Structural optimization problem with stress constraint using the level set method

Hélio Emmendoerfer Junior, hemjr@ig.com.br
Eduardo Alberto Fancello, fancello@grante.ufsc.br
Group of Mechanical Analysis and Design - GRANTE, Department of Mechanical Engineering, Federal University of Santa Catarina, 88040-900 Florianópolis, SC, Brazil

Abstract. *This work presents an approach to structural topology optimization. In this study the problem of mass minimization is formulated under local material failure constraint based on stress levels. This constraint is imposed using an approach based on Augmented Lagrangian technique in such a way that the constraints are incorporated into the objective function by penalization. The convergence of the updating sequence of the Lagrange parameters corresponds to the satisfaction of the Karush-Kuhn-Tucker conditions of the original problem. The level set method is used to control the domain topology whose boundary is represented implicitly by level curves of a higher dimensional scalar function. The minimizing sequence is achieved by updating the body boundary using a gradient direction of the objective function. This is obtained through shape sensitivity analysis, providing a velocity field that updates the boundary by the solution of the Hamilton-Jacobi partial differential equation embedded in a fixed domain. Some numerical examples are shown in order to verify the behavior of the proposed technique, including examples in which the analytical solution is known a priori.*

Keywords: *Level set method; Topology optimization; Structural optimization*

1. INTRODUCTION

The level set method was firstly introduced into the structural topology optimization by Sethian and Wiegmann (2000). The papers of M.Y. Wang (2003) and Allaire *et al.* (2004) may be considered as starting works combining topology optimization, shape derivatives and the level set model. From these works, several other publications have emerged in this direction (M.Y. Wang (2004), Wang *et al.* (2004), Xia *et al.* (2006), Luo *et al.* (2008), Rong and Liang (2008), Challis (2010) and others), most of them concerning the mean compliance problem.

The problem of structural topology optimization considering failure constraints has received increasing attention in recent years. A topology optimization problem that takes local stresses in a continuum structure as constraints was defined by Duysinx and Bendsoe (1998). Many different approaches are seen in literature to treat the stress constraints, but most of them employ the well-known SIMP method to solve the topology optimization of continuum structures (Duysinx and Bendsoe (1998), Pereira *et al.* (2004), París *et al.* (2009)). One of the main difficulties introduced by this kind of approach on the stress constrained problem is the *Stress Singularity Phenomenon* among others like the local nature of the constraints and the highly non-linear stress behavior.

Recently, other methods were successfully employed to solve the stress-based topology optimization of continuum structures, as such the topological asymptotic analysis that provides the sensitivity of a given shape functional with respect to an infinitesimal domain perturbation (Amstutz and Novotny (2010)).

The present paper addresses the problem of mass minimization with local failure (stress) constraints using the *level set* method. The formulation eliminates the existence of singular optima by using the ϵ-regularization proposed by Cheng and Guo (1997). The formulation of the problem in combination with a penalization-type (Augmented Lagrangian) approach and its numerical examples are presented.

2. FORMULATION OF THE PROBLEM

Let Ω be an open domain with boundary Γ belonging to the physical space $\Re^n (n = 1, 2, 3)$ occupied by an isotropic linear elastic solid material. It is assumed that the boundary of Ω is smooth enough and contains three disjoint parts:

$$\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_H,$$

with Dirichlet boundary conditions on Γ_D, Neumann boundary conditions on Γ_N where a traction τ is applied and homogeneous Neumann boundary conditions on Γ_H, i.e., traction free. Γ_H is the only part subject to optimization, i.e., the only part free to move during optimization. It is assumed that admissible shapes Ω belong to a fixed reference domain D.

Classical linear elasticity is assumed to be a good representation of the body behavior. Thus, a set of constitutive, kinematic and equilibrium equations relate the constitutive elasticity tensor $\mathbf{C}$, the stress tensor σ, the deformation tensor ε, the body forces $\mathbf{b}$ and the displacement vector $\mathbf{u}$ as follows:

$$\sigma = \mathbf{C}\varepsilon, \qquad \varepsilon = \nabla^S \mathbf{u} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla^T \mathbf{u}\right) \qquad \text{and} \qquad \operatorname{div}(\sigma) + \mathbf{b} = \mathbf{0} \quad \forall \mathbf{x} \in \Omega \tag{1}$$

The following boundary conditions also apply:

$$\sigma\mathbf{n} = \tau \quad \forall\mathbf{x} \in \Gamma_N, \qquad \sigma\mathbf{n} = \mathbf{0} \quad \forall\mathbf{x} \in \Gamma_H \qquad \text{and} \qquad \mathbf{u} = \bar{\mathbf{u}} \quad \forall\mathbf{x} \in \Gamma_D. \tag{2}$$

The weighted-residual method applied to Eq. (1) and (2) leads to the classical variational expression in which the equilibrium displacements $\mathbf{u} \in U$ satisfy the equation:

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad \forall\mathbf{v} \in V, \tag{3}$$

where $a(\cdot, \cdot) \colon U \times V \to \Re$ is a symmetric bilinear form, $l(\cdot) \colon V \to \Re$ is a linear functional associated with the external loads, U and V are the kinematic admissible fields of the displacements and variations, respectively. This variational forms can be written as:

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{C}\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v}) d\Omega \quad \text{and} \quad l(\mathbf{v}) = \int_\Omega \mathbf{b} \cdot \mathbf{v} d\Omega + \int_{\Gamma_N} \tau \cdot \mathbf{v} d\partial\Omega.$$

With these assumptions in hand, the structural optimization problem of the present work is to minimize the mass with local stress constraints and satisfying equilibrium equations. This may be written as Problem P1:

$$\begin{aligned} &\underset{\Omega}{\text{Minimize}} \;\; m(\mathbf{u}) = \int_\Omega \rho d\Omega \\ &\text{subject to:} \;\; \begin{cases} a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0 \;\; \forall\mathbf{v} \in V, \\ g(\mathbf{u}) \leq 0, \end{cases} \end{aligned} \tag{4}$$

where $g(\mathbf{u})$ is the material failure function.

