TERRESTRIAL ARTIFICIAL SATELLITES DYNAMICS: RESONANT EFFECTS

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Abstract. *In this work, the resonance problem in the artificial satellites motion is studied. The development of the geopotential includes the zonal harmonics J_{20} and J_{40} and the tesseral harmonics J_{22} and J_{42}. Through successive Mathieu transformations, the order of dynamical system is reduced and the final system is solved by numerical integration. In the dynamical model, six critical angles, associated to the tesseral harmonics J_{22} and J_{42}, are studied together. Numerical results show the time behavior of the semi-major axis, argument of pericentre and of the eccentricity.*

Keywords: *Artificial Satellites, Orbits, Resonance*

1. INTRODUCTION

Synchronous satellites in circular or elliptical orbits have been extensively used for navigation, communication and military missions. This fact justifies the great attention that has been given in literature to the study of resonant orbits characterizing the dynamics of these satellites since the 60's (Blitzer, 1963; Ely and Howell, 1996; Garfinkel, 1965, 1966; Gedeon et al., 1967; Gedeon, 1969; Jupp, 1969; Lane, 1988; Morando, 1963). For example, Molniya series satellites used by the old Soviet Union for communication form a constellation of approximately 110 satellites, launched since 1965, which have highly eccentric orbits with periods of 12 hours. Another example of missions that use eccentric orbits, inclined and synchronous, include satellites to investigate the solar magnetosphere, launched in the 90's (Neto, 2006).

The dynamics of synchronous satellites is very complex. The tesseral harmonics of the geopotential produce multiple resonances which interact resulting significantly nonlinear motions, when compared to non-resonant orbits. It has been found that the orbital elements show relatively large oscillation amplitudes differing from neighboring trajectories, they are in fact chaotic (Ely and Howell, 1996). It should also be noted that the characteristics of several missions involving such orbits require that they are kept to a minimum fuel consumption. Geographic requirements determined by the missions and spatial maneuvers of minimum cost demand precise control of the trajectories that are subjected to significant nonlinearities during the satellite lifetime.

In this paper, the 2:1 resonance is considered; in other words, the satellite completes two revolutions while the Earth carries one.

2. HAMILTONIAN AND MOTION EQUATIONS

In this section, a Hamiltonian describing the resonant problem is derived through sucessive Mathieu transformations. Consider Eq. (1) to the Earth gravitational potential written in classical orbital elements (Osorio, 1973; Kaula, 1966)

$$V = \frac{\mu}{2a} + \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=+\infty}^{-\infty} \frac{\mu}{a} \left(\frac{a_e}{a}\right)^l J_{lm} F_{lm}(I) G_{lpq}(e) cos(\phi_{lmpq}(M, \omega, \Omega, \Theta)). \tag{1}$$

where μ is the Gaussian constant, μ=3.986009 x 10^{14} m^3/s^2, a, e, I, Ω, ω, M are the classical keplerian elements: a is the semi-major axis, e is the eccentricity, I is the inclination of the orbit plane with the equator, Ω is the longitude of the ascending node, ω is the argument of pericentre and M is the mean anomaly, respectively; a_e is the Earth mean equatorial radius, a_e=6378.140 km, J_{lm} is the spherical harmonic coefficient of degree l and order m, $F_{lmp}(I)$ and $G_{lpq}(e)$ are Kaula's inclination and eccentricity functions, respectively. The argument $\phi_{lmpq}(M, \omega, \Omega, \Theta)$ is defined by

$$\phi_{lmpq}(M, \omega, \Omega, \Theta) = qM + (l - 2p)\omega + m(\Omega - \Theta - \lambda_{lm}) + (l - m)\frac{\pi}{2} \ .$$

where Θ is the Greenwich sidereal time and λ_{lm} is the corresponding reference longitude along the equator.

In order to describe the problem in Hamiltonian form, Delaunay canonical variables are introduced

$$L = \sqrt{\mu a} \qquad G = \sqrt{\mu a(1-e^2)} \qquad H = \sqrt{\mu a(1-e^2)}cos(I)$$

$$l = M \qquad g = \omega \qquad h = \Omega. \tag{2}$$

Using the canonical variables, one gets the Hamiltonian $\hat{F}$,

$$\hat{F} = \frac{\mu^2}{2L^2} + \sum_{l=2}^{\infty}\sum_{m=0}^{l} R_{lm} \ , \tag{3}$$

with the disturbing potential R_{lm} given by

$$R_{lm} = \sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty} B_{lmpq}(L,G,H)cos(\phi_{lmpq}(l,g,h,\Theta)) \ . \tag{4}$$

