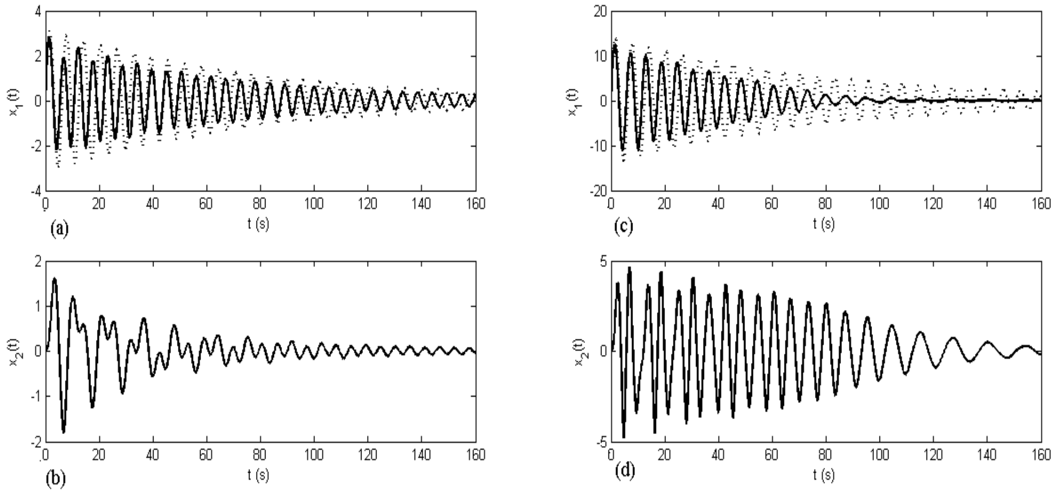

Figure 3. (a) Displacement $x_1(t)$ of the linear oscillator ($h = 5$): dotted line denotes the case without coupling ($\gamma = 0$) and the solid line denotes the case with coupling ($\gamma = 250$). (b) Displacement $x_2(t)$ of the nonlinear oscillator ($h = 5$). (c) Displacement $x_1(t)$ of the linear oscillator ($h = 100$). (d) Displacement $x_2(t)$ of the nonlinear oscillator ($h = 100$).

The total energy related to the nonlinear oscillator will be denoted by ET_2 and it will be given by Eq. 3:

$$ET_2 = \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k_2\frac{x^4}{4} + c_2\int_0^t \dot{x}_2^2(t)dt \tag{3}$$

It is important to say that the term related to the energy stored by the coupling stiffness is not considered because it should be present in both equations and no to add it will be cause the same effect in terms of energy pumping measure.

Three different values of the initial energy are considered for the simulations $h = 5$, $h = 14$ and $h = 100$ and the corresponding plots are shown in Fig. 4.

