OPTIMAL LOW-THRUST LIMITED-POWER TRANSFERS BETWEEN COPLANAR ORBITS WITH VERY SMALL ECCENTRICITIES

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Abstract. *In this paper an analytical first order solution for optimal low-thrust limited power transfers (no rendezvous) between coplanar orbits with very small eccentricities in an inverse-square force field is presented. This analytical solution is determined through canonical transformation theory and is expressed in terms of non-singular elements. A preliminary analysis of interplanetary transfers is performed and the analytical results are compared to the ones obtained through a numerical technique.*

Keywords*: Optimal low-thrust limited power trajectories, transfers between circular coplanar orbits.*

1. INTRODUCTION

The purpose of this paper is to present an analytical study of optimal low-thrust limited power trajectories for transfers (no rendezvous) between coplanar orbits very small eccentricities in an inverse-square force field. The study of these transfers is particularly interesting because the orbits found in practice often have a small eccentricity and the problem of transferring a vehicle from a low orbit to a high orbit is frequently met. Besides, the analysis has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions verified in the last two decades (Edelbaum, 1965; Marec and Vinh, 1977; Marec, 1979; Haissig et al, 1992; Kechichian, 1996, 1997; Vasile et al, 2000; Sukhanov and Prado, 2001; Kiforenko, 2005).

The optimization problem associated to the space transfer problem is formulated as a Mayer problem of optimal control with “Cartesian” elements – radial component of position vector and radial and transverse components of velocity vector – as state variables. After applying the Pontryagin Maximum Principle (Pontryagin et al, 1962), successive Mathieu transformations are performed and suitable sets of orbital elements are introduced. The short periodic terms are eliminated from the maximum Hamiltonian function through an infinitesimal canonical transformation built through Hori method (Hori, 1966) – a perturbation canonical method based on Lie series. The new Hamiltonian function, resulting from the infinitesimal canonical transformation, describes the extremal trajectories associated with the long duration maneuvers for simple transfers (no rendez-vous). The separation of variables technique is applied to solve the Hamilton-Jacobi equation associated to the average canonical system and closed-form analytical solution is obtained. A first order analytical solution for the non-singular orbital elements is then obtained applying the transformation equations of the algorithm of Hori method.

Finally, the analytical solution is applied in preliminary analysis of interplanetary transfers and the results are compared to the ones obtained through a numerical technique.

2. OPTIMAL LOW-THRUST TRAJECTORIES

Low-thrust power-limited systems or, simply, LP system, are characterized by low-thrust acceleration level and high specific impulse (Marec, 1979). The ratio between the maximum thrust acceleration and the gravity acceleration on the ground, $\gamma_{\max}/g_0$, is between 10^{-4} and 10^{-2}. For such system, the fuel consumption is described by the variable J defined as

$$J = \frac{1}{2}\int_{t_0}^{t_f} \gamma^2 dt\ , \tag{1}$$

where γ is the magnitude of the thrust acceleration vector $\boldsymbol{\Gamma}$, used as control variable. The consumption variable J is a monotonic decreasing function of the mass m of the space vehicle,

$$J = P_{\max}\left(\frac{1}{m} - \frac{1}{m_0}\right), \tag{2}$$

where $P_{\max}$ is the maximum power and m_0 is the initial mass. The minimization of the final value of the fuel consumption J_f is equivalent to the maximization of m_f .

The optimization problem concerning with simple transfers (no rendezvous) between coplanar orbits will be formulated as a Mayer problem of optimal control by using Cartesian elements as state variables. At time *t*, the state of a space vehicle *M* is defined by the radial distance *r* from the center of attraction, the radial and transverse components of the velocity, *u* and *v*, and the fuel consumption *J*. The geometry of the transfer problem is illustrated in Fig.1.