3. SOLUTION BY THE AUGMENTED LAGRANGIAN METHOD

The Augmented Lagrangian (AL) technique has proven to be an efficient approach for solving different constrained minimization problems (Martínez and Santos (1995), Martínez (2006)). It is well known that the apparent simplicity of the proposed problem hides a great number of mathematical and numerical difficulties: stress singularities, big number of constraints among others. In the present approach the augmented Lagrangian method was chosen in an effort to stabilize the optimization process through appropriate choices of penalization factors as well as schemes for updating Lagrangian multipliers.

For the Problem P1, we added a slack function $s \geq 0$ on the inequality constraint $g(\mathbf{u})$, then the corresponding optimization problem can be rewritten as Problem P2:

$$\begin{aligned} &\underset{s,\Omega}{\text{Minimize}} \;\; m(\mathbf{u}) = \int_\Omega \rho d\Omega \\ &\text{subject to:} \;\; \begin{cases} a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0 \;\; \forall\mathbf{v} \in V, \\ h(\mathbf{u}, s) = g(\mathbf{u}) + s = 0, \\ s \geq 0. \end{cases} \end{aligned} \tag{5}$$

Note that the minimum found in Problem P1 is the same of problem P2.

The material failure constraint can be embedded into the functional $J(\mathbf{u})$ through Lagrange multiplier function α, so the Problem P3 is defined as:

$$\begin{aligned} &\underset{s,\Omega}{\text{Minimize}} \;\; J(\mathbf{u}) = \int_\Omega \rho d\Omega + \int_\Omega \alpha h(\mathbf{u}, s) d\Omega \\ &\text{subject to:} \;\; \begin{cases} a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0 \;\; \forall\mathbf{v} \in V, \\ h(\mathbf{u}, s) = g(\mathbf{u}) + s = 0, \\ s \geq 0. \end{cases} \end{aligned} \tag{6}$$

Again Problems P1, P2 and P3 are equivalent.

Problem P3 may be solved in an aproximate way by penalization. Thus, the Problem P4 can be formulated as:

$$\begin{aligned} &\underset{s,\Omega}{\text{Minimize}} \;\; J(\mathbf{u}) = \int_\Omega \rho d\Omega + \int_\Omega \alpha h(\mathbf{u}, s) d\Omega + \frac{c}{2} \int_\Omega h^2(\mathbf{u}, s) d\Omega \\ &\text{subject to:} \;\; \begin{cases} a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0 \;\; \forall\mathbf{v} \in V, \\ s \geq 0. \end{cases} \end{aligned} \tag{7}$$

The minimization of the variable s can be done in explicit form. The function s which minimizes the functional is given by (Bertsekas (1996)):

$$s = \max\left\{0; -(\frac{\alpha}{c} + g(\mathbf{u}))\right\}. \tag{8}$$

In according with Eq. (6), we have

$$h(\mathbf{u}) \equiv g(\mathbf{u}) + \max\left\{0; -(\frac{\alpha}{c} + g(\mathbf{u}))\right\} \equiv \max\left\{g(\mathbf{u}); -\frac{\alpha}{c}\right\}. \tag{9}$$

With this results in hands, we re-define the optimization problem. The cost function consists of three terms. The first term corresponds to the original cost function, while the last terms are associated with the penalization. Therefore, the Problem P5 can be stated as

$$\begin{aligned} &\underset{\Omega}{\text{Minimize}} \;\; J(\mathbf{u}) = \int_\Omega \rho d\Omega + \int_\Omega \alpha h(\mathbf{u}) d\Omega + \frac{c}{2}\int_\Omega h^2(\mathbf{u}) d\Omega \\ &\text{subject to:} \;\; a(\mathbf{u},\mathbf{v}) - l(\mathbf{v}) = 0 \;\; \forall \mathbf{v} \in V, \end{aligned} \tag{10}$$

where $h(\mathbf{u})$ is given by Eq. (9). Once the minimization of P5 is achieved, the Lagrange multiplier α is updated following the rule (Bertsekas (1996)).

4. LEVEL SET MODEL

The basic idea behind the level set representation is expressing a curve or surface as the zero level set of a high-dimensional function in an implicit manner, and then tracking the deformation of the curve or surface via the evolution of the higher-dimensional level set function (Sethian (1999)).