The argument ϕ_{lmpq} is defined by

$$\phi_{lmpq}(l,g,h,\Theta) = ql + (l-2p)g + m(h - \Theta - \lambda_{lm}) + (l-m)\frac{\pi}{2} \ , \tag{5}$$

and the coefficient $B_{lmpq}(L,G,H)$ by

$$B_{lmpq} = \sum_{l=2}^{\infty}\sum_{m=0}^{l}\sum_{p=0}^{l}\sum_{q=+\infty}^{-\infty} \frac{\mu^2}{L^2}\left(\frac{\mu a_e}{L^2}\right)^l J_{lm}F_{lmp}(L,G,H)H_q^{-(l+1),(l-2p)}(L,G). \tag{6}$$

The Hamiltonian $\hat{F}$ depends explicitly on the time through the Greenwich sidereal time Θ, where $\Theta = \Omega_e t$ (Ω_e is the Earth's angular velocity and t is the time). A new variable θ, conjugated to Θ, is introduced in order to extend the phase space. In the extended phase space, the extended Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \hat{F} + \omega_e\theta. \tag{7}$$

For resonant orbits, it is convenient to use a new set of canonical variables. Consider the canonical transformation of variables defined by the following relations

$$X = L \qquad Y = G - L \qquad Z = H - G \qquad \Theta = \Theta$$

$$x = l + g + h \qquad y = g + h \qquad z = h \qquad \theta = \theta \ , \tag{8}$$

where $X, Y, Z, \Theta, x, y, z, \theta$ are the modified Delaunay variables.

The new Hamiltonian $\hat{H}'$, resulting from the canonical transformation defined by Eqn (8), is given by

$$\hat{H}' = \frac{\mu^2}{2X^2} + \omega_e\theta + \sum_{l=2}^{\infty}\sum_{m=0}^{l} R'_{lm} \ , \tag{9}$$

where the disturbing potential R'_{lm} is given by

$$R'_{lm} = \sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty} B'_{lmpq}(X,Y,Z)cos(\phi_{lmpq}(x,y,z,\Theta)). \tag{10}$$

Consider the resonance to be studied in this work; that is, the commensurability between the Earth rotation angular velocity Ω_e and the mean motion n. This commensurability can be expressed as

$$qn - m\omega_e \cong 0 \tag{11}$$

considering q and m as integers. The commensurability of the resonance studied, q/m, is defined by α. When this commensurability ocurrs, small divisors, associated to the tesseral harmonics, arise in the integration of the equations of motion (Lane, 1988). These terms are called resonants.

The short and long period terms can be eliminated from the Hamiltonian $\hat{H}'$ by applying an averaging method. A reduced Hamiltonian $\hat{H}_r$ is obtained from the Hamiltonian $\hat{H}'$ when only secular and resonant terms are considered. Several authors, Lima Jr. (1996), Grosso (1989), Ely and Howell (1996) and Neto (2006) also use this simplified Hamiltonian to study the resonance. The reduced Hamiltonian $\hat{H}_r$ is given by

$$\begin{aligned} \hat{H}_r &= \frac{\mu^2}{2X^2} + \omega_e\theta + \sum_{j=1}^{\infty} B'_{2j,0,j,0}(X,Y,Z) + \\ &+ \sum_{l=2}^{\infty}\sum_{m=2}^{l}\sum_{p=0}^{l} B'_{lmp(\alpha m)}(X,Y,Z)cos(\phi_{lmp(\alpha m)}(x,y,z,\Theta)). \end{aligned} \tag{12}$$

The canonical system of differential equations governed by $\hat{H}_r$ has the first integral

$$\left(1 - \frac{1}{\alpha}\right) X + Y + Z = C_1 \tag{13}$$

where C_1 is an integration constant.

Using this first integral, a Mathieu transformation

$$(X, Y, Z, \Theta, x, y, z, \theta) \rightarrow (X_1, Y_1, Z_1, \Theta_1, x_1, y_1, z_1, \theta_1)$$

can be defined.

This transformation is given by the following equations

$$X_1 = X \qquad Y_1 = Y \qquad Z_1 = \left(1 - \frac{1}{\alpha}\right) X + Y + Z \qquad \Theta_1 = \Theta$$

$$x_1 = x - \left(1 - \frac{1}{\alpha}\right) z \qquad y_1 = y - z \qquad z_1 = z \qquad \theta_1 = \theta. \tag{14}$$

The subscript 1 denotes the new set of canonical variables. Note that Z_1=C_1 and the z_1 is an ignorable variable. So, the order of the dynamical system is reduced in one degree of freedom.