In the two-dimension optimization problem, the state equations are given by

$$\frac{du}{dt}=\frac{v^2}{r}-\frac{\mu}{r^2}+R \qquad \frac{dv}{dt}=-\frac{uv}{r}+S$$

$$\frac{dr}{dt}=u \qquad \frac{dJ}{dt}=\frac{1}{2}\left(R^2+S^2\right), \tag{3}$$

where μ is the gravitational parameter, *R* and *S* are the components of the thrust acceleration vector in a moving frame of reference, that is, $\boldsymbol{\Gamma}=R\boldsymbol{e}_r+S\boldsymbol{e}_s$, with the unit vector $\boldsymbol{e}_r$ pointing radially outward and the unit vector $\boldsymbol{e}_s$ perpendicular to $\boldsymbol{e}_r$ in the direction of the motion and in the plane of orbit.

The optimization problem is stated as: it is proposed to transfer a space vehicle *M* from the initial conditions at $t_0=0$,

$$u(0)=0 \qquad v(0)=1 \qquad r(0)=1 \qquad J(0)=0\,, \tag{4}$$

to the final state at the prescribed final time t_f,

$$u(t_f)=0 \qquad v(t_f)=\sqrt{\mu/r_f} \qquad r(t_f)=r_f\,, \tag{5}$$

such that J_f is a minimum. In the formulation of the boundary conditions, all variables are expressed in canonical units, such that $\mu=1$.

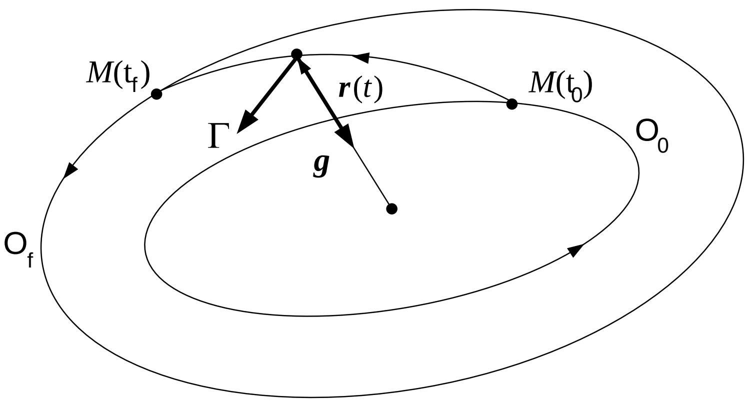

Figure 1. Geometry of transfer problem.

Following the Pontryagin Maximum Principle (Pontryagin et al, 1962), the adjoint variables λ_u, λ_v, λ_r and λ_J are introduced and the Hamiltonian function $H(u,v,r,J,\lambda_u,\lambda_v,\lambda_r,\lambda_J,R,S)$ is formed using the right-hand side of Eqs (3),

$$H=\lambda_u\left(\frac{v^2}{r}-\frac{\mu}{r^2}+R\right)+\lambda_v\left(-\frac{uv}{r}+S\right)+\lambda_r u+\frac{\lambda_J}{2}\left(R^2+S^2\right). \tag{6}$$

The control variables *R* and *S* must be selected from the admissible controls such that the Hamiltonian function reaches its maximum along the optimal trajectory. Thus,

$$R^*=-\frac{\lambda_u}{\lambda_J} \qquad S^*=-\frac{\lambda_v}{\lambda_J}\,. \tag{7}$$

Since λ_J is a first integral and $\lambda_J(t_f) = -1$ (this results follows from transversality condition) one finds

$$\lambda_J(t) = -1. \tag{8}$$

Thus, from Eqs (7), the optimal thrust acceleration is given by

$$R^* = \lambda_u \qquad\qquad S^* = \lambda_v. \tag{9}$$

Introducing these equations into the Eq. (6), one finds

$$H^* = u\lambda_r + \left(\frac{v^2}{r} - \frac{\mu}{r^2}\right)\lambda_u - \frac{uv}{r}\lambda_v + \frac{1}{2}\left(\lambda_u^2 + \lambda_v^2\right). \tag{10}$$

The problem of determining a first order analytical solution of the system of differential equations governed by the Hamiltonian H^* is solved by means of the theory of canonical transformations as it will be described in the next section.