Let D be a fixed design domain in which all admissible shapes Ω are included, i.e., $\Omega \subseteq D$ and $\partial\Omega$ represents its boundary. Supposing the level set function $\phi(\mathbf{x})$ is defined as

$$\begin{aligned} &\phi(\mathbf{x}) > 0 \qquad \forall \mathbf{x} \in \Omega\setminus\partial\Omega \qquad \text{(solid material)},\\ &\phi(\mathbf{x}) = 0 \qquad \forall \mathbf{x} \in \partial\Omega \qquad \text{(boundary)},\\ &\phi(\mathbf{x}) < 0 \qquad \forall \mathbf{x} \in D\setminus\Omega \qquad \text{(void material)}. \end{aligned}$$

The boundary $\partial\Omega(\mathbf{x})$ of $\Omega(\mathbf{x})$ is represented as the zero level set

$$\partial\Omega(\mathbf{x}) = \left\{\mathbf{x} \in \Re^d \mid \phi(\mathbf{x}) = 0\right\} \qquad (d = 2 \text{ or } 3).$$

Letting the level set function evolve dynamically in time t with a normal velocity, then the motion of the structural boundary can be expressed as follows:

$$\partial\Omega(\mathbf{x},t) = \left\{\mathbf{x}(t) \in \Re^{d+1} \mid \phi(\mathbf{x},t) = 0\right\} \qquad \forall \mathbf{x}(t) \in \partial\Omega(\mathbf{x},t). \tag{11}$$

Differentiating Eq. (11) with respect to t on both sides yields

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t} + \nabla\phi \cdot \frac{d\mathbf{x}}{dt} = \frac{\partial\phi}{\partial t} + \nabla\phi\cdot\mathbf{v} = 0. \tag{12}$$

Consider a normal velocity $v_n = \mathbf{v}\cdot\mathbf{n}$ with an outward defined direction $\mathbf{n} = \nabla\phi/\|\nabla\phi\|$, then the Hamilton–Jacobi PDE can be defined as (Sethian (1999),Osher and Fedkiw (2002))

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t} + v_n\|\nabla\phi\| = 0. \tag{13}$$

If the velocity v_n on the boundary is known, transporting ϕ by the level set model is equivalent to moving the boundary $\partial\Omega$ along the normal direction n.

The *level set* function ϕ is generally constructed to be a signed distance function and it is updated according to a design boundary velocity field defined via shape sensitivity analysis. This velocity is derived from the governing equations in a continuum formulation. It should be noted that in the *level set* method the design velocity v_n defined on the free boundary Γ_H of a structure must be extended to the whole reference domain D or a narrow band around the free boundary (Osher and Fedkiw (2002)). In the topology optimization of continuum structures, since a fixed mesh is used for the finite element analysis, and voids are represented by artificial weak material, the normal velocity which is defined on the traction free boundary Γ_H can be "naturally" extended to the entire reference domain D.

5. THE LEVEL SET BASED OPTIMIZATION METHOD

With the level set model we can describe the geometry of the body in terms of the scalar function ϕ. It is also convenient to use the Heaviside function H and the Dirac delta function δ defined as

$$H(\phi) = \begin{cases} 1 & \text{if } \phi > 0 \\ 0 & \text{if } \phi < 0 \end{cases} \qquad \text{and} \qquad \delta(\phi) = \frac{dH}{d\phi}. \tag{14}$$

It is a well-known procedure in topology optimization the use of intermediate densities for the material. In numerical practice via Finite Element Analysis, the *level set* function needs to be mapped to a field of "Heaviside densities", where the element densities are usually averaged values of the Heaviside function applied on the level-set function $H(\phi)$ per element (van Dijk *et al.* (2008)),

$$\rho_e = \frac{\int_{D_e} H(\phi) dD}{\int_{D_e} dD}, \tag{15}$$

where D_e is the material domain of element e. Many forms to approximate the Heaviside can be found in the literature (Osher and Fedkiw (2002)), which is in fact the material volume fraction per element.

Thus, with the *level set* model, the optimization problem of the Eq. (10) can be formulated as follows:

$$\begin{aligned} &\underset{\Omega}{\text{Minimize}} \;\; J_\phi(\mathbf{u}) = \int_D \left[\rho + \alpha h(\mathbf{u}) + \frac{c}{2} h^2(\mathbf{u})\right] H(\phi) dD \\ &\text{where } \mathbf{u}(\phi) \text{ is solution of:} \\ &a_\phi(\mathbf{u}, \mathbf{v}) = l_\phi(\mathbf{v}) \quad \forall \mathbf{v} \in V. \end{aligned} \tag{16}$$

The energy bilinear form $a_\phi(\mathbf{u}, \mathbf{v})$ and the load linear form $l_\phi(\mathbf{v})$ are described by

$$a_\phi(\mathbf{u}, \mathbf{v}) = \int_D \mathbf{C}\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v}) H(\phi) dD, \tag{17}$$

$$l_\phi(\mathbf{v}) = \int_D \mathbf{b} \cdot \mathbf{v} H(\phi) dD + \int_D \tau \cdot \mathbf{v} \delta(\phi) \|\nabla \phi\| dD = \int_D \left[\mathbf{b} \cdot \mathbf{v} + \operatorname{div}((\tau \cdot \mathbf{v})\mathbf{n})\right] H(\phi) dD. \tag{18}$$

5.1 Sensitivity analysis and velocity field

The Lagrangian functional of the optimization problem of Eq. (16) is given by

$$\pounds(\phi, \mathbf{u}, \lambda) = \int_D \left[\rho + \alpha h(\mathbf{u}) + \frac{c}{2} h^2(\mathbf{u})\right] H(\phi) dD + a_\phi(\mathbf{u}, \lambda) - l_\phi(\lambda) \quad \forall \lambda \in V, \tag{19}$$

where λ is the Lagrange multiplier for the state equation of linear elasticity Eq. (3).