Substituting the new set of canonical variables, $X_1, Y_1, Z_1, \Theta_1, x_1, y_1, z_1, \theta_1$, in the reduced Hamiltonian given by Eqn. (12), one gets the resonant Hamiltonian. The word "resonant" is used to denote the Hamiltonian $\hat{H}_{1,rs}$ which is valid for any resonance. The periodic terms in this Hamiltonian are resonant terms. The Hamiltonian $\hat{H}_{1,rs}$ is given by

$$\begin{aligned} \hat{H}_{1,rs} &= \frac{\mu^2}{2X_1^2} + \omega_e\theta_1 + \sum_{j=1}^{\infty} B_{1,2j,0,j,0}(X_1,Y_1,Z_1) + \\ &+ \sum_{l=2}^{\infty}\sum_{m=2}^{l}\sum_{p=0}^{l} B_{1,lmp,(\alpha m)}(X_1,Y_1,Z_1)cos(\phi_{1,lmp(\alpha m)}(x_1,y_1,z_1,\Theta_1)). \end{aligned} \tag{15}$$

The Hamiltonian $\hat{H}_{1,rs}$ has all resonant frequencies, relative to the commensurability α, where the $\phi_{1,lmp(\alpha m)}$ argument is given by

$$\phi_{1,lmp(\alpha m)} = m(\alpha x_1 - \Theta_1) + (l - 2p - \alpha m)y_1 - \phi_{1,lmp(\alpha m)0} \ , \tag{16}$$

with

$$\phi_{1,lmp(\alpha m)0} = m\lambda_{lm} - (l-m)\frac{\pi}{2}. \tag{17}$$

The secular and resonant terms are given, respectively, by $B_{1,2j,0,j,0}(X_1, Y_1, Z_1)$ and $B_{1,lmp(\alpha m)}(X_1, Y_1, Z_1)$.

Each one of the frequencies contained in $\frac{dx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{d\Theta_1}{dt}$ is related, through the coefficients l, m, to a tesseral harmonic J_{lm}. By varying the coefficients l, m, p and keeping q/m fixed, one find, all frequencies $\frac{d\phi_{1,lmp(\alpha m)}}{dt}$ concerning to a specified resonance.

Now considering, *j*=1,2, *l*=2,4, *m*=2, $\alpha = 1/2$ and *p*=0,1,2,3, from $\hat{H}_{1,rs}$, one gets

$$\begin{aligned}
\hat{H}_1 \quad = \quad & \frac{\mu^2}{2X_1^2} + \omega_e\theta_1 + B_{1,2010}(X_1, Y_1, C_1) + B_{1,4020}(X_1, Y_1, C_1) + \\
& + B_{1,2201}(X_1, Y_1, C_1)cos(x_1 - 2\Theta_1 + y_1 - 2\lambda_{22}) + \\
& + B_{1,2211}(X_1, Y_1, C_1)cos(x_1 - 2\Theta_1 - y_1 - 2\lambda_{22}) + \\
& + B_{1,2221}(X_1, Y_1, C_1)cos(x_1 - 2\Theta_1 - 3y_1 - 2\lambda_{22}) + \\
& + B_{1,4211}(X_1, Y_1, C_1)cos(x_1 - 2\Theta_1 + y_1 - 2\lambda_{42} + \pi) + \\
& + B_{1,4221}(X_1, Y_1, C_1)cos(x_1 - 2\Theta_1 - y_1 - 2\lambda_{42} + \pi) + \\
& + B_{1,4231}(X_1, Y_1, C_1)cos(x_1 - 2\Theta_1 - 3y_1 - 2\lambda_{42} + \pi).
\end{aligned} \tag{18}$$

A new transformation is considered

$$X_1' = X_1 \qquad Y_1' = Y_1 \qquad \Theta_1' = -\theta_1$$

$$x_1' = x_1 \qquad y_1' = y_1 \qquad \theta_1' = \Theta_1. \tag{19}$$

From Equations (18) and (19) the Hamiltonian $\hat{H}_1'$ is obtained

$$\begin{aligned}
\hat{H}_1' \quad = \quad & \frac{\mu^2}{2X_1'^2} - \omega_e\Theta_1' + B_{1,2010}(X_1', Y_1', C_1) + B_{1,4020}(X_1', Y_1', C_1) + \\
& + B_{1,2201}(X_1', Y_1', C_1)cos(x_1' - 2\theta_1' + y_1' - 2\lambda_{22}) + \\
& + B_{1,2211}(X_1', Y_1', C_1)cos(x_1' - 2\theta_1' - y_1' - 2\lambda_{22}) + \\
& + B_{1,2221}(X_1', Y_1', C_1)cos(x_1' - 2\theta_1' - 3y_1' - 2\lambda_{22}) + \\
& + B_{1,4211}(X_1', Y_1', C_1)cos(x_1' - 2\theta_1' + y_1' - 2\lambda_{42} + \pi) + \\
& + B_{1,4221}(X_1', Y_1', C_1)cos(x_1' - 2\theta_1' - y_1' - 2\lambda_{42} + \pi) + \\
& + B_{1,4231}(X_1', Y_1', C_1)cos(x_1' - 2\theta_1' - 3y_1' - 2\lambda_{42} + \pi).
\end{aligned} \tag{20}$$