3. FIRST ORDER ANALYTICAL SOLUTION

Consider the Hamiltonian function describing a null thrust arc in the two-dimensional formulation of the optimization problem:

$$\mathrm{H} = u\lambda_r + \left(\frac{v^2}{r} - \frac{\mu}{r^2}\right)\lambda_u - \frac{uv}{r}\lambda_v + \frac{v}{r}\lambda_\theta. \tag{11}$$

Note that H is obtained from Eq. (6) taking $R = S = 0$ and adding the last term concerning to the differential equation of the angular variable θ, which defines the position of the space vehicle with respect to a reference axis in the plane of motion. This variable is important for rendez-vous problems and plays no special role for simple transfer problems, but it is necessary to define the canonical transformations described below.

In the transformation theory described in the next paragraphs, it is assumed that the Hamiltonian H^* is augmented in order to include term associated to the angular variable θ; that is,

$$H^* = u\lambda_r + \left(\frac{v^2}{r} - \frac{\mu}{r^2}\right)\lambda_u - \frac{uv}{r}\lambda_v + \frac{v}{r}\lambda_\theta + \frac{1}{2}\left(\lambda_u^2 + \lambda_v^2\right). \tag{12}$$

H is the undisturbed part of H^* and plays a fundamental role in the theory.

The general solution of the system of differential equations governed by the Hamiltonian H can be expressed in terms of a fast phase and is given by (da Silva Fernandes, 1999a, 1999b)

$$u = \sqrt{\frac{\mu}{p}}e\sin f$$

$$v = \sqrt{\frac{\mu}{p}}\left(1 + e\cos f\right)$$

$$r = \frac{p}{1 + e\cos f}$$

$$\theta = \omega + f$$

$$\lambda_u = \sqrt{\frac{p}{\mu}}\sin f\,\lambda_e + \sqrt{\frac{p}{\mu}}\frac{\cos f}{e}\left(\lambda_f - \lambda_\omega\right)$$

$$\lambda_v = 2\sqrt{\frac{p}{\mu}}r\lambda_p + \sqrt{\frac{p}{\mu}}\left(2\cos f + e\cos^2 f + e\right)\frac{r}{p}\lambda_e - \sqrt{\frac{p}{\mu}}\frac{\sin f}{e}\left[1+\frac{r}{p}\right]\left(\lambda_f - \lambda_\omega\right)$$

$$\lambda_r = 2\frac{p}{r}\lambda_p + \frac{\cos f + e}{r}\lambda_e - \frac{\sin f}{re}\left(\lambda_f - \lambda_\omega\right)$$

$$\lambda_\theta = \lambda_\omega , \qquad (13)$$

where p is the semi-latus rectum, e is the eccentricity, ω is the pericenter argument and f is the true anomaly (fast phase) and $(\lambda_p, \lambda_e, \lambda_f, \lambda_\omega)$ are adjoint variables to (p, e, f, ω).

Equations (13) define a Mathieu transformation,

$$\left(u, v, r, \theta, \lambda_u, \lambda_v, \lambda_r, \lambda_\theta\right) \xrightarrow{MATHIEU} (p, e, f, \omega, \lambda_p, \lambda_e, \lambda_f, \lambda_\omega) .$$

The undisturbed Hamiltonian function H is invariant with respect to this canonical transformation and is written in terms of the new variables as

$$\mathbf{H} = \frac{\sqrt{\mu p}}{r^2}\lambda_f . \qquad (14)$$

Equations (13) have singularities for circular orbits ($e = 0$). In order to avoid this drawback, a set of nonsingular elements is introduced. The transformation equations between the singular orbital elements and the nonsingular ones are given by

$$a = \frac{p}{\left(1-e^2\right)} \qquad h = e\cos\omega$$

$$k = e\sin\omega \qquad L = f + \omega . \qquad (15)$$

These equations define a Lagrange point transformation. Following the properties of generalized canonical systems, the Jacobian of the inverse of this transformation must be computed in order to get the transformation equations between the corresponding adjoint variables. Thus,

$$\lambda_a = \left(1-e^2\right)\lambda_p$$

$$\lambda_h = \left(\lambda_e - \frac{2ep}{\left(1-e^2\right)}\lambda_p\right)\cos\omega + \left(\frac{\lambda_L - \lambda_\omega}{e}\right)\sin\omega$$

