IMPROVED ESTIMATES OF PLASTIC ZONES AROUND CRACK TIPS PART 1: THE EFFECTS OF THE T-STRESSES AND OF THE WESTERGAARD STRESS FUNCTION

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Abstract. *Stress intensity factors (SIF), the fulcrum for linear elastic fracture mechanics (LEFM) predictions, quantify LE stress fields around crack tips, except very near the tips, in which they predict singular stresses. Hence, SIF-based analysis cannot describe stresses exactly at the cracked piece critical point. However, real materials are neither linear nor elastic at high stresses. Thus all loaded cracked pieces must have a nonlinear plastic zone (pz) close to their crack tips. If this (pz) size is small in relation to the piece dimensions, the stress field remains predominantly LE, hence controlled by the SIF. In such cases, the SIF can then be used to estimate the (pz) size by locating up to where the material yields in front of the crack tip. This means that LEFM predictions can be self-validated by the (pz) size estimated from a LE stress field. In other words, if (pz) is small, SIF can be used to describe crack effects. Therefore, the precise estimation of (pz) is a problem of major practical importance for crack analysis and structural integrity evaluations.The first classical (pz) estimates proposed by Irwin and by Dugdale are based only on the SIF value, but it has long been recognized their precision is quite limited to very low nominal stresses. Improved estimates have been proposed considering the T-stress, the name given by Irwin for the Williams series constant or zero order term. However, neither the SIF nor the T-stress can reproduce LE stress fields which obey all boundary conditions in cracked components. In particular, they cannot reproduce the nominal stress far from the crack tip. It is quite surprising that such a fact has not been well treated in the literature so far, since it has a major influence on the LE predicted (pz) size and shape. Indeed, using the correct LE stress field in the Griffith plate, generated by its complete Westergaard stress function (which of course not only reproduces the nominal stress that loads it, but is also confirmed by the Inglis plate solution when its elliptical notch root is supposed equal to half the crack tip opening displacement), it is showed that the nominal stress to yielding strength ratio has a major influence on the (pz) size and shape. This first part of this two-paper work presents the complete LE stress field solution for the Griffith plate, and compares the (pz) estimates generated from it with the classical and the T-stress corrected (pz) estimates, demonstrating the importance of using correct stress fields to evaluate LEFM limitations.*

Keywords*: Complete Westergaard stress function; T-stress; crack tip plastic zone*

1. INTRODUCTION

Analyzing a Griffith plate (1920) with a $2a$ crack, loaded in mode I by a nominal stress σ_n and with the linear elastic (LE) stress field generated by a SIF $K_I = \sigma_n\sqrt{\pi a}$ according Irwin (1957) and Williams (1957), one may note that the expected response is not met, since $\sigma_{yy}(SIF, r \to \infty, 0) = 0$, instead of $\sigma_{yy}(SIF, r \to \infty, 0) = \sigma_n$ as needed, where r is the distance from the tip, θ is the angle measured from the crack plane, and $g_{ij}(\theta)$ are the Irwin θ-functions. This fact indicates that the SIF alone cannot model well some simple crack problems, especially at high stress levels. Moreover, LE analysis cannot describe stresses and strains inside plastic zones $pz(\theta)$ around crack tips. However, for design purposes, $pz(\theta)$ are traditionally estimated from simplified LE analysis according was done by Irwin (1958) and Dugdale (1960), assuming they depend only on K_I (in mode I). Indeed, equating the LE Mises stress to S_Y, the yielding strength, the simplest mode I elastic-plastic frontiers in plane stress (pl-σ), Eq. (1), and in plane strain (pl-ε), Eq. (2), are estimated by (Unger, 2001)

$$pz(\theta)_{pl-\sigma} = \left(K_I{}^2/2\pi S_Y{}^2\right)\cos^2(\theta/2)\left(1+3\sin^2(\theta/2)\right) \tag{1}$$

$$pz(\theta)_{pl-\varepsilon} = \left(K_I{}^2 / 2\pi\, S_Y{}^2\right)\cos^2(\theta/2)\left((1-2\nu)^2 + 3\sin^2(\theta/2)\right) \tag{2}$$