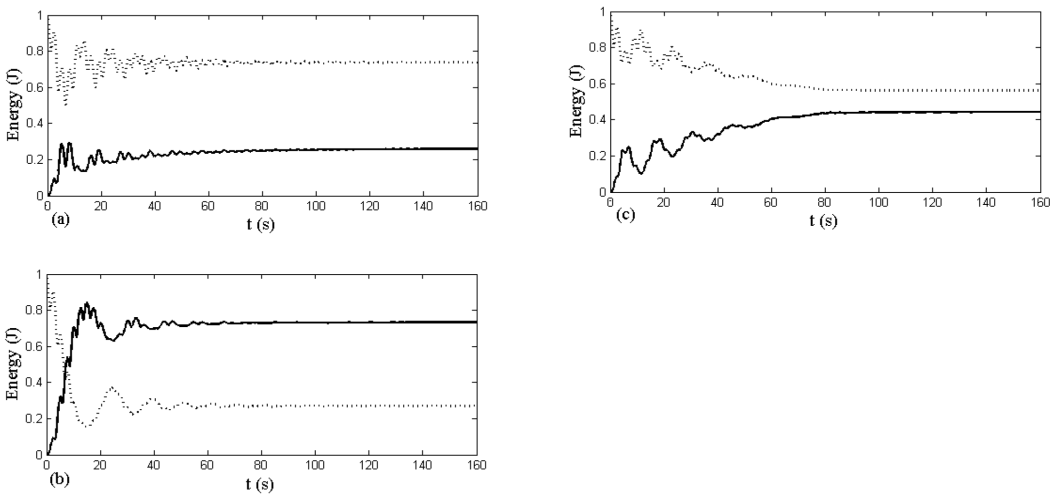

Figure 4. Total energy related to the linear and to the nonlinear system, considering three different cases for the initial energy given by h. ET_1 is represented by the dotted line and ET_2 by the solid line. (a) $h = 5$, no energy pumping. (b) $h = 14$ energy pumping occurs. (c) $h = 100$, no energy pumping.

It is not enough that the nonlinear oscillator move itself to say that the energy pumping is ocurring. To characterize the energy pumping phenomenon, it is important to observe that the displacement of the linear mass is largerly diminished whereas the displacement of the nonlinear mass is largerly increased, even during a short period of time. Considering the values of ET_1 and ET_2, it can be noted that when no energy pumping occurs, the value of ET_1 is greater than the value of ET_2, during all the time, which does not happen when the energy pumping phenomenon is present, as shown in Fig. 4 (b). Perhaps, a simpler way to discuss the relation between the energies (ET_1 and ET_2) is to evaluate the ratio $\frac{ET_2}{ET_1}$, which corresponding plots are shown in Fig. 5.

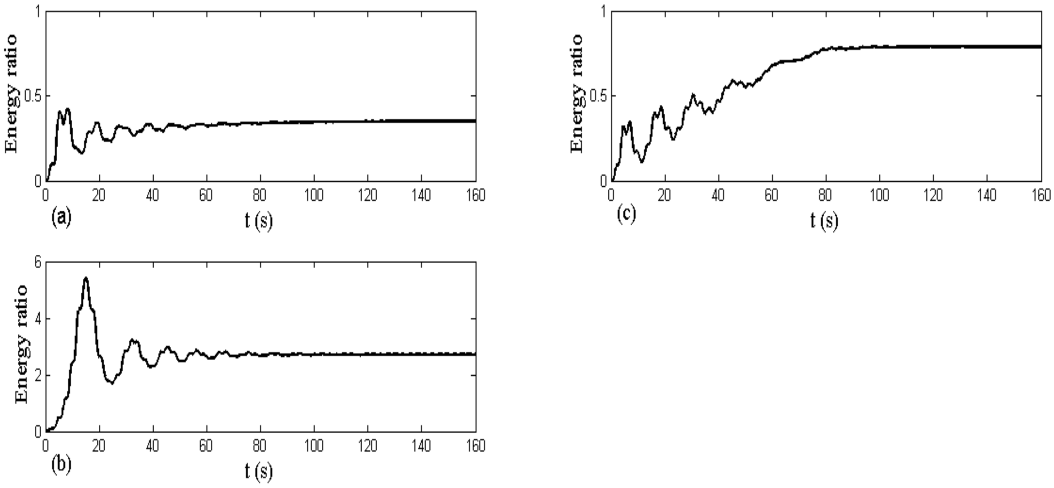

Figure 5. Ratio between energies $\frac{ET_2}{ET_1}$, considering three different cases for the initial energy. (a) $h = 5$, no energy pumping. (b) $h = 14$ energy pumping occurs. (c) $h = 100$, no energy pumping.

It can be observed that in Fig. 5, cases (a) and (c) the graph is always below the value 1, case where no energy pumping occurs, and in Fig. 5, case (b), the graph crosses the value 1.

Thus, verifying the robustness of the energy pumping, taking into account the uncertainties in some parameters, seems to be a good idea.