$$\lambda_k = \left(\lambda_e - \frac{2ep}{\left(1-e^2\right)}\lambda_p\right)\sin\omega - \left(\frac{\lambda_L - \lambda_\omega}{e}\right)\cos\omega$$

$$\lambda_L = \lambda_f . \qquad (16)$$

Equations (15) and (16) define a new Mathieu transformation between singular and nonsingular elements,

$$(p, e, f, \omega, \lambda_p, \lambda_e, \lambda_f, \lambda_\omega) \xrightarrow{MATHIEU} \left(a, h, k, L, \lambda_a, \lambda_h, \lambda_k, \lambda_L\right).$$

Substituting Eqns (15) and (16) into the Eqns (13), one finds

$$u = \sqrt{\frac{\mu}{a\left(1-h^2-k^2\right)}}\left(h\sin L - k\cos L\right)$$

$$v = \sqrt{\frac{\mu}{a\left(1-h^2-k^2\right)}}\left(1+h\cos L+k\sin L\right)$$

$$r = \frac{a\left(1-h^2-k^2\right)}{1+h\cos L+k\sin L}$$

$$\theta = L$$

$$\lambda_u = \sqrt{\frac{a}{\mu}}\left\{2a\lambda_a \frac{\left(h\sin L-k\cos L\right)}{\sqrt{1-h^2-k^2}}+\sqrt{1-h^2-k^2}\left(\lambda_h \sin L-\lambda_k \cos L\right)\right\}$$

$$\lambda_v = \sqrt{\frac{a}{\mu}}\left\{2a\lambda_a\sqrt{1-h^2-k^2}\left(\frac{a}{r}\right)+\frac{1}{\sqrt{1-h^2-k^2}}\left(\frac{r}{a}\right)\left\{\left[\frac{3}{2}h+2\cos L+\frac{h}{2}\cos 2L\right.\right.\right.$$
$$\left.\left.\left.+\frac{k}{2}\sin 2L\right]\lambda_h+\left[\frac{3}{2}k+2\sin L-\frac{k}{2}\cos 2L+\frac{h}{2}\sin 2L\right]\lambda_k\right\}\right\}$$

$$\lambda_r = 2\left(\frac{a}{r}\right)^2\lambda_a+\frac{1}{r}\left[\left(h+\cos L\right)\lambda_h+\left(k+\sin L\right)\lambda_k\right]$$

$$\lambda_\theta = -k\lambda_h+h\lambda_k+\lambda_L\,. \tag{17}$$

Equations (17) are valid for all orbits. For quasi circular orbits, that is, orbits with very small eccentricities, these equations can be greatly simplified if higher order terms in eccentricity are neglected. Considering first order terms in eccentricity, one finds

$$u = na\left(h\sin\ell-k\cos\ell\right)$$

$$v = na\left(1+h\cos\ell+k\sin\ell\right)$$

$$r = \frac{a}{1+h\cos\ell+k\sin\ell}$$

$$\theta = \ell$$

$$\lambda_u = \sqrt{\frac{a}{\mu}}\left\{2a\left[h\sin\ell-k\cos\ell\right]\lambda_a+\left[-k+\sin\ell+h\sin 2\ell-k\cos 2\ell\right]\lambda_h\right.$$
$$\left.+\left[h-\cos\ell-h\cos 2\ell-k\sin 2\ell\right]\lambda_k+\left[-2+\frac{3}{2}h\cos\ell+\frac{3}{2}k\sin\ell\right]\lambda_\ell\right\},$$

$$\lambda_v = \sqrt{\frac{a}{\mu}}\left\{2a\left[1+h\cos\ell+k\sin\ell\right]\lambda_a+\left[-\frac{3}{2}h+2\cos\ell+\frac{3}{2}h\cos 2\ell+\frac{3}{2}k\sin 2\ell\right]\lambda_h\right.$$
$$\left.+\left[-\frac{3}{2}k+2\sin\ell+\frac{3}{2}h\sin 2\ell-\frac{3}{2}k\cos 2\ell\right]\lambda_k+\left[h\sin\ell-k\cos\ell\right]\lambda_\ell\right\},$$