in which ν is Poisson's coefficient. Thus, according to this classical estimate, the $pz(\theta)$ size directly ahead of crack tips in $pl\text{-}\sigma$ should be $pz(0)_{pl\text{-}\sigma} = pz_0 = (1/2\pi)(K_I/S_Y)^2$. The pz_0 size is used here as a reference to normalize pz plots. However, the SIF-based stress field, $\sigma_{ij} = f(K_I)$, is exact only when $r \to 0$, where the assumed LE behavior has no sense. Singular elastic-plastic (EP) estimates, such as the HRR field, do not solve this problem either. As the pz border may not be too close to crack tips, it is worth to at least estimate the effect of σ_n/S_Y on $pz(\theta)$, where S_Y is the yielding strength. This task has already been fulfilled by Rodriguez *et al*. (2008). This author showed that the Mises plastic zone $pz(\theta)_M$ are insensitive to the increase of σ_n/S_Y when they are estimated using K_I alone to describe the stress field, which components are obtained according to Eq. (3):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi\, r}}\cos\left(\frac{\theta}{2}\right)\begin{Bmatrix} 1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \\ 1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \end{Bmatrix} \tag{3}$$

The r values that satisfy $\sigma_{Mises}(\theta) = S_Y$ using these (Irwin) stress components can be used to estimate the elastic-plastic frontier of the Mises plastic zone around the crack tip $pz(\theta)^{Irw}_{M,pl-\sigma}$ in plane stress or $pz(\theta)^{Irw}_{M,pl-\varepsilon}$ in plane strain. Such classical $pz(\theta)_M$ estimates are quite reasonable for very low nominal stresses, but they are insensible to the nominal stress to yielding strength σ_n/S_Y ratio. In addition, as the stresses inside the plastic zone are limited by the yielding strength, the truncated LE stress field cannot satisfy equilibrium conditions.

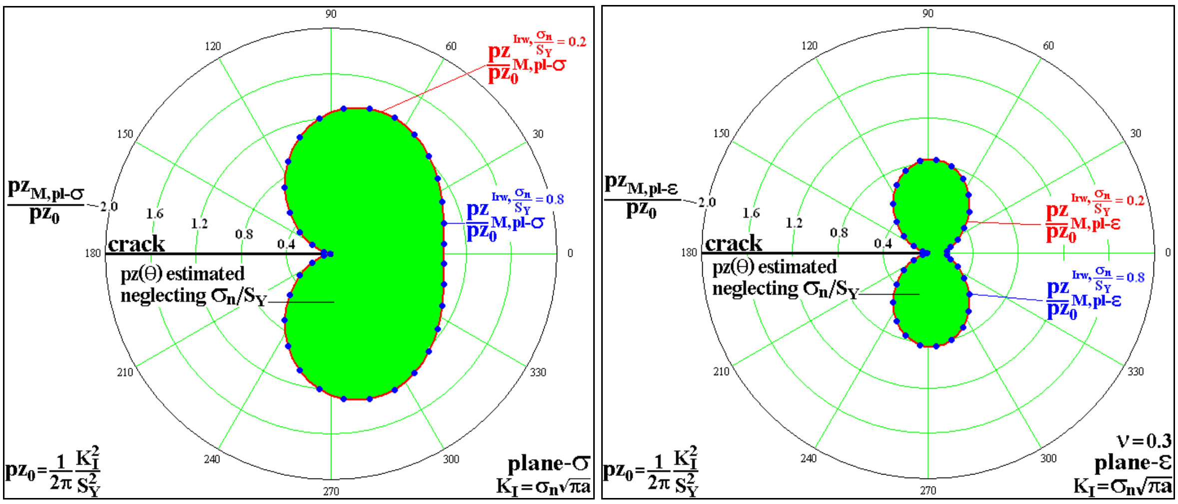

Figure 1. The $pz(\theta)_M$, insensitive to σ_n/S_Y, estimated for the Griffith plate loaded in mode I from the Irwin stress field (which depends only on K_I) in plane stress and in plane strain

However, improved $pz(\theta)$ estimates for real components are very sensitive to the σ_n/S_Y ratio, since such components are usually designed for much higher stresses, typically with yield safety factors $1.2 < \phi_Y < 3$. To prove this affirmative, this work is divided in two parts: Part 1 deals with $pz(\theta)$ estimated from the simplified stress field described by (3), from the stress field generated by SIF plus *T-stress*, or from the complete stress field generated from the Westergaard stress function (Westergaard, 1939). Part 2 describes $pz(\theta)$ estimates obtained from stress fields generated from the Williams series, and demonstrates that these estimates are practically identical to the results obtained from the Westergaard stress function, increasing the number of terms in the series.