Finally, a last transformation of variables is done,

$$X_4 = X_1' \qquad Y_4 = Y_1' \qquad \Theta_4 = -\Theta_1' + 2X_1'$$

$$x_4 = x_1' - 2\theta_1' \qquad y_4 = y_1' \qquad \theta_4 = \theta_1'. \tag{21}$$

So, the Hamiltonian H_4 is found, considering the Equations (20) and (21)

$$\begin{aligned}
H_4 \quad = \quad & \frac{\mu^2}{2X_4^2} - \omega_e(\Theta_4 - 2X_4) + B_{4,2010}(X_4, Y_4, C_1) + B_{4,4020}(X_4, Y_4, C_1) + \\
& + B_{4,2201}(X_4, Y_4, C_1)cos(x_4 + y_4 - 2\lambda_{22}) + \\
& + B_{4,2211}(X_4, Y_4, C_1)cos(x_4 - y_4 - 2\lambda_{22}) + \\
& + B_{4,2221}(X_4, Y_4, C_1)cos(x_4 - 3y_4 - 2\lambda_{22}) + \\
& + B_{4,4211}(X_4, Y_4, C_1)cos(x_4 + y_4 - 2\lambda_{42} + \pi) + \\
& + B_{4,4221}(X_4, Y_4, C_1)cos(x_4 - y_4 - 2\lambda_{42} + \pi) + \\
& + B_{4,4231}(X_4, Y_4, C_1)cos(x_4 - 3y_4 - 2\lambda_{42} + \pi),
\end{aligned} \tag{22}$$

with θ_4 ignorable and $\omega_e\Theta_4$ constant. Since the term $\omega_e\Theta_4$ is constant, it plays no role in the equations of motion and a new Hamiltonian can be introduced,

$$\hat{H}_4 = H_4 + \omega_e\Theta_4,$$

with $\hat{H}_4$ given by

$$\begin{aligned}
\hat{H}_4 \quad = \quad & \frac{\mu^2}{2X_4^2} + 2\omega_e X_4 + B_{4,2010}(X_4, Y_4, C_1) + B_{4,4020}(X_4, Y_4, C_1) + \\
& + B_{4,2201}(X_4, Y_4, C_1)cos(x_4 + y_4 - 2\lambda_{22}) + \\
& + B_{4,2211}(X_4, Y_4, C_1)cos(x_4 - y_4 - 2\lambda_{22}) + \\
& + B_{4,2221}(X_4, Y_4, C_1)cos(x_4 - 3y_4 - 2\lambda_{22}) + \\
& + B_{4,4211}(X_4, Y_4, C_1)cos(x_4 + y_4 - 2\lambda_{42} + \pi) + \\
& + B_{4,4221}(X_4, Y_4, C_1)cos(x_4 - y_4 - 2\lambda_{42} + \pi) + \\
& + B_{4,4231}(X_4, Y_4, C_1)cos(x_4 - 3y_4 - 2\lambda_{42} + \pi).
\end{aligned} \tag{23}$$

The dynamical system described by $\hat{H}_4$,

$$\frac{d(X_4, Y_4)}{dt} = \frac{\partial \hat{H}_4}{\partial(x_4, y_4)} \qquad \frac{d(x_4, y_4)}{dt} = -\frac{\partial \hat{H}_4}{\partial(X_4, Y_4)}, \tag{24}$$

is given explicitly by

$$\begin{aligned}
\frac{dX_4}{dt} \quad = \quad & -B_{4,2201}(X_4, Y_4, C_1)sin(x_4 + y_4 - 2\lambda_{22}) - \\
& -B_{4,2211}(X_4, Y_4, C_1)sin(x_4 - y_4 - 2\lambda_{22}) - \\
& -B_{4,2221}(X_4, Y_4, C_1)sin(x_4 - 3y_4 - 2\lambda_{22}) - \\
& -B_{4,4211}(X_4, Y_4, C_1)sin(x_4 + y_4 - 2\lambda_{42} + \pi) - \\
& -B_{4,4221}(X_4, Y_4, C_1)sin(x_4 - y_4 - 2\lambda_{42} + \pi) - \\
& -B_{4,4231}(X_4, Y_4, C_1)sin(x_4 - 3y_4 - 2\lambda_{42} + \pi),
\end{aligned} \tag{25}$$