$$\lambda_r = 2\left(1+2h\cos\ell+2k\sin\ell\right)\lambda_a+\frac{1}{a}\left[\left(\frac{3}{2}h+\cos\ell+\frac{h}{2}\cos 2\ell+\frac{k}{2}\sin 2\ell\right)\lambda_h\right.$$
$$\left.+\left(\frac{3}{2}k+\sin\ell+\frac{k}{2}\sin 2\ell-\frac{k}{2}\cos 2\ell\right)\lambda_k\right],$$

$$\lambda_\theta = -k\lambda_h + h\lambda_k + \lambda_\ell , \tag{18}$$

$n = \sqrt{\mu/a^3}$ is the mean motion and $\ell = \omega + M$ is the mean latitude.

Introducing Eqns (18) into the expression for H^* and considering first order terms in eccentricity, it results

$$\begin{aligned}
H^* = n\lambda_\ell + \frac{1}{n^2 a^2}\Big\{ & 2[1 + 2h\cos\ell + 2k\sin\ell]a^2\lambda_a^2 + [4a\cos\ell + 4ak\sin 2\ell + 4ah\cos 2\ell]\lambda_a\lambda_h \\
& + [4a\sin\ell + 4ah\sin 2\ell - 4ak\cos 2\ell]\lambda_a\lambda_k + [2ak\cos\ell - 2ah\,\mathrm{sen}\,\ell]\lambda_a\lambda_\ell \\
& + \left[\frac{5}{4} - h\cos\ell + k\sin\ell + \frac{3}{4}\cos 2\ell + h\cos 3\ell + k\sin 3\ell\right]\lambda_h^2 \\
& + \left[-2h\sin\ell - 2k\cos\ell + \frac{3}{2}\sin 2\ell + 2h\sin 3\ell - 2k\cos 3\ell\right]\lambda_h\lambda_k + \left[\frac{7}{4}k - 2\sin\ell + \frac{1}{4}k\cos 2\ell - \frac{1}{4}h\sin 2\ell\right]\lambda_h\lambda_\ell \\
& + \left[\frac{5}{4} + h\cos\ell - k\sin\ell - \frac{3}{4}\cos 2\ell - h\cos 3\ell - k\sin 3\ell\right]\lambda_k^2 \\
& + \left[-\frac{7}{4}h + 2\cos\ell + \frac{1}{4}h\cos 2\ell + \frac{1}{4}k\sin 2\ell\right]\lambda_k\lambda_\ell + [2 - 3k\sin\ell - 3h\cos\ell]\lambda_\ell^2\Big\}.
\end{aligned} \tag{19}$$

In order to get a first order analytical solution for the system of differential equations described by the maximum Hamiltonian function H^*, defined by Eqn (19), Hori method (Hori, 1966) is applied.

According to da Silva Fernandes (2009a), the new Hamiltonian F_1 and the generating function S_1 are given by the following equations:

$$F_1 = \frac{a}{2\mu}\left\{4a'^2\lambda_a'^2 + \frac{5}{2}\left(\lambda_h'^2 + \lambda_k'^2\right)\right\}, \tag{20}$$

$$\begin{aligned}
S_1 = \sqrt{\frac{a'^5}{\mu^3}}\Big\{ & [4h'\sin\ell' - 4k'\cos\ell']a'^2\lambda_a'^2 + [4a'\sin\ell' - 2a'k'\cos 2\ell' + 2a'h'\sin 2\ell']\lambda_a'\lambda_h' \\
& + [-4a'\cos\ell' - 2a'h'\cos 2\ell' - 2a'k'\sin 2\ell']\lambda_a'\lambda_k' \\
& + \left[-h'\sin\ell' - k'\cos\ell' + \frac{3}{8}\sin 2\ell' + \frac{1}{3}h'\sin 3\ell' - \frac{1}{3}k'\cos 3\ell'\right]\lambda_h'^2 \\
& + \left[2h'\cos\ell' - 2k'\sin\ell' - \frac{3}{4}\cos 2\ell' - \frac{2}{3}h'\cos 3\ell' - \frac{2}{3}k'\sin 3\ell'\right]\lambda_h'\lambda_k' \\
& + \left[h'\sin\ell' + k'\cos\ell' - \frac{3}{8}\sin 2\ell' - \frac{1}{3}h'\sin 3\ell' + \frac{1}{3}k'\cos 3\ell'\right]\lambda_k'^2\Big\}.
\end{aligned} \tag{21}$$