2. THE T-STRESS TERM

Irwin (1958) proposed adding a constant term to the stress component σ_{xx} parallel to the crack direction given by (3), based on photoelastic stress field around crack tips measurements performed by Weels and Post (1958), naming it the *T-stress*. Larsson and Carlsson (1973), investigating the limits recommended by ASTM (1970) for the SIF use, said that the *T-stress* adjusted the shapes of the plastic zones estimated from LE analysis, approximating them to the $pz(\theta)$ shapes obtained from their nonlinear Finite Element (FE) numerical analysis. After that, the *T-stress* has been widely explored to model some interesting problems by Rice (1974), Leevers and Radon (1982) and others. The *T-stress* addition in the SIF-based LE field results in:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \begin{Bmatrix} 1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \\ 1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \end{Bmatrix} + \begin{Bmatrix} T_{stress} \\ 0 \\ 0 \end{Bmatrix} \tag{4}$$

Fett (1998) reviewed the theoretical foundation and listed *T-stress* values for several geometries. However, despite being able to make LE $pz(\theta)$ estimates sensitive to σ_n/S_Y, such K_I +*T-stress* solutions are not complete, thus cannot be used carelessly. For example, they do not satisfy the expected response of $\sigma_{yy}^{\infty} = \sigma_n$ at $x = \pm\infty$ in the Griffith plate. Indeed, the next section shows that the correct LE solution for the Griffith plate requires that its the complete stress field be determined from its Westergaard stress function.

3. THE WESTERGAARD STRESS FUNCTION

Irwin (1957) and Williams (1957) solved independently the stress field around a crack tip problem, starting the modern Fracture Mechanics era by introducing the idea that the linear elastic (LE) field in any cracked component around the crack tip is controlled by their so-called stress intensity factors (SIF). The traditional notation for SIF is K_I, K_{II} and K_{III}, in mode I, II and III, respectively. Williams used an expansion in infinite series and Irwin used the Westergaard stress function to solve this problem. In this case, the stress field is obtained using complex variables as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} \mathrm{Re}(Z(z)) - y\,\mathrm{Im}(Z'(z)) \\ \mathrm{Re}(Z(z)) + y\,\mathrm{Im}(Z'(z)) \\ -y\,\mathrm{Re}(Z'(z)) \end{Bmatrix} \tag{5}$$

in which $z = x + iy$, $i = \sqrt{-1}$ and $Z(z)$ is the Westergaard stress function for a biaxially loaded plate:

$$Z = \frac{\sigma_n z}{\sqrt{z^2 - a^2}} \tag{6}$$

For the the Griffith plate, in particular, it is necessary to add a $-\sigma_n$ term to the σ_{xx} component to force the boundary condition $\sigma_{xx}(\infty) = 0$, an adequate mathematical trick, since a constant stress in the x direction does not affect the stress field near the crack tip. Eftis and Liebowitz (1972) presented a similar Westergaard stress function for a finite rectangular plate with a central crack:

$$Z_{RtgPlate} = \frac{\sigma_n \sin\left(\frac{z\pi}{W}\right)\left[\frac{a\pi}{W}\csc\left(\frac{a\pi}{W}\right)\right]^{0,5}}{\left[\sin^2\left(\frac{z\pi}{W}\right) - \sin^2\left(\frac{a\pi}{W}\right)\right]^{0,5}} \tag{7}$$

Sousa *et al.* (2009) showed, when studying improved estimates for plastic zones, that (7) generates a LE stress field for a finite rectangular plate with $a/W = 0.1$ which is very close to the Griffith plate, as it could be expected.