Considering a deterministic system, it was possible to discuss the principles of the energy pumping. However, the main objective here is to discuss the robustness of energy pumping taking into account the uncertainties of some parameters of the system. Then, the concepts discussed will be used when the system is not deterministic anymore, but when parameters are substituted by random variables, in order to take into account uncertainties of the parameters.

3. Energy pumping robustness

3.1 General considerations

The energy pumping phenomenon happens due to a resonance 1:1 (the two structures oscillate at the same frequency), that is, a resonance state is produced between the nonlinear oscillator with one linear mode and then the energy is transferred to the nonlinear system in a irreversible way.

Hence, it is important to know if the energy pumping can be produced even when the parameters are uncertain, in order to apply the theory to real structures, where the nonlinear structure annexed does not reflect perfectly the theoretical conception and the problems related to the nonlinear identification appear.

The aim of this section is to study the energy pumping when uncertainties are present; that is, when the parameters are not well known. For example, the damping (in general, damping is not well known) plays an important role in the development of the energy pumping phenomenon. As the nonlinear system does not reflect perfectly the concepts and the problems related to the nonlinear identification appear, the uncertainties stay inside the nonlinearities Gourdon (2006).

3.2 Stochastic modeling

Herein, three system parameters will be considered as uncertain: (i) k_2, the nonlinear stiffness, (ii) c_1, the damping corresponding to the linear system, and (iii) c_2 the damping corresponding to the nonlinear system. Random variables K, C_1 and C_2 will be associated to the uncertain parameters and probability density functions constructed to these random variables using the Maximum Entropy Principle, which states that out of all probability density distributions consistent with a given set of available information, the one with the maximum uncertainty (entropy) is chosen.

Let X be a random variable. The entropy associated to X is defined by Eq. 4:

$$S(p_X) = -\int_{-\infty}^{+\infty} p_X(x)\ell n(p_X(x))dx. \tag{4}$$

The goal is to maximize S under the constraints defined by some available information on the random variable X.

3.3 Stochastic solver for the uncertain system

The stochastic system is constructed from the corresponding deterministic one substituting k_2, c_1 or c_2 by the random variable K, C_1 or C_2, respectively. The stochastic solver used is based on the Monte Carlo method.

Each random variable will be substituted separately by the corresponding uncertain parameter in the determinist system.

Let X be each one of the random variables K, C_1 or C_2. The following steps will be performed for each random variable.

(1) A probability density function is constructed using the Maximum Entropy Principle.

(2) Independent realizations $X(\theta)$ of the random variable X are constructed using the associated probability density function obtained in step (1). For each realization $X(\theta)$, the system of differential equation given by Eq. 1 is solved and the stochastic processes $X_1(t)$ and $X_2(t)$, associated to the displacements $x_1(t)$ and $x_2(t)$ of the masses m_1 and m_2, are obtained.

(3) Plots of the realizations $X_1(t,\theta)$ and $X_2(t,\theta)$ are constructed and also their mean value and standard deviation.

(4) Confidence intervals corresponding to the ratio $\dfrac{ET_2}{ET_1}$ are plotted considering different values for the dispersion coefficient. The confidence interval associated with a specific probability level is constructed using quantiles **?**Cataldo *et al.* (2009). Herein, the value used for the confidence interval is 0.95.

3.3.1 Parameter k_2 chosen as uncertain

The first parameter to be considered as uncertain is the nonlinear stiffness k_2. The random variable K is associated to this parameter and the following information is considered as available: (1) the support of the probability density function is $]0,+\infty[$, (2) the mean value which is known, $E[K]=\underline{K}$ and (3) the condition $E\{\ell n(K)\}<+\infty$ which implies that zero is a repulsive value.