The sensitivity analysis of the Lagrangian $\pounds$ with respect to pseudo-time t to deviations of ϕ, u and λ, applying the chain rule, can be written as:

$$\frac{d\pounds(\phi, \mathbf{u}, \lambda)}{dt} = \frac{\partial \pounds}{\partial \phi}[\dot{\phi}] + \frac{\partial \pounds}{\partial \mathbf{u}}[\dot{\mathbf{u}}] + \frac{\partial \pounds}{\partial \lambda}[\dot{\lambda}], \tag{20}$$

where

$$\dot{\phi} = \frac{d\phi}{dt} \qquad \dot{\mathbf{u}} = \frac{du}{dt} \in V \qquad \text{and} \qquad \dot{\lambda} = \frac{d\lambda}{dt} \in V.$$

For any stationary point $(\phi, \mathbf{u}^*, \lambda^*)$ with respect to u and λ, we have

$$\left.\frac{\partial \pounds}{\partial \mathbf{u}}[\dot{\mathbf{u}}]\right|_{u=u^*} = 0, \tag{21}$$

$$\left.\frac{\partial \pounds}{\partial \lambda}[\dot{\lambda}]\right|_{\lambda=\lambda^*} = 0, \tag{22}$$

that correspond, as we show bellow, to the adjoint and state equations respectively. If the state equation is satisfied, we also conclude that

$$\frac{d\pounds(\phi, \mathbf{u}, \lambda)}{dt} = \frac{dJ_\phi(\mathbf{u})}{dt}. \tag{23}$$

Thus, if Eq. (21) and Eq. (22) are satisfied, and using the Eq. (23), the time derivative of the objective function Eq. (16) is given by

$$\frac{dJ_\phi(\mathbf{u})}{dt} = \frac{\partial \pounds}{\partial \phi}[\dot{\phi}]. \tag{24}$$

The adjoint solution λ is obtained solving the Eq. (21):

$$\begin{aligned}\frac{\partial \pounds(\phi,\mathbf{u},\lambda)}{\partial \mathbf{u}}[\dot{\mathbf{u}}] &= \int_D \alpha \frac{\partial h(\mathbf{u})}{\partial \mathbf{u}}[\dot{\mathbf{u}}]\, H(\phi) dD + \int_D c h(\mathbf{u}) \frac{\partial h(\mathbf{u})}{\partial \mathbf{u}}[\dot{\mathbf{u}}]\, H(\phi) dD \\ &\quad - \int_D \mathbf{C}\frac{\partial \varepsilon(\mathbf{u})}{\partial \mathbf{u}}[\dot{\mathbf{u}}]\cdot \varepsilon(\lambda) H(\phi) dD = 0 \qquad \forall \dot{\mathbf{u}} \in V.\end{aligned} \tag{25}$$

It is an usual approach in elastoplasticity to write the failure function of any isotropic material as function of the invariants of the tensor σ, i.e., $h = f(I_1, J_2, J_3)$. Thus, the derivative of the failure function may be written using the chain rule as

$$\frac{\partial h(\mathbf{u})}{\partial \mathbf{u}}[\dot{\mathbf{u}}] = \frac{\partial h(\mathbf{u})}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{u}}[\dot{\mathbf{u}}] + \frac{\partial h(\mathbf{u})}{\partial J_2}\frac{\partial J_2}{\partial \mathbf{u}}[\dot{\mathbf{u}}] + \frac{\partial h(\mathbf{u})}{\partial J_3}\frac{\partial J_3}{\partial \mathbf{u}}[\dot{\mathbf{u}}]. \tag{26}$$

In terms of Fréchet derivatives, we obtain:

$$\frac{\partial I_1}{\partial \mathbf{u}}[\dot{\mathbf{u}}] = \mathbf{C}\mathbf{I}\cdot\varepsilon(\dot{\mathbf{u}}), \quad \frac{\partial J_2}{\partial \mathbf{u}}[\dot{\mathbf{u}}] = \mathbf{C}\mathbf{P}^T\mathbf{S}(\mathbf{u})\cdot\varepsilon(\dot{\mathbf{u}}) \quad \text{and} \quad \frac{\partial J_3}{\partial \mathbf{u}}[\dot{\mathbf{u}}] = \mathbf{C}\mathbf{P}^T\mathbf{S}(\mathbf{u})\mathbf{S}(\mathbf{u})\cdot\varepsilon(\dot{\mathbf{u}}), \tag{27}$$

where

$$\mathbf{S}(\mathbf{u}) = \mathbf{P}\sigma(\mathbf{u}), \qquad \mathbf{P} = \mathbf{II} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}. \tag{28}$$

Here, $\mathbf{S}$ denotes the deviatoric stress, and $\mathbf{I}$ and $\mathbf{II}$ are the second and fourth order identity tensors. Substituting the Eqs.(27) into Eq. (26) leads to

$$\frac{\partial h(\mathbf{u})}{\partial \mathbf{u}}[\dot{\mathbf{u}}] = \mathbb{C}\mathbf{A}(\mathbf{u})\cdot\varepsilon(\dot{\mathbf{u}}), \tag{29}$$

where

$$\mathbf{A}(\mathbf{u}) = \frac{\partial h(\mathbf{u})}{\partial I_1}\mathbf{I} + \frac{\partial h(\mathbf{u})}{\partial J_2}\mathbf{P}^T\mathbf{S}(\mathbf{u}) + \frac{\partial h(\mathbf{u})}{\partial J_3}\mathbf{P}^T\mathbf{S}(\mathbf{u})\mathbf{S}(\mathbf{u}) \quad \text{and} \quad \frac{\partial \varepsilon(\mathbf{u})}{\partial \mathbf{u}}[\dot{\mathbf{u}}] = \varepsilon(\dot{\mathbf{u}}). \tag{30}$$