$$\begin{aligned}
\frac{dY_4}{dt} \quad = \quad & -B_{4,2201}(X_4, Y_4, C_1)sin(x_4 + y_4 - 2\lambda_{22}) + \\
& +B_{4,2211}(X_4, Y_4, C_1)sin(x_4 - y_4 - 2\lambda_{22}) + \\
& +3B_{4,2221}(X_4, Y_4, C_1)sin(x_4 - 3y_4 - 2\lambda_{22}) - \\
& -B_{4,4211}(X_4, Y_4, C_1)sin(x_4 + y_4 - 2\lambda_{42} + \pi) + \\
& +B_{4,4221}(X_4, Y_4, C_1)sin(x_4 - y_4 - 2\lambda_{42} + \pi) + \\
& +3B_{4,4231}(X_4, Y_4, C_1)sin(x_4 - 3y_4 - 2\lambda_{42} + \pi),
\end{aligned} \tag{26}$$

$$
\begin{aligned}
\frac{dx_4}{dt} &= \frac{\mu^2}{X_4^3} - 2\omega_e - \frac{\partial B_{4,2010}(X_4,Y_4,C_1)}{\partial X_4} - \frac{\partial B_{4,4020}(X_4,Y_4,C_1)}{\partial X_4} - \\
&- \frac{\partial B_{4,2201}(X_4,Y_4,C_1)}{\partial X_4} cos(x_4 + y_4 - 2\lambda_{22}) - \\
&- \frac{\partial B_{4,2211}(X_4,Y_4,C_1)}{\partial X_4} cos(x_4 - y_4 - 2\lambda_{22}) - \\
&- \frac{\partial B_{4,2221}(X_4,Y_4,C_1)}{\partial X_4} cos(x_4 - 3y_4 - 2\lambda_{22}) - \\
&- \frac{\partial B_{4,4211}(X_4,Y_4,C_1)}{\partial X_4} cos(x_4 + y_4 - 2\lambda_{42} + \pi) - \\
&- \frac{\partial B_{4,4221}(X_4,Y_4,C_1)}{\partial X_4} cos(x_4 - y_4 - 2\lambda_{42} + \pi) - \\
&- \frac{\partial B_{4,4231}(X_4,Y_4,C_1)}{\partial X_4} cos(x_4 - 3y_4 - 2\lambda_{42} + \pi), \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy_4}{dt} &= - \frac{\partial B_{4,2010}(X_4,Y_4,C_1)}{\partial Y_4} - \frac{\partial B_{4,4020}(X_4,Y_4,C_1)}{\partial Y_4} - \\
&- \frac{\partial B_{4,2201}(X_4,Y_4,C_1)}{\partial Y_4} cos(x_4 + y_4 - 2\lambda_{22}) - \\
&- \frac{\partial B_{4,2211}(X_4,Y_4,C_1)}{\partial Y_4} cos(x_4 - y_4 - 2\lambda_{22}) - \\
&- \frac{\partial B_{4,2221}(X_4,Y_4,C_1)}{\partial Y_4} cos(x_4 - 3y_4 - 2\lambda_{22}) - \\
&- \frac{\partial B_{4,4211}(X_4,Y_4,C_1)}{\partial Y_4} cos(x_4 + y_4 - 2\lambda_{42} + \pi) - \\
&- \frac{\partial B_{4,4221}(X_4,Y_4,C_1)}{\partial Y_4} cos(x_4 - y_4 - 2\lambda_{42} + \pi) - \\
&- \frac{\partial B_{4,4231}(X_4,Y_4,C_1)}{\partial Y_4} cos(x_4 - 3y_4 - 2\lambda_{42} + \pi). \qquad (28)
\end{aligned}
$$

The $B_{4,2010}$, $B_{4,4020}$, $B_{4,2201}$, $B_{4,2211}$, $B_{4,2221}$, $B_{4,4211}$, $B_{4,4221}$ and $B_{4,4231}$ terms are given by

$$B_{4,2010} = \frac{\mu^4}{X_4^6} {a_e}^2 J_{20} \left(-\frac{3}{4}\,\frac{(C_1 + 2\,X_4)^2}{(X_4+Y_4)^2} + \frac{1}{4} \right) \left(1 + \frac{3}{2}\,\frac{-{Y_4}^2 - 2\,X_4 Y_4}{{X_4}^2} \right), \qquad (29)$$

$$B_{4,4020} = \frac{\mu^6}{X_4^{10}} {a_e}^4 J_{40} \left(\frac{105}{64} \left(1 - \frac{(C_1 + 2\,X_4)^2}{(X_4+Y_4)^2} \right)^2 - \frac{3}{2} + \frac{15}{8}\,\frac{(C_1 + 2\,X_4)^2}{(X_4+Y_4)^2} \right) \left(1 + 5\,\frac{-{Y_4}^2 - 2\,X_4 Y_4}{{X_4}^2} \right), \qquad (30)$$