The probability density function p_K has then to verify the following constraint equations (Soize, 2001; Cataldo *et al.*, 2009):

$$\int_{-\infty}^{+\infty} p_K(k)dk = 1 \ , \ \int_{-\infty}^{+\infty} k p_K(k)dk = \underline{K} \ , \ \int_{-\infty}^{+\infty} \ell n(K) p_K(k)dk < +\infty \tag{5}$$

Applying the Maximum Entropy Principle yields the following probability density function for K:

$$p_K(k) = \mathbf{1}_{]0,+\infty[}(k)\frac{1}{\underline{K}}\left(\frac{1}{\delta_K^2}\right)^{\frac{1}{\delta_K^2}} \times \frac{1}{\Gamma\left(1/\delta_K^2\right)}\left(\frac{q}{\underline{K}}\right)^{\frac{1}{\delta_K^2}-1} exp\left(-\frac{k}{\delta_K^2\underline{K}}\right) \ , \tag{6}$$

where $\delta_K = \dfrac{\sigma_K}{\underline{K}}$ is the coefficient of dispersion of the random variable K such that $\delta_K < \dfrac{1}{\sqrt{2}}$ and σ_K is the standard deviation of K. It can be verified that K is a second-order random variable and that $E\{1/K^2\} < +\infty$.

Figure 6 shows the realizations of the displacement of the two masses for $\delta_K = 0.05$.

It can be noted that the energy pumping phenomenon is produced and the energy pumping is effective for all the realizations (see Fig. 6 (top)). The mean values of $x_1(t)$ and $x_2(t)$ are characterized by two different behaviors. In the initial phase (0 < t < 15), the mean displacements of the two oscillators are almost in-phase, the oscillation amplitude of the mean displacement of the NES is large and the decay of the primary system amplitude is linear and very fast compared with the single mass-spring-damper system . This behavior recalls the energy pumping condition, characterized by an irreversible transfer of energy from the primary system to the nonlinear subsystem (energy localization in the NES), where it is dissipated. In the second phase (t > 15), the behavior of the two oscillators is similar to that of the weakly excited case, showing out-of-phase displacements. Note that in this case, the standard deviations of $x_1(t)$ and $x_2(t)$ are small, although near $t = 20s$ their values are higher.

It is interesting to analyze the confidence interval of the energy ratio (ET_2/ET_1) when the dispersion coefficient varies (Fig. 7).

When the dispersion coefficient has a low value (0.05) it can be noted that all the realizations cross the value 1, approximately after the first 15 seconds, and remain above the value 1 during all the time indicating that the energy pumping occurs. Increasing the dispersion coefficient, just a little (0.1), it can be observed that the confidence interval is larger and some realizations remain under the value 1 during all the time, indicating that the energy pumping does not

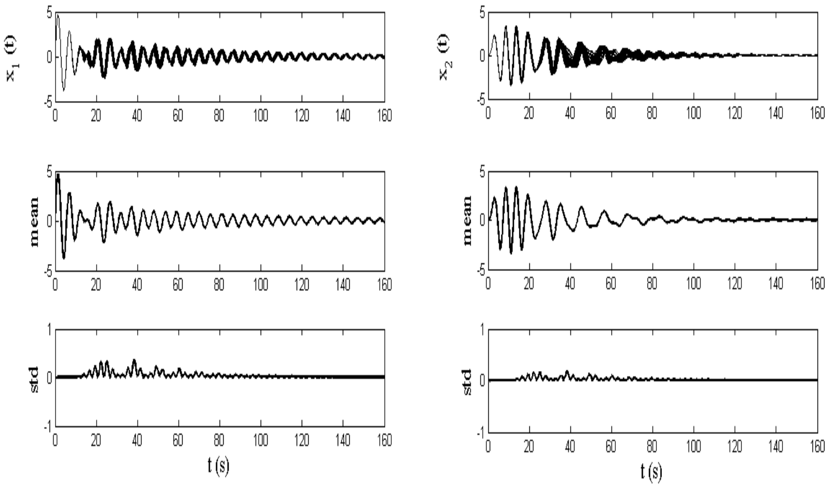

Figure 6. Numerical results for $\underline{K} = 150$ and $\delta_K = 0.05$. (a) Realizations of the displacement of the mass. (b) Mean value of the realizations. (c) Standard deviation of the realizations. Left: nonlinear system. Right: linear system.