Terms factored by λ_ℓ' have been omitted in equations above, since only transfers (no rendez-vous) are considered. Prime denotes the new variables resulting from the canonical transformation.

The general solution of the canonical system governed by the new Hamiltonian function F_1 can be obtained applying Hamilton-Jacobi theory as described in (da Silva Fernandes, 2009a). For a given set of initial conditions, one gets

$$a'(t) = \frac{a_0'}{1 + \dfrac{4a_0'}{\mu}\left(\dfrac{1}{2}\mathrm{E}t^2 - a_0' p_{a_0}' t\right)}, \tag{22}$$

$$h' = h_0' + \sqrt{\frac{5}{2}}\frac{\lambda_h'}{C}\left\{\tan^{-1}\left(\left(\frac{4\mu\mathrm{E}}{5C^2 a_0'} - 1\right)^{1/2}\right) - \tan^{-1}\left(\left(\frac{4\mu\mathrm{E}}{5C^2 a'} - 1\right)^{1/2}\right)\right\}, \tag{23}$$

$$k' = k_0' + \sqrt{\frac{5}{2}}\frac{\lambda_k'}{C}\left\{\tan^{-1}\left(\left(\frac{4\mu\mathrm{E}}{5C^2a_0'}-1\right)^{1/2}\right)-\tan^{-1}\left(\left(\frac{4\mu\mathrm{E}}{5C^2a'}-1\right)^{1/2}\right)\right\}, \tag{24}$$

$$\lambda_a'^2 = \left(\frac{a_0'}{a'}\right)^3\lambda_{a_0}'^2 + \frac{5}{8}C^2\left(\frac{a_0'}{a'^3}-\frac{1}{a'^2}\right), \tag{25}$$

$$\lambda_h' = \lambda_{h_0}', \tag{26}$$

$$\lambda_k' = \lambda_{k_0}', \tag{27}$$

with C and E given in terms of the initial conditions by

$$C^2 = \lambda_{h_0}'^2 + \lambda_{k_0}'^2$$

$$4\mu\mathrm{E} = a_0'\left(8\left(a_0'\lambda_{a_0}'\right)^2 + 5\left(\lambda_{h_0}'^2+\lambda_{k_0}'^2\right)\right).$$

The initial conditions for the state variables (generalized coordinates) are given by $a'(0)=a_0'$, $h'(0)=h_0'$ and $k'(0)=k_0'$.

Equations (22) through (27) represent the solution of the canonical system concerning the problem of optimal long duration low-thrust limited-power transfers between elliptic coplanar orbits with very small eccentricities in an inverse-square force field.

For maneuvers with arbitrary duration, the periodic terms must be included. Following Hori method (Hori, 1966) and applying the initial conditions, one gets a first order analytical solution:

$$\begin{aligned} a(t) = a'(t) + \sqrt{\frac{a'^5}{\mu^3}}\Big\{&\left[8h'\sin\ell' - 8k'\cos\ell'\right]a'^2\lambda_a' + \left[4a'\sin\ell' - 2a'k'\cos 2\ell' + 2a'h'\sin 2\ell'\right]\lambda_h' \\ &+\left[-4a'\cos\ell' - 2a'h'\cos 2\ell' - 2a'k''\sin 2\ell'\right]\lambda_k'\Big\}\Big|_{\ell_0'}^{\ell_f'}, \end{aligned} \tag{28}$$