$$B_{4,2201} = \frac{21}{8X_4^7}\,\mu^4 {a_e}^2 J_{22} \left(1 + \frac{C_1 + 2\,X_4}{X_4 + Y_4} \right)^2 \sqrt{-{Y_4}^2 - 2\,X_4 Y_4}, \qquad (31)$$

$$B_{4,2211} = \frac{3}{2X_4^7}\,\mu^4 {a_e}^2 J_{22} \left(\frac{3}{2} - \frac{3}{2}\,\frac{(C_1 + 2\,X_4)^2}{(X_4+Y_4)^2} \right) \sqrt{-{Y_4}^2 - 2\,X_4 Y_4}, \qquad (32)$$

$$B_{4,2221} = -\frac{3}{8X_4^7}\,\mu^4 {a_e}^2 J_{22} \left(1 - \frac{C_1 + 2\,X_4}{X_4 + Y_4} \right)^2 \sqrt{-{Y_4}^2 - 2\,X_4 Y_4}, \qquad (33)$$

$$B_{4,4211} = \frac{9}{2X_4^{11}}\,\mu^6 {a_e}^4 J_{42}\Big(\frac{35}{27}\left(1-\frac{(C_1+2\,X_4)^2}{(X_4+Y_4)^2}\right)(C_1+2\,X_4)\left(1+\frac{C_1+2\,X_4}{X_4+Y_4}\right)(X_4+Y_4)^{-1} - -\frac{15}{8}\left(1+\frac{C_1+2\,X_4}{X_4+Y_4}\right)^2\Big)\sqrt{-{Y_4}^2-2\,X_4Y_4}, \quad (34)$$

$$B_{4,4221} = \frac{5}{2X_4^{11}}\,\mu^6 {a_e}^4 J_{42}\Big(\frac{105}{16}\left(1-\frac{(C_1+2\,X_4)^2}{(X_4+Y_4)^2}\right)\left(1-3\,\frac{(C_1+2\,X_4)^2}{(X_4+Y_4)^2}\right)+\frac{15}{4} - -\frac{15}{4}\,\frac{(C_1+2\,X_4)^2}{(X_4+Y_4)^2}\Big)\sqrt{-{Y_4}^2-2\,X_4Y_4}, \quad (35)$$

$$B_{4,4231} = \frac{\mu^6}{X_4^{10}}\,{a_e}^4 J_{42}\Big(-\frac{35}{27}\left(1-\frac{(C_1+2\,X_4)^2}{(X_4+Y_4)^2}\right)(C_1+2\,X_4)\left(1-\frac{C_1+2\,X_4}{X_4+Y_4}\right)(X_4+Y_4)^{-1} - -\frac{15}{8}\left(1-\frac{C_1+2\,X_4}{X_4+Y_4}\right)^2\Big)\left(\frac{1}{2}\,\frac{\sqrt{-{Y_4}^2-2\,X_4Y_4}}{X_4}+\frac{33}{16}\,\frac{-{Y_4}^2-2\,X_4Y_4}{{X_4}^2}\right). \quad (36)$$

The zonal harmonics used in the Eqs. (29) and (30) are $J_{20} = 1.0826 \times 10^{-3}$ and $J_{40} = -1.6204 \times 10^{-6}$ and the tesseral harmonics used in the Eqs. (31) to (36) are $J_{22} = 1.8154 \times 10^{-6}$ and $J_{42} = 1.6765 \times 10^{-7}$.

The term C_1 used in the Eqs. (29) to (36) is given by

$$C_1 = \sqrt{\mu a}(\sqrt{1-e^2}cos(I) - 2). \quad (37)$$

In the numerical integration of the Eqs. (25) to (28) have been used as initial conditions, the following conditions

$$X_4 = \sqrt{\mu a} \qquad Y_4 = \sqrt{\mu a}[(1-e^2)^{1/2} - 1]$$

$$x_4 = M + \omega + 2\Omega - 2\Theta \qquad y_4 = \omega \quad (38)$$

In the next section are shown some results of the numerical integration of the Eqs. (25) to (28).

3. RESULTS

Figures 1 to 4 show the time behavior of the semi-major axis, x_4 and y_4 angles and of the eccentricity, according to the numerical integration of the motion equations, (25) to (28). The initial conditions, in the Figs. 1 to 4, for inclination is 55^0, and eccentricity is 0.001. The initial values of the x_4 and y_4 angles are 0^o and 0^o, respectively, and the initial values of the semi-major axis are shown in the figures.