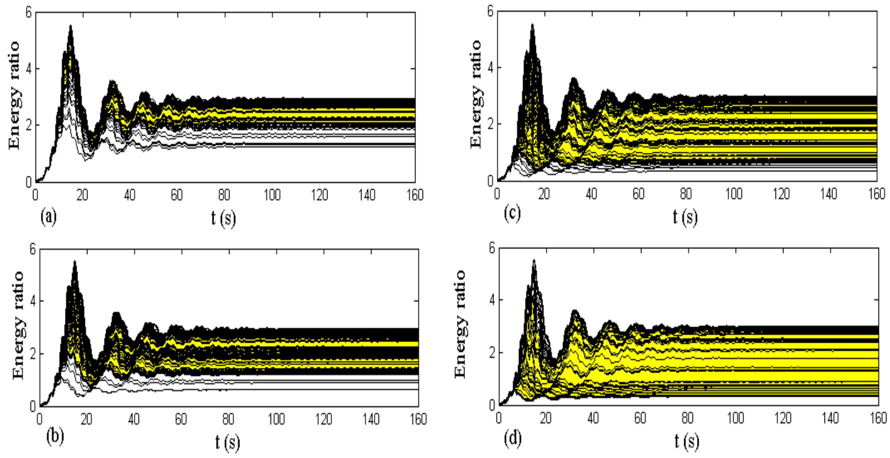

Figure 7. Confidence interval related to the energy ratio for different values of the dispersion coefficient. (a) $\delta_K = 0.05$. (b) $\delta_K = 0.1$. (c) $\delta_K = 0.2$. (d) $\delta_K = 0.3$

occur anymore. Increasing a little bit more the dispersion coefficient (0.2 and 0.3) more realizations appear under the value 1.

3.3.2 Parameter c_1 chosen as uncertain

The next parameter consider as uncertain is the linear damping c_1 and the random variable C_1 is assigned to it. The available information is the same as considered for K and, consequently, using the Maximum Entropy Principle, the corresponding probability density function constructed will have the same expression as the one constructed for K (Eq. 6), substituting K by C_1.

Figure 8 shows the realizations of the displacement of each mass and the corresponding mean value and standard deviation, considering $\delta_{C_1} = 0.5$ as the level of dispersion.

In this case, it can be observed that the energy pumping occur for all of the realizations and the standard deviation has a very low value, indicating that the system is very robust for this level of uncertainty for C_1. Then, to enable a comparison, the level of uncertainty is increased up to 0.3 and the realizations obtained are shown in Fig. 9 . The corresponding mean value and standard deviation are shown as well.

With this level of uncertainty, it is not difficult to observe that the standard deviation has its value increased with a significant value, but it is not so clear to verify if the energy pumping occurs or not for all the realizations. The confidence interval for the realizations of the energy ratio is plotted for different values of the dispersion coefficient and the results