Finally, by substituting Eq.(29) and (30) into Eq.(25), we obtain the variational expression that allow the calculation of the adjoint solution λ:

$$\int_D \mathbf{C}\varepsilon(\lambda)\cdot\varepsilon(\dot{\mathbf{u}})H(\phi)dD = -\int_D [\alpha + ch(\mathbf{u})]\,\mathbf{C}\mathbf{A}(\mathbf{u})\cdot\varepsilon(\dot{\mathbf{u}})H(\phi)dD \qquad \forall \dot{\mathbf{u}} \in V. \tag{31}$$

We now solve the Eq. (22) as follow:

$$\begin{aligned}\frac{\partial \pounds(\phi,\mathbf{u},\lambda)}{\partial \lambda}[\dot{\lambda}] &= \int_D \mathbf{C}\varepsilon(\mathbf{u})\cdot\frac{\partial \varepsilon(\lambda)}{\partial \lambda}[\dot{\lambda}]H(\phi)dD - \int_D \mathbf{b}\cdot\frac{\partial \lambda}{\partial \lambda}[\dot{\lambda}]H(\phi)dD \\ &\quad - \int_D \frac{\partial(\tau.\lambda)}{\partial \lambda}[\dot{\lambda}]\delta(\phi)\|\nabla\phi\|dD = 0 \qquad \forall \dot{\lambda} \in V,\end{aligned} \tag{32}$$

that after conventional derivative operations leads to

$$\int_D \mathbf{C}\varepsilon(\mathbf{u})\cdot\varepsilon(\dot{\lambda})H(\phi)dD = \int_D \left[\mathbf{b}.\dot{\lambda} + \operatorname{div}((\tau\cdot\dot{\lambda})\mathbf{n})\right] H(\phi)dD \quad \forall \dot{\lambda} \in V, \tag{33}$$

which is exactly the state equation.

Considering the Eq. (14), the shape derivative of the objective function $J_\phi(\mathbf{u})$ is given by the expression

$$\begin{aligned}\frac{\partial \pounds(\phi,\mathbf{u},\lambda)}{\partial \phi}[\dot{\phi}] = \frac{dJ_\phi(\mathbf{u})}{dt} = \int_D \Big\{ &\rho + \alpha h(\mathbf{u}) + \frac{c}{2}h^2(\mathbf{u}) + \mathbf{C}\varepsilon(\mathbf{u}).\varepsilon(\lambda) \\ &- \mathbf{b}\cdot\lambda - \operatorname{div}((\tau\cdot\lambda)\mathbf{n})\delta(\phi)[\dot{\phi}]\Big\}\, dD \qquad \forall \dot{\phi} \in V,\end{aligned} \tag{34}$$

where $\mathbf{u}$ and λ are the solution of Eq. (33) and (31) respectively.

Using Eq. (13) we have that $\dot{\phi} = \frac{\partial \phi}{\partial t} = v_n \|\nabla \phi\|$. Then the sensitivity of the objective function for a stationary point $(\phi, \mathbf{u}^*, \lambda^*)$ can be written as

$$\frac{dJ_\phi(\mathbf{u})}{dt} = \int_D G(\phi)\delta(\phi)\|\nabla\phi\| v_n dD, \tag{35}$$

where:

$$G(\phi) = \rho + \alpha h(\mathbf{u}) + \frac{c}{2}h^2(\mathbf{u}) + \mathbf{C}\varepsilon(\mathbf{u}) \cdot \varepsilon(\lambda) - [\mathbf{b}\cdot\lambda + \mathrm{div}((\tau\cdot\lambda)\mathbf{n})], \tag{36}$$

is known as the *shape gradient density* (Haug *et al.* (1986)).

To ensure the decrease of the objective function, a proper velocity field v_n should be selected for the *level set* function. As indicated in the literature (M.Y. Wang (2003), Allaire *et al.* (2004)), the simplest way is to directly choose a steepest descent direction by letting,

$$v_n = -G(\phi). \tag{37}$$

Thus, the shape derivative can guarantee a descent direction of the objective function

$$\frac{dJ_\phi(\mathbf{u})}{dt} = -\int_D G^2(\phi)\delta(\phi)\|\nabla\phi\| dD \leq 0. \tag{38}$$

In addition, to enhance the *level set* regularity, the normal velocity field can be modified by adding an artificial regularization term for smoothing purpose

$$v_N = v_n + I(x)\,\mathrm{div}\, n, \tag{39}$$

where $\mathrm{div}\, n$ is the mean curvature κ of the *level set* and $I(x) > 0$ corresponds to a geometric metric function. In (Wang *et al.* (2004)) is used the geometric metric as follows:

$$I(x) = \frac{c_1}{1 + c_2 v_n^2}, \tag{40}$$

where c_1 and c_2 are two positive constants used according to the problem.