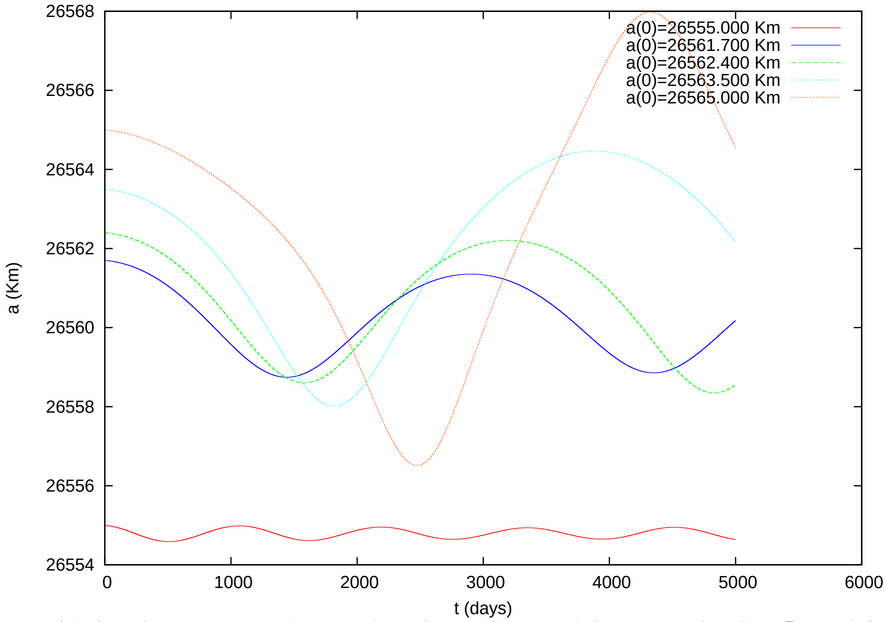

Figure 1. a versus t, considering six resonant angles together; ϕ_{2201}, ϕ_{2211} and ϕ_{2221} associated to J_{22} and ϕ_{4211}, ϕ_{4221} and ϕ_{4231} associated to J_{42}. The initial conditions for inclination and eccentricity are $I = 55^o$ and e=0.001, respectively.

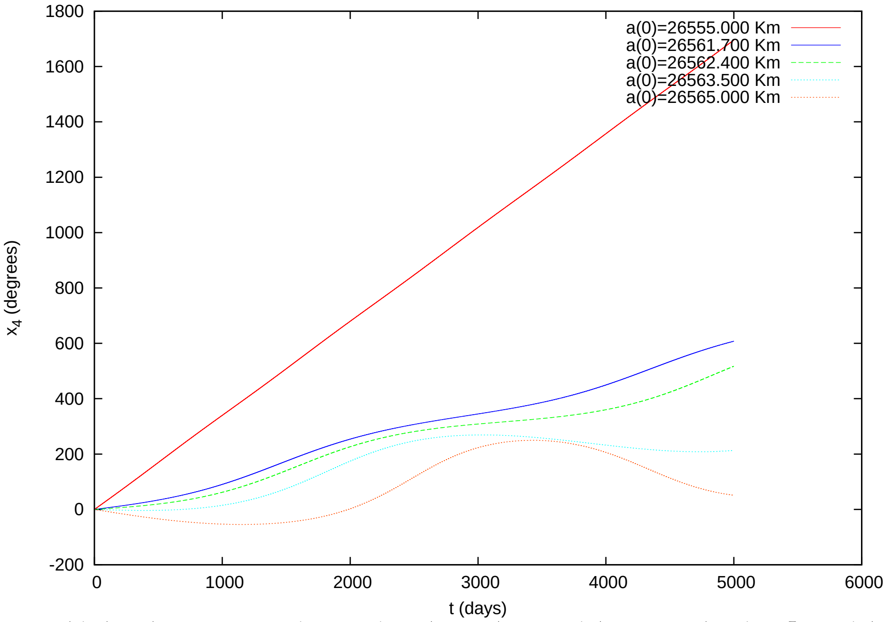

Figure 2. x_4 versus t, considering six resonant angles together; ϕ_{2201}, ϕ_{2211} and ϕ_{2221} associated to J_{22} and ϕ_{4211}, ϕ_{4221} and ϕ_{4231} associated to J_{42}. The initial conditions for inclination and eccentricity are $I = 55^o$ and e=0.001, respectively.