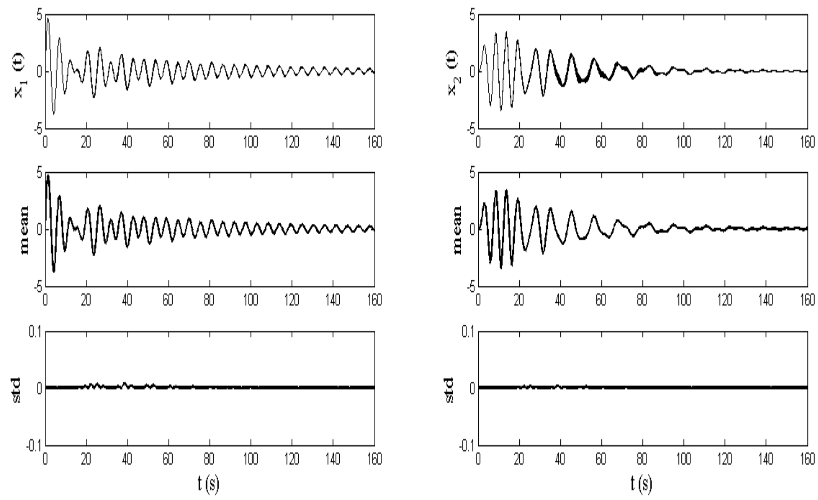

Figure 8. Numerical results for $\underline{C_1} = 25$ and $\delta_{C_1} = 0.05$. (a) Realizations of the displacement of the mass. (b) Mean value of the realizations. (c) Standard deviation of the realizations. Left: nonlinear system. Right: linear system.

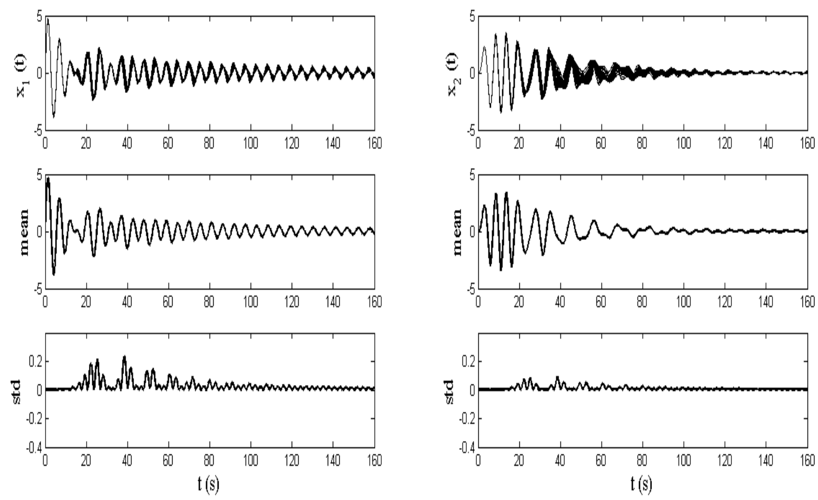

Figure 9. Numerical results for $\underline{C_1} = 25$ and $\delta_{C_1} = 0.3$. (a) Realizations of the displacement of the mass. (b) Mean value of the realizations. (c) Standard deviation of the realizations. Left: nonlinear system. Right: linear system.

are shown in Fig. 10.

In this case, the confidence interval seem to be considerable large only when the dispersion coefficient value is 0.3 and maybe in this case the energy pumping cannot occur for some realizations, indicating that the system is very robust in relation to variations of the uncertain parameter c_1.

3.3.3 Parameter c_2 chosen as uncertain

The next parameter to be considered as uncertain is the linear damping c_1 associated to the nonlinear subsystem, and the random variable C_2 is assigned to it. The available information is the same as considered for C_1 and, consequently, using the Maximum Entropy Principle, the corresponding probability density function constructed will have the same expression as the one constructed for K (Eq. 6), substituting K by C_2.

Figure 11 shows the realizations of the displacement of each mass and the corresponding mean value and standard deviation, considering $\delta_{C_2} = 0.5$ as the level of dispersion.

The system appear also to be robust to this level of variations of C_2, as it happened to C_1, but even at this low level of dispersion it is possible to observe that the standard deviation has a value a little bit larger, when compared with C_1. The level of uncertainty is then increased and the results are shown in Fig. 12. In this case, it is not difficult to observe that the standard deviation has a value much larger then the one obtained for C_1 and that for some realizations the energy pumping does not occur.