6. MATERIAL FAILURE CRITERION

It is necessary to define an appropriate failure criterion for the intermediate materials. Based on Eq. (15) we define a relative density that controls the elastic constitutive equation through the following expressions:

$$\mathbf{C}_\phi = H(\phi)\mathbf{C}, \quad \sigma = \mathbf{C}_\phi\varepsilon, \quad \text{and} \quad \bar{\sigma} = \mathbf{C}\varepsilon. \tag{41}$$

The effective stress tensor $\bar{\sigma}$ for an arbitrary intermediate material depends on the original (solid material) elasticity tensor and the apparent (homogenized) value of deformation (Duysinx and Bendsoe (1998)). An appropriate definition of the effective stress $\bar{\sigma}$ is a key aspect in the present problem due to its relation with the failure function. For a given effective stress tensor $\bar{\sigma}$, an equivalent (von Mises) scalar stress σ_e is computed and a failure function is defined:

$$F(\bar{\sigma}) = \frac{\sigma_e}{\sigma_{adm}} - 1 \leq 0, \tag{42}$$

where σ_{adm} is the material yielding stress. To overcome the *Stress Singularity* phenomenon, the ϵ-regularization technique (Cheng and Guo (1997)) is used and the failure function is re-defined:

$$g_\sigma(\mathbf{u}) \equiv H(\phi)F(\bar{\sigma}(\mathbf{u})) - \epsilon(1 - H(\phi)) \leq 0, \qquad \text{a.e. in } \Omega \tag{43}$$
$$0 < \epsilon^2 \leq H_{\min} \leq H(\phi) \leq 1,$$

where ϵ is a relaxation parameter.

7. NUMERICAL EXAMPLES

For all examples, the Finite Element Method (FEM) with a fixed mesh and artificial weak material is employed for the elasticity analysis. We use a mesh with uniform four-node rectangular finite elements for both the *level set* function and the elastic displacement, where the *level set* function is dicretized with gridpoints centered on the elements of the mesh. The Young modulus E of solid material is normalized to 1 and 1×10^{-6} for the weak material and the Poisson ratio ν is fixed to $0,3$. The examples are in 2D under plane stress condition and it is assumed that the body force is neglected ($\mathbf{b} = 0$). The first order upwind scheme is utilized to solve the Hamilton–Jacobi PDE (Sethian (1999)) and a reinitialization is performed after each optimization step. The time step used in the upwind scheme is based in the CFL condition (Sethian (1999)). The initial parameters used in examples have been taken from Pereira (2001) and Pereira *et al.* (2004).

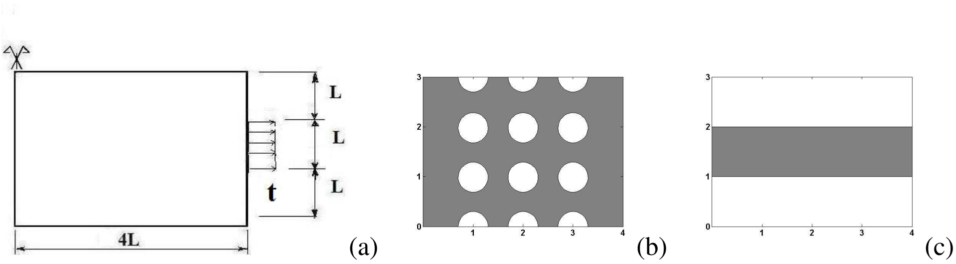


Figure 1. (a) Bar in traction, (b) Initial *level set*, (c) optimal structure

7.1 Bar in traction

This example discusses the case of a bar submitted to a simple traction (see Fig. 1a). The loading applied t is exactly the same value of material yielding stress (σ_{adm}). Naturally, the solution of the problem is a bar in traction whose transverse section is equal to transverse section where is applied the load and it is fully stressed, as expected (Fig. 1c). The initial design is showed in Fig. 1b. The design domain is discretized by 60 x 45 mesh. Symmetry conditions are used. The initial parameters were the following:
$\sigma_{adm} = 1,0$ Pa; $t = 1,0$ Pa; $L = 1,0$ m.

The Fig. 2 shows the convergence history of the objective function. The Fig. 2a to show the convergence and stability of our algorithm. The first term of the objective function J corresponds to the original cost function in wich is the mass minimization and can be checked in Fig. 2b. The terms associated with the penalization (linear and quadratic terms) must converge to zero (see Fig. 2c and Fig. 2d). The Fig. 3 shows the ϵ-relaxed failure function.It is possible to see that the constraints are satisfied within a small error.