The following sections shows the size and shape of the plastic zones estimated from LE stress fields based on K_I alone and K_I + *T-stress* for the circular disk and for a finite rectangular plate. In the latter example, the complete stress field generated from Eq. (7) is also considered.

4. PLASTIC ZONES ESTIMATED FROM LINEAR ELASTIC ANALYSIS

As shown in Fig. 1, the plastic zone shapes estimated from K_I-based LE stress fields expressed by Eq. (3) are insensible to the σ_n/S_Y ratio and to the crack tip distance from the cracked component border, or to its residual ligament, since such LE stress fields depend on the SIF K_I alone. In other words, although K_I surely increases when the nominal load augments, this load increase has no effect on the stress field θ-component, thus the σ_n/S_Y ratio has no influence on the $pz(\theta)$ shape. This same observation is also valid for modes II and III, but such problems are considered beyond this paper scope. This section shows improved plastic zones estimates for the cases of a circular disk and a finite rectangular plate with a central crack, also based on their LE stress fields, but considering terms which are neglected to obtain their K_I expressions. For the circular disk with an internal crack, the plastic zone frontiers $pz(\theta)$ estimated from its K_I + *T-stress* LE stress fields are sensible to σ_n/S_Y and to its residual ligament, meaning to the relationship between the crack length and the diameter of the disk. For the rectangular plate, the $pz(\theta)$ estimated from K_I + *T-stress*, although sensible to σ_n/S_Y, is not its complete LE solution, because such improved stress field still does not satisfy neither the horizontal traction boundary condition at its lateral borders, nor the expected response for the σ_{yy} stress component at these borders.

4.1. Circular disk with an internal crack

Figure 2 shows the circular disk with diameter 2R and a central crack of length 2a under a constant radial nominal stress σ_n.

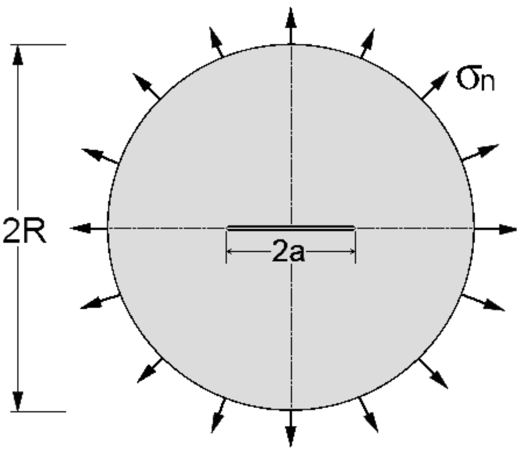

Figure 2. Circular disk with an internal crack

According to Fett (1998), the *T-stress* for this cracked disk depends on σ_n and on its a/R ratio as expressed in Eq. (8), and its K_I is given by Eq. (9) and Table 1 (Tada *et al.*, 1985):

$$T_{stress} = \sigma_n \frac{-1 + a/R - 2.34\, a/R^2 + 4.27^3 - 3.326\, a/R^4 + 0.9824^5}{1 - a/R} \tag{8}$$

$$K_I = \sigma_n \sqrt{\pi a}\, F(a/R) \tag{9}$$

Table 1. $F(a/R)$

a/R	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$F(a/W)$	1.000	1.022	1.062	1.135	1.252	1.393	1.513	1.597	2.236	3.036

From these K_I and *T-stress* values, it is possible to estimate the Mises plastic zones $pz(\theta)^{K_I+T}_{M,pl-\sigma}$ and $pz(\theta)^{K_I+T}_{M,pl-\varepsilon}$ in plane stress and plane strain for this disk (induced by its LE stress field expressed by Eq. (4),) see Fig. 3. This can be done by finding the r values which satisfy $\sigma_{Mises}(\theta) = S_Y$.

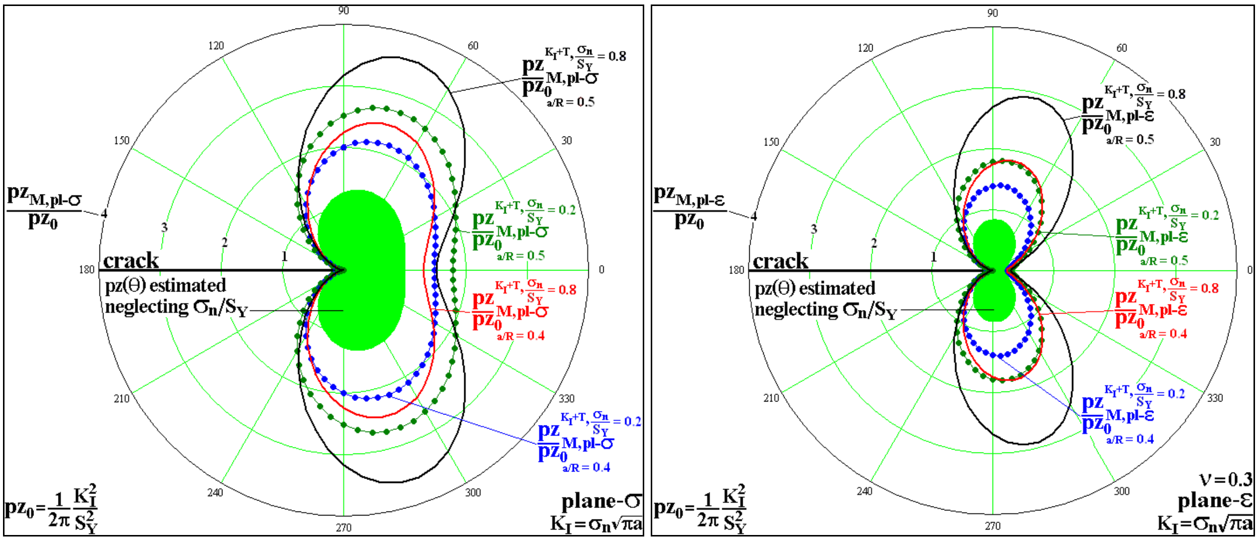

Figure 3. Mises $pz(\theta)_M$ frontiers estimated in plane stress and in plane strain for the circular disk under uniform nominal radial stress with an internal crack from the LE stress field generated by its K_I + *T-stress*, as described in (4).

Fig. 3 shows that with the addition of the disk *T-stress* to its SIF-based σ_{xx} component, $pz(\theta)_M$ become sensitive to σ_n/S_Y and to its residual ligament. The effect of σ_n/S_Y can be seen by fixing a/R = 0.4 or 0.5 and increasing σ_n/S_Y from 0.2 to 0.8. Similarly, the effect of a/R can be seen by fixing σ_n/S_Y = 0.2 or 0.8 and increasing a/R from 0.4 to 0.5. These estimated plastic zones not only increase in size, but also change their shapes when σ_n/S_Y or a/R augment.

4.2. Rectangular plate with a central crack

Figure 4 shows a rectangular plate of width 2*W* and high 2*H* with a central crack of length 2*a,* loaded in mode I by a nominal stress σ_n.

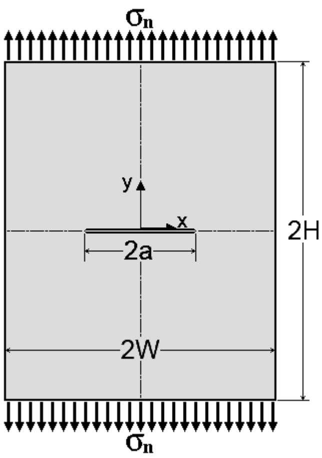

Figure 4. Rectangular plate with a central crack

According to Fett (1998), this plate *T-stress* depends on σ_n and on the ratios a/W and H/W:

$$T_{stress} = \alpha \cdot \sigma_n / (1 + a/W) \tag{10}$$

Table 2. α(a/W, H/W = 1.25).

a/W	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
H/W = 1.25	-1.0	-0.9	-0.83	-0.777	-0.716	-0.656	-0.596	-0.53	-0.47	-0.43	-0.413

According Tada *et al* (1985), this plate SIF K_I can also be obtained from Eq. (9), using $F(a/W)$ given in Tab. 3.

Table 3. $F(a/W)$ values for, the plate with a central crack loaded in mode I.

a/W	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$F(a/W)$	1.0000	1.0060	1.0246	1.0577	1.1094	1.1867	1.3033	1.4882	1.8160	2.5776

Plastic zones estimated from 3 LE stress fields (K_I, K_I + *T-stress*, and the complete field induced by the Westergaard stress function, Eq. (7)) are shown below. Figure 5 shows $pz(\theta)_M$ estimates, in plane stress, for σ_n/S_Y = 0.2, 0.4, 0.5, 0.6, 0.7, and 0.8, for a/W = 0.1. Figure 6 shows $pz(\theta)_M$ estimated in plane strain under same these same conditions.

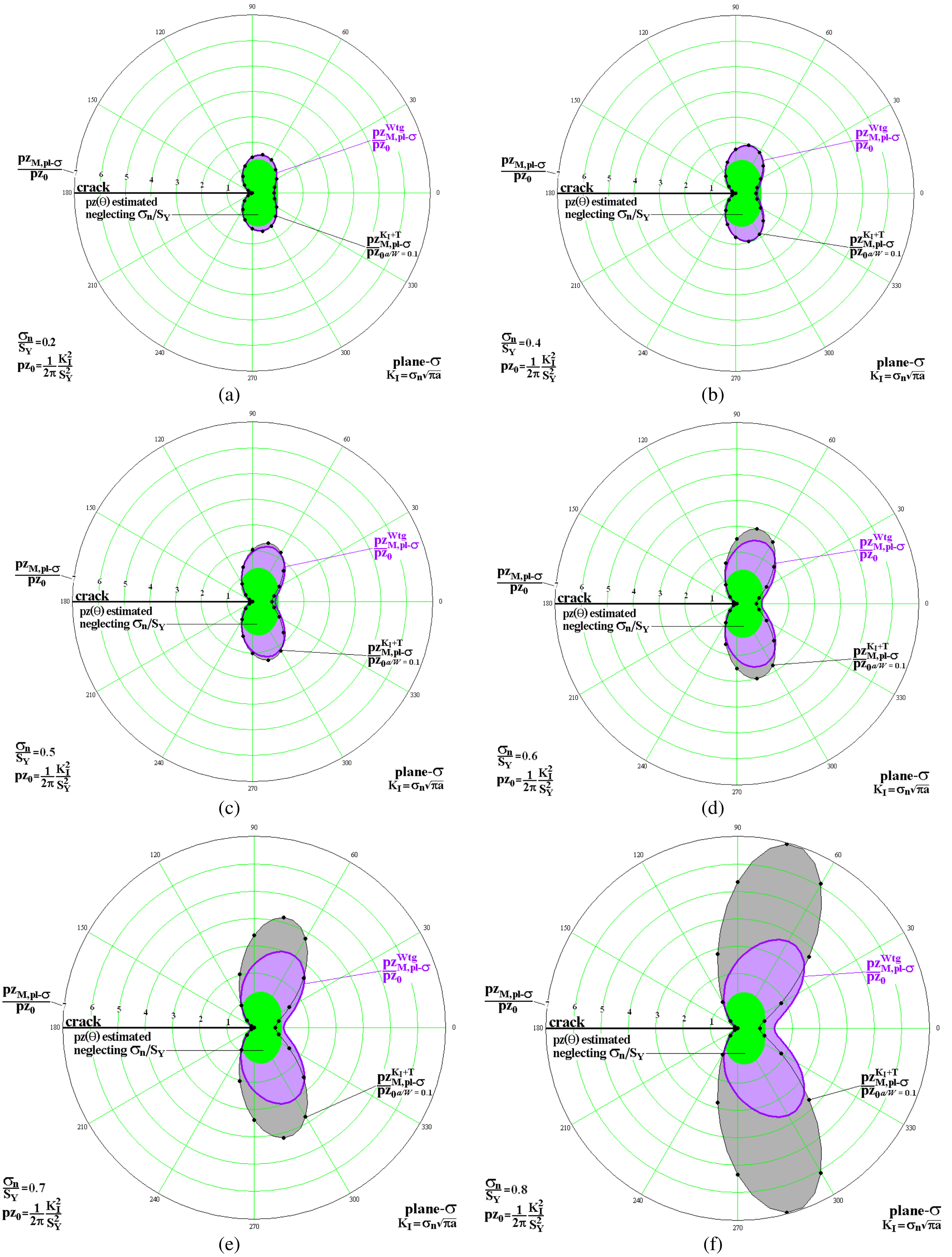

Figure 5. The $pz(\theta)_M$ estimated for the finite rectangular plate with an internal crack in plane stress, for a/W = 0.1 and (a) σ_n/S_Y = 0.2, (b) σ_n/S_Y = 0.4, (c) σ_n/S_Y = 0.5, (d) σ_n/S_Y = 0.6, (e) σ_n/S_Y = 0.7, and (f) σ_n/S_Y = 0.8.

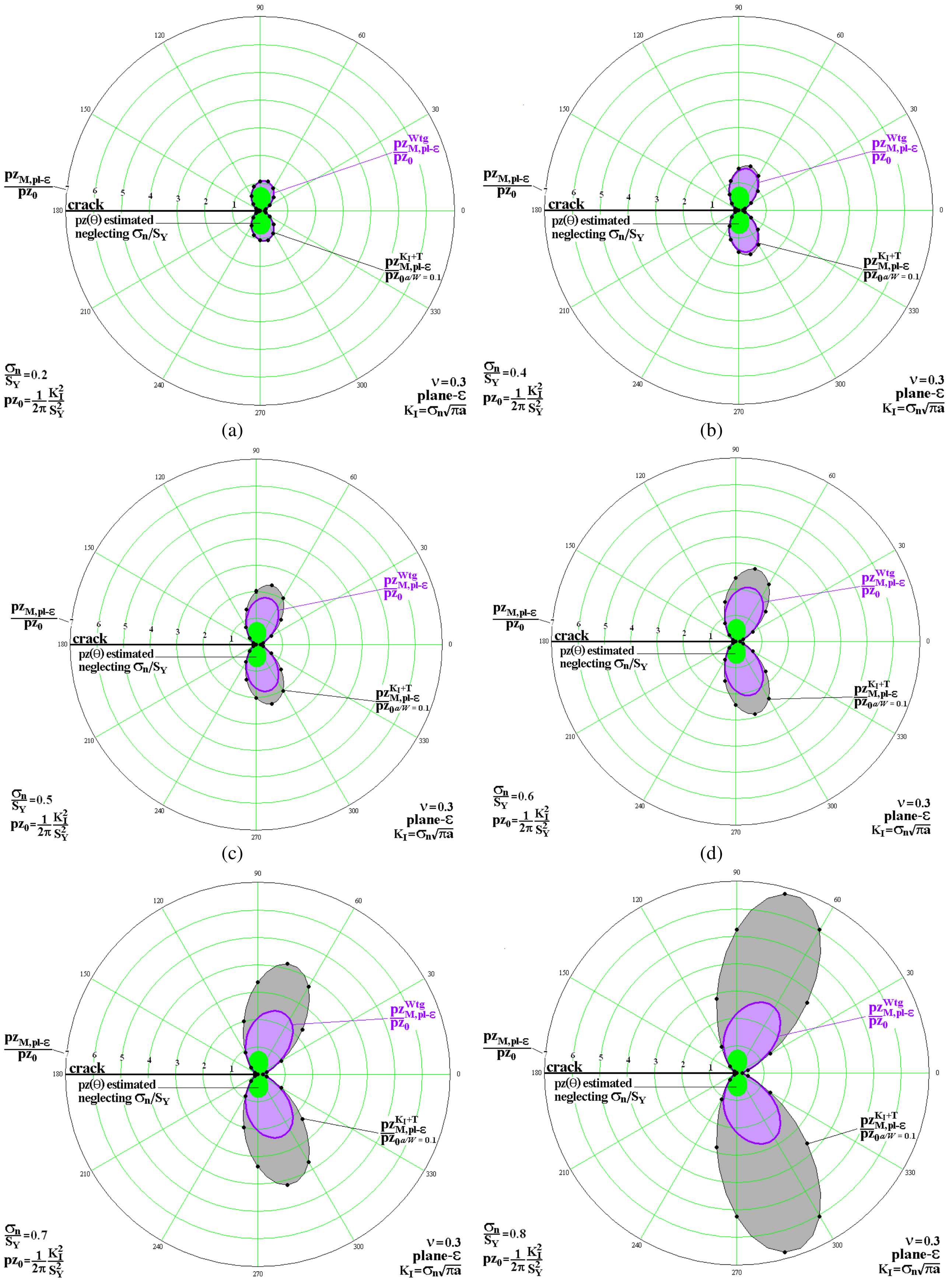

Figure 6. The $pz(\theta)_M$ estimated for the finite rectangular plate with internal crack in plane strain, for $a/W = 0.1$ and (a) $\sigma_n/S_Y = 0.2$, (b) $\sigma_n/S_Y = 0.4$, (c) $\sigma_n/S_Y = 0.5$, (d) $\sigma_n/S_Y = 0.6$, (e) $\sigma_n/S_Y = 0.7$, and (f) $\sigma_n/S_Y = 0.8$.

Fig. 5 and Fig. 6 show that when the *T-stress* is added to σ_{xx} stress component generated from K_I alone, the elastic-plastic Mises frontiers estimated from the resulting LE stress field capture the effects of σ_n/S_Y in the size and shape of the plastic zones, exactly as observed in previous example of the circular disk with an internal crack. In addition, these figures show that the important σ_n/S_Y effect can also be seen in $pz(\theta)_M$ estimated from the complete LE stress field generated from the cracked finite rectangular plate Westergaard stress function. It is important to note how theses improved plastic zones estimates increase and change shape as the σ_n/S_Y ratio augments. Moreover, it is still more important to note that the plastic zone sizes estimated from the K_I + *T-stress* LE stress field can be considerably larger than the $pz_M(\theta)$ elastic-plastic frontiers estimated from the plate complete LE stress field generated from the plate Westergaard stress function, for larger σ_n/S_Y ratios. As this is the ***correct*** LE field for that plate, this result highlights the importance of using estimates based on $K_I + T$ with caution.

5. CONCLUSION

This work deals with plastic zones sizes and shapes estimated from three different linear elastic stress fields: (i) from the traditional field based solely on the cracked component SIF; (ii) from the improved field generated by adding the cracked component *T-stress* to the its σ_{xx} stress generated from its SIF K_I; and (iii) from the correct LE stress field calculated from the component Westergaard stress function (for the finite rectangular plate with a central crack case). The SIF-based LE stress field, besides not satisfying boundary conditions, generates $pz(\theta)_M$ that are insensitive to σ_n/S_Y. This is not the case for the $pz_M(\theta)$ estimated from the LE stress field obtained by adding the *T-stress* to the σ_{xx} stress component generated from K_I, which are sensitive to both the σ_n/S_Y ratio and to the residual ligament size. However, since the *T-stress* term is added only to the σ_{xx} stress component, the resulting LE stress field cannot obey the cracked component boundary conditions. Therefore, such an approximation cannot generate the "best" LE estimates for the plastic zones sizes and shapes. However, the complete LE stress field can be generated from the cracked component Westergaard stress function, when it is available. Therefore, one may argue that the estimates of plastic zones based in such correct LE solutions are better than the estimates based on partial stress fields solely from K_I or from K_I + *T-stress.* As expected, for low ratios of σ_n/S_Y, the plastic zones estimated from K_I plus *T-stress* are very close to the ones obtained by the Westergaard stress function. But higher yield safety factors, of the order of those used in most structural components (which are typically designed to sustain maximum loads which induce $0.3 < \sigma_n/S_Y < 0.85$), improved $pz_M(\theta)$ estimates based on $K_I + T$ should be used with caution. Since it is the plastic zone size and shape which controls the validity of traditional Fracture Mechanics estimates, including in particular critical loads and crack sizes, this problem should not be overlooked by structural engineers.

6. ACKNOWLEDGEMENT

The first author acknowledge the financial support by Tecgraf (Group of Technology in Computer Graphics), PUC-Rio, Rio de Janeiro.

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