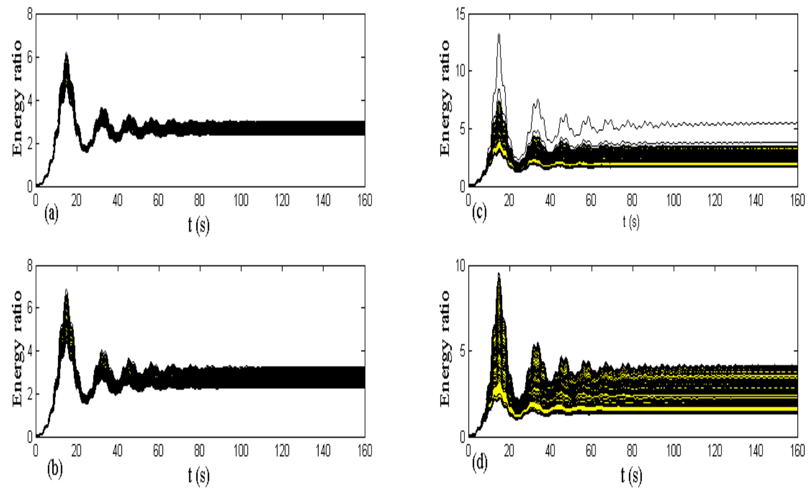

Figure 10. Confidence interval related to the energy ratio for different values of the dispersion coefficient. (a) $\delta_{C_1} = 0.05$. (b) $\delta_{C_1} = 0.1$. (c) $\delta_{C_1} = 0.2$. (d) $\delta_{C_1} = 0.3$

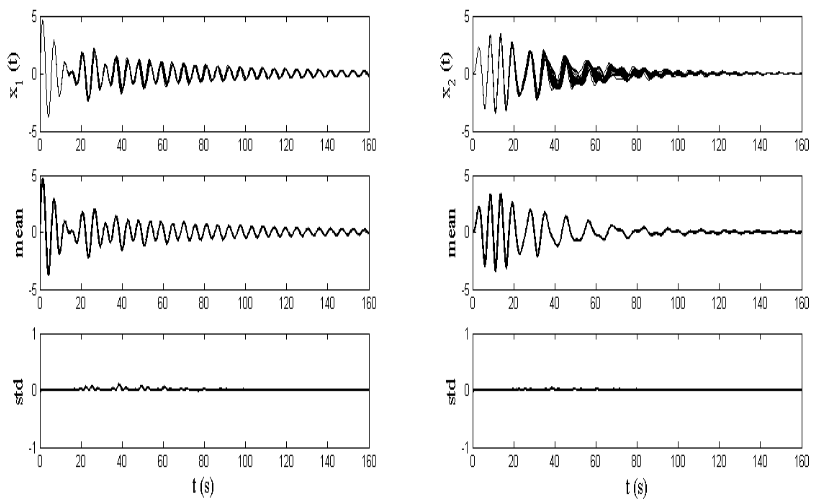

Figure 11. Numerical results for $\underline{C_2} = 75$ and $\delta_{C_2} = 0.05$. (a) Realizations of the displacement of the mass. (b) Mean value of the realizations. (c) Standard deviation of the realizations. Left: nonlinear system. Right: linear system.

Figure 13 shows the confidence interval related to the energy ratio for different values of the dispersion coefficient δ_{C_2}.

One can say that the system is still robust considering the uncertainties of this parameter, but less robust when compared with values obtained when the random variable C_1 is considered.

4. CONCLUSIONS

This paper analyzes energy pumping using a system composed of two subsystems, one linear and another nonlinear, to discuss the energy pumping phenomenon. The main objective was to take into account uncertainties in some parameters of the system and to analyze the robustness of the energy pumping.

Three parameters were considered as uncertain: the coupling stiffness and the two dampers of the linear and the nonlinear subsystems. Probability density functions were assigned to random variables, related to these parameters.