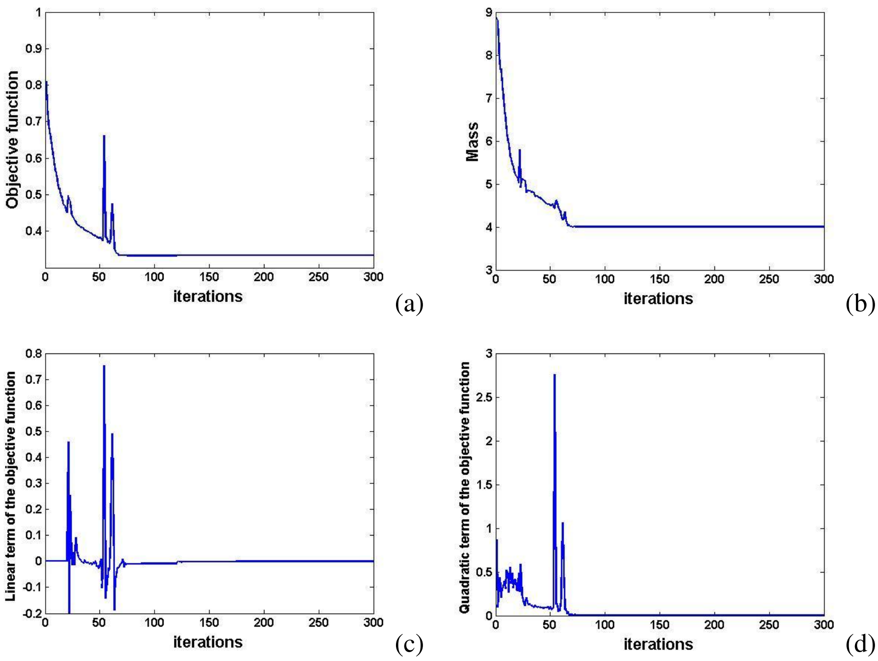


Figure 2. Traction load case: Convergence history of the objective function

7.2 Bar in flexion

This example discusses the case of a bar submitted to a bending moment due to a linear distribution of stress along the right vertical boundary. The domain is decomposed in two regions. The right side is fixed, while the left one is optimized. The load is set to a distribution of normal stress from $t = 30$ Pa at the bottom line to $t = -30$ Pa at the top line (Fig. 4a). A minimum mass problem is run subject to a $\sigma_{adm} = 35$ Pa. A mesh of 60 x 45 elements is used for the finite element analysis. Symmetry conditions are used. The initial *level set* design is displayed in Fig. 4b. The final design in which the material is concentrated at the "flange" as expected is showed in Fig. 4c. The results about the convergence history of the objective function and the ϵ-relaxed failure function also are presented (Fig. 5).

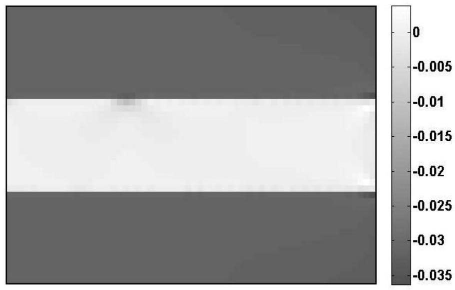


Figure 3. Traction load case: ϵ-relaxed failure function

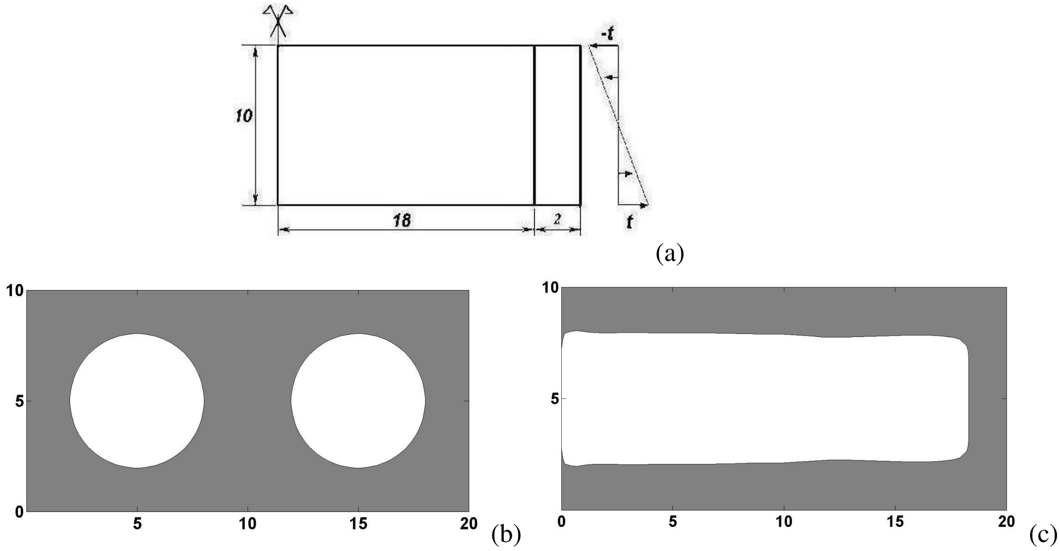


Figure 4. Bending load case: (a) model, (b) Initial *level set*, (c) optimal structure

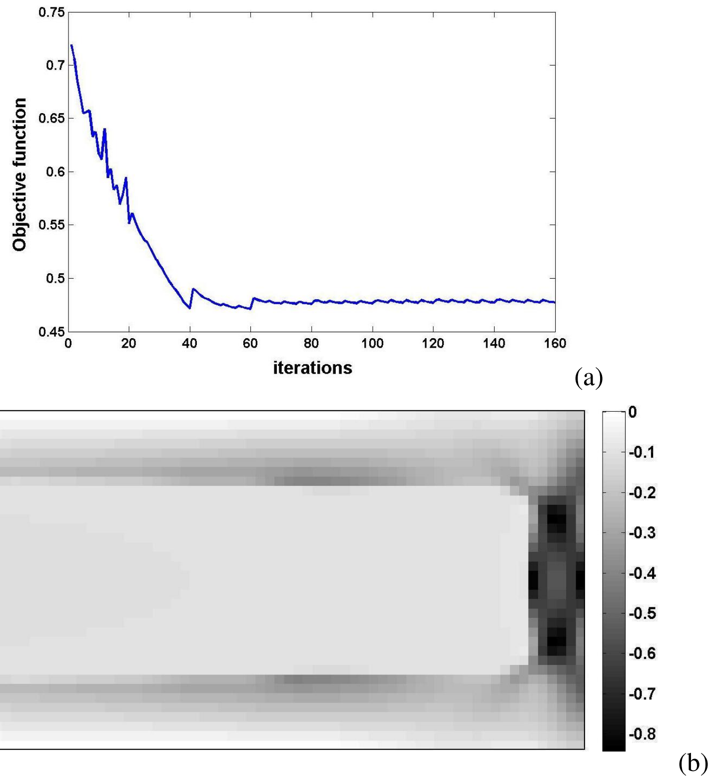


Figure 5. Bending load case. (a) Convergence history of the objective function, (b) ϵ-relaxed failure function

7.3 MBB-beam

The MBB-beam is one of the classic benchmarks of topology optimization. It consists of a simply supported beam, loaded with a vertical force centered on its top boundary (Fig. 6). Symmetry conditions are used. The initial parameters were the following:
$\sigma_{adm} = 17,8$ kPa; $P = 2,0$ kN; $L = 1,0$ m; $\Delta L = 0,2$ m.