$$\begin{aligned} h(t) = h'(t) + \sqrt{\frac{a'^5}{\mu^3}}\Big\{&\left[4a'\sin\ell' - 2a'k'\cos 2\ell' + 2a'h'\sin 2\ell'\right]\lambda_a' \\ &+\left[-2h'\sin\ell' - 2k'\cos\ell' + \frac{3}{4}\sin 2\ell' + \frac{2}{3}h'\sin 3\ell' - \frac{2}{3}k'\cos 3\ell'\right]\lambda_h' \\ &+\left[2h'\cos\ell' - 2k'\sin\ell' - \frac{3}{4}\cos 2\ell' - \frac{2}{3}h'\cos 3\ell' - \frac{2}{3}k'\sin 3\ell'\right]\lambda_k'\Big\}\Big|_{\ell_0'}^{\ell_f'}, \end{aligned} \tag{29}$$

$$\begin{aligned} k(t) = k'(t) + \sqrt{\frac{a'^5}{\mu^3}}\Big\{&\left[-4a'\cos\ell' - 2a'h'\cos 2\ell' - 2a'k'\sin 2\ell'\right]\lambda_a' \\ &+\left[2h'\cos\ell' - 2k'\sin\ell' - \frac{3}{4}\cos 2\ell' - \frac{2}{3}h'\cos 3\ell' - \frac{2}{3}k'\sin 3\ell'\right]\lambda_h' \\ &+\left[2h'\sin\ell' + 2k'\cos\ell' - \frac{3}{4}\sin 2\ell' - \frac{2}{3}h'\sin 3\ell' + \frac{2}{3}k'\cos 3\ell'\right]\lambda_k'\Big\}. \end{aligned} \tag{30}$$

Equations (28) – (30) represents a complete first order analytical solution, including short period terms, for optimal low-thrust limited-power trajectories concerning with the transfer problem between coplanar orbits with very small eccentricities an inverse-square force field. This solution involves three unknowns, the initial values of the adjoint variables, which must be determine to satisfy the two-point boundary value problem of going from the initial orbit $O_0 : (a_0, h_0, k_0))$ to the final orbit $O_f : (a_f, h_f, k_f))$. This point boundary value problem can be solved through a Newton-Raphson algorithm as described in da Silva Fernandes and Carvalho (2008) for transfers between arbitrary elliptical coplanar orbits.

4. PRELIMINARY ANALYSIS OF INTERPLANETARY MISSION

In this section, the analytical first order solution previously derived is applied in a preliminary analysis of interplanetary missions considering Earth-Venus, Earth-Mars and Earth-Asteroids belt transfers. The following assumptions are employed:

1. The orbits of the planets are circular;
2. The orbits of the planets lie in the plane of the ecliptic;
3. The flight of the space vehicle takes place in the plane of the ecliptic;
4. Only the heliocentric phase is considered; that is, the attraction of planets is neglected.

Table 1 shows a comparison between the results given by the analytical solution with numerical results obtained through a neighboring extremals algorithm based on state transition matrix (da Silva Fernandes, 2009b). This algorithm is applied to solve the two-point boundary value problem determined by the application of the Pontryagin Maximum Principle to the Mayer formulation of the optimization problem with the original state variables r, u, v and J , presented in Section 2. One sees the good agreement between the analytical and numerical results, mainly for transfers with long duration and short amplitude.

Table 1 – Comparison between and numerical results (in canonical units)

r_f/r_0	Duration	$J_{analytical}$	$J_{numerical}$
0.727 (Venus)	25	0.0005973	0.0005985
	50	0.0002987	0.0002990
1.523 (Mars)	25	0.0007208	0.0007246
	50	0.0003604	0.0003609
2.500 (Asteroids)	25	0.0027018	0.0028897
	50	0.0013509	0.0013859

5. CONCLUSION

In this paper an analytical first order solution for optimal low-thrust limited power trajectories for simple transfer (no rendezvous) between circular coplanar orbits with very small eccentricities in an inverse-square force field is presented. This analytical solution has been obtained through classical mathematical methods of Analytical Mechanics – canonical transformation theory, Lie-Hori perturbation method and separation of variables technique – and is expressed in terms of non-singular elements. A preliminary analysis of interplanetary transfers is performed and the analytical results are compared to the ones obtained through a numerical technique.

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