The main conclusions obtained were that the system is more robust when uncertainties related to the dampers are taken into account, because with a greater level of dispersion in this parameter, the energy pumping phenomenon could still be observed. For the same level of dispersion, the effects of the three random variables were compared and the system results were more sensitive to variations of the random variable associated to the stiffness coupling.

The displacements of the linear subsystem are less sensitive to the variations of the uncertain parameter than the displacements of the nonlinear subsystem.

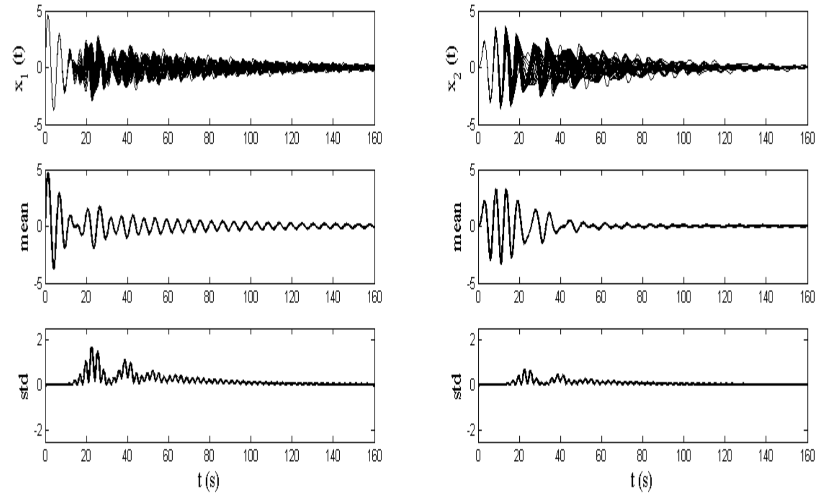

Figure 12. Numerical results for $\underline{C_2} = 75$ and $\delta_{C_2} = 0.3$. (a) Realizations of the displacement of the mass. (b) Mean value of the realizations. (c) Standard deviation of the realizations. Left: nonlinear system. Right: linear system.

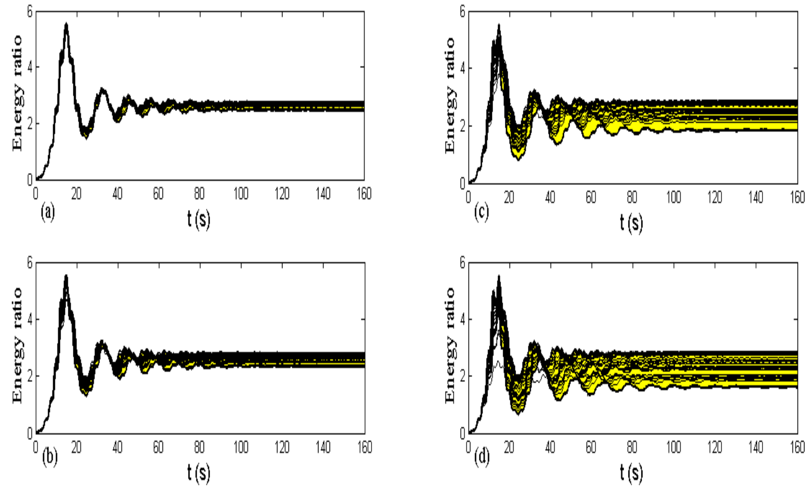

Figure 13. Confidence interval related to the energy ratio for different values of the dispersion coefficient. (a) $\delta_{C_2} = 0.05$. (b) $\delta_{C_2} = 0.1$. (c) $\delta_{C_2} = 0.2$. (d) $\delta_{C_2} = 0.3$

An idea for a future study is to consider other parameters as uncertain and employ other methodologies to take it into account. Other quantities can also be analyzed, for example, the energy variation of the system.

5. ACKNOWLEDGMENTS

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