Due to symmetry conditions, only half of the domain is meshed with 150 x 50 elements The initial design and the final optimization results are shown in Fig. 7. The final optimum solution is similar to results obtained by using the SIMP method (Pereira *et al.* (2004)). The convergence of the optimization process is shown in Fig. 8a. The ϵ-relaxed failure function of this example is shown in Fig. 8b.

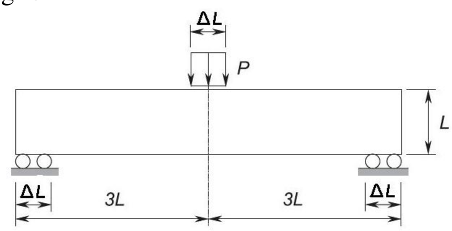
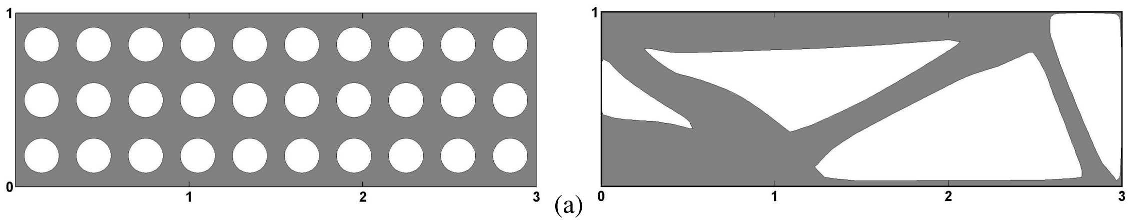
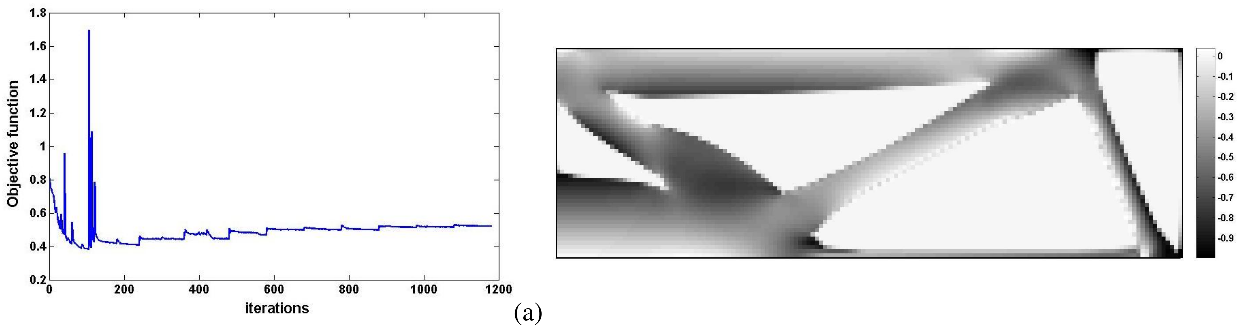


Figure 6. MBB-beam: model



(a) (b)

Figure 7. MBB-beam. (a) Initial *level set*, (b) optimal structure



(a) (b)

Figure 8. MBB-beam: (a) Convergence history of the objective function, (b) ϵ-relaxed failure function

8. CONCLUSION

In the present study, a *level set* based method is proposed for the topology optimization of continuum structures, whose objective is the minimization of the mass constrained by a material failure criterion. The *level set* method relies on a representation of the design boundaries, in which the movements of the boundaries of the structure are driven by a transformation of the objective function and constraints into speed functions that govern the *level set* propagation. This technique demonstrated flexibility of handling topological changes and fidelity of boundary representation if compared with other methods of topology optimization presents in the literature.

The results obtained in this work show final designs comparable to those obtained in Pereira *et al.* (2004) using the SIMP method. The failure ϵ-relaxed criterion is (numerically) satisfied in most of the mesh. Also the velocity defined trought of sensitivity analysis used in the optimization algorithm proved to be a proper descent direction to describe a gradient method for minimization of the objective function.

The explicit first order upwind scheme was used to solve the Hamilton-Jacobi equation. To improve the numerical accuracy and efficiency of *level set* evolution, we believe that other numerical techniques (Xia *et al.* (2006), Luo *et al.* (2008)) may have a significant impact about the results obtained mainly in the final of the optimization process. Also other methods of reinitialization of *level set* to avoid induced numerical errors can be adopted to overcome numerical instabilities (Li *et al.* (2010)). Finally more apropriate evaluation of stress or stress recovering techniques should be tested. It is worth mentioning that the algorithm is just a steepest descent gradient method, and its convergence could be speed up by using, for example, a quasi-Newton concepts. Due to the stress calculated only in the center of elements, leading to a poor estimate of the stress on structure, it is necessary to improve the stress calculation using other methods for obtaining the structural response as such as X-FEM proposed in Wei and Wang (2008).

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10. Responsibility notice

The authors are the only responsible for the printed material included in this paper.