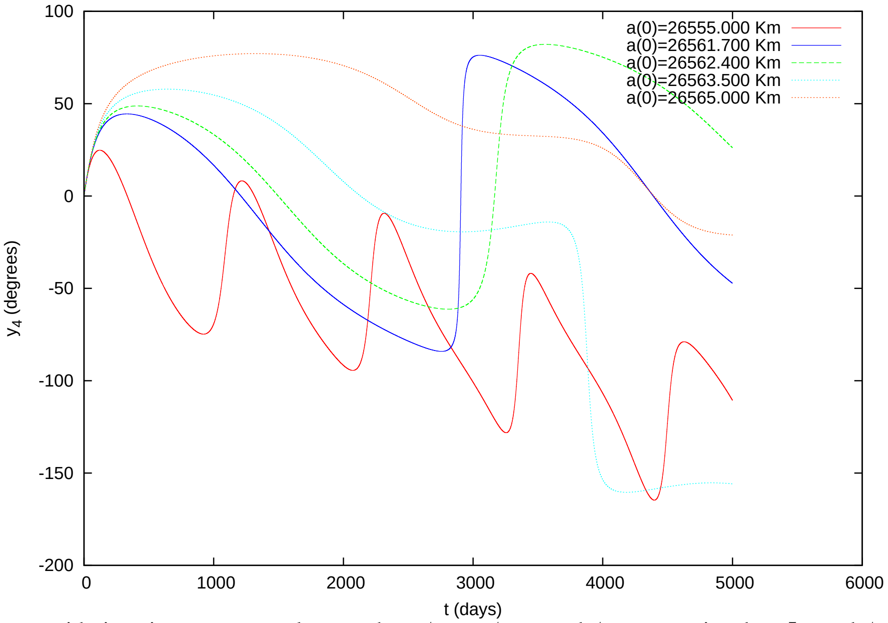

Figure 3. y_4 versus t, considering six resonant angles together; ϕ_{2201}, ϕ_{2211} and ϕ_{2221} associated to J_{22} and ϕ_{4211}, ϕ_{4221} and ϕ_{4231} associated to J_{42}. The initial conditions for inclination and eccentricity are $I = 55^o$ and e=0.001, respectively.

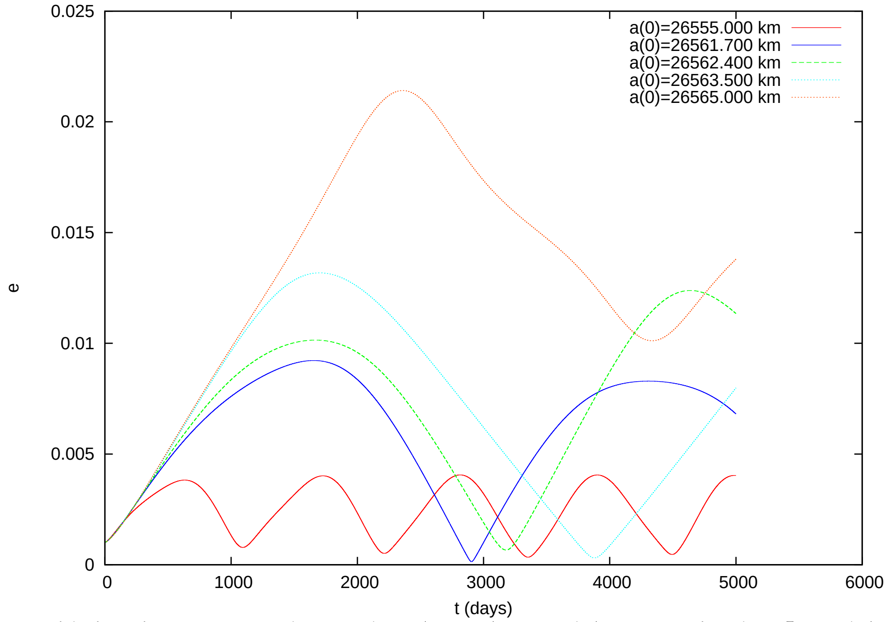

Figure 4. e versus t, considering six resonant angles together; ϕ_{2201}, ϕ_{2211} and ϕ_{2221} associated to J_{22} and ϕ_{4211}, ϕ_{4221} and ϕ_{4231} associated to J_{42}. The initial conditions for inclination and eccentricity are $I = 55^o$ and e=0.001, respectively.

4. CONCLUSIONS

In this work, the dynamical behavior of six critical angles associated to the 2:1 resonance problem in the artificial satellites motion have been investigated.

The results show the time behavior of the semi-major axis, x_4 and y_4 angles and of the eccentricity. In the numerical integration, the initial conditions used are 55^0 for inclination and 0.001 for eccentricity. The initial values of the x_4 and y_4 angles are 0^o and 0^o, respectively. The six critical angles studied together are ϕ_{2201}, ϕ_{2211} and ϕ_{2221} associated to J_{22} and ϕ_{4211}, ϕ_{4221} and ϕ_{4231} associated to J_{42}.

Inside the region where the resonances are found, the motion can be chaotic, because it shows sensibility to initial conditions.

5. ACKNOWLEDGEMENTS

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