# IMPROVED ESTIMATES OF PLASTIC ZONES AROUND CRACK TIPS PART 2: PZ ESTIMATES USING STRESS FIELDS GENERATED BY THE COMPLETE WILLIAMS SERIES

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***Abstract.*** *In Part 1 of this paper it was seen estimates of plastic zone from two linear elastic (LE) stress fields, which are: the stress field generated from Stress intensity factors (SIF), which are the fulcrum for linear elastic fracture mechanics (LEFM) predictions, and the stress field obtained from SIF plus T-stress, the name given by Irwin for the Williams series constant, or zero order, term parallel to the crack plane. However, it was showed in the Part 1 that the LE stress field generated by SIF besides not satisfying boundary conditions, it generates plastic zones (pz) that are insensitive to ratio between nominal stress and yielding strength ($\sigma_n/S_Y$). This is not the case for the LE stress field generated by SIF plus T-stress. Nevertheless, these two LE stress fields cannot replicate the nominal stress far from the crack tip in the component* $\sigma_{yy}$. *Indeed, using the correct LE stress field in the Griffith plate, generated by its complete Westergaard stress function, it is showed that the nominal stress to yielding strength ratio has a major influence on the pz size and shape. In addition, the Westergaard stress function may be confirmed by the Inglis plate solution when its elliptical notch root is supposed equal to half the crack tip opening displacement. This second part of this two-paper work shows that it is possible to use the complete Williams series to reproduce the correct stress field generated by the Westergaard stress function. Therefore, as presented in the part 1 paper, an appropriated (pz) estimative could be obtained from the complete William series for the cracked component.*

***Keywords****: Complete Williams series, Westergaard stress function, Plastic zones around crack tips*

## 1. INTRODUCTION

Irwin (1957) and Williams (1957) introduced the stress intensity factors (SIF) $K_I$, $K_{II}$, or $K_{III}$ as the parameters responsible for characterizing the severity of the stress field around crack tips in mode I, II or III. However, the linear elastic (LE) stress field generated from the SIF alone is correct only very close of the crack tips. This affirmation can be confirmed by the proper analysis of the Griffith plate (1920) (the infinite plate with a central 2*a* crack, loaded in mode I by a nominal stress $\sigma_n$ perpendicular to the crack plane). The LE stress field generated from its SIF $K_I = \sigma_n\sqrt{\pi a}$ does not satisfy boundary conditions. Indeed, that simplified stress field predicts that $\sigma_{yy}(r \to \infty) = 0$ instead of $\sigma_{yy}(r \to \infty) = \sigma_n$ as it should, where $\sigma_{yy}$ is the stress component parallel to $\sigma_n$ and $r$ is the distance from the crack tip. Due to that, LE stress fields obtained from $K_I$ cannot estimate well the plastic zone $pz(\theta)$ around crack tips frontiers, except for very low nominal stress to yielding strength $\sigma_n/S_Y$ ratios, much lower than those actually used in most structural components. However, for teaching and designing purposes, $pz(\theta)$ are traditionally estimated from simplified LE analysis, assuming that they depend only on $K_I$ (in mode I) according was done by Irwin (1958).

Part 1 of this work showed estimated plastic zones from stress fields generated from SIF, SIF + *T-stress*, and Westergaard stress function, which was presented by Westergaard (1939), demonstrating that the latter two generate *pz* that depend on the $\sigma_n/S_Y$ ratio. It also demonstrated that the (constant) *T-stress* addition to the $\sigma_{xx}$ stress component does not satisfy the boundary conditions either. Analyzing the Griffith plate, this work shows that the plastic zone estimates obtained from a stress field generated from the Williams series are identical to the estimates obtained from the complete stress field generated from its Westergaard stress function, when increasing the number of terms in the series. Since the complete stress field obtained from the Westergaard stress function satisfies all boundary conditions, it is expected that more consistent *pz* estimates can also be obtained from the William series which reproduces it.

This article first presents a brief development of the Williams series (Lopes, 2002), indicating that stress fields generated using only the first term of the series are identical to the SIF stress field given by Eq. (3). Next, it shows how to get the SIF stress field from the Westergaard stress function (Castro and Meggiolaro, 2009). Finally, it shows through two examples that the estimates of the plastic zones are sensitive to $\sigma_n/S_Y$ when the number of terms in the Williams series is increased. Moreover, these examples demonstrate that the *pz* estimates based on the Williams series converges to the estimates based on the complete stress field obtained from Westergaard stress function, when an adequate number of terms is considered in the series.

## 2. THE WILLIAMS SERIES

Williams (1957) proposed the following stress function to solve LE stress analysis problems:

$$\Phi = r^{\lambda+1}\left[\mathrm{c}_1 \sin(\lambda+1)\theta^* + \mathrm{c}_2 \cos(\lambda+1)\theta^* + \mathrm{c}_3 \sin(\lambda-1)\theta^* + \mathrm{c}_4 \cos(\lambda-1)\theta^*\right] \tag{1}$$

in which $c_1$, $c_2$, $c_3$ and $c_4$ are constants and $\lambda$ is an exponent to be determined according the boundary conditions of the problem. Eq. (1) can be written in a more compact form as:

$$\Phi = r^{\lambda+1}\, F(\theta^*, \lambda) \tag{2}$$

In order to be a stress function, $\Phi = \Phi(r,\theta)$ must satisfy the following equation:

$$\nabla^4 \Phi(r,\theta) = 0 \tag{3}$$

The stress components in polar coordinates may be obtained from the William stress function:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{r^2}\dfrac{\partial^2\Phi}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial\Phi}{\partial r} \\ \dfrac{\partial^2\Phi}{\partial r^2} \\ \dfrac{1}{r^2}\dfrac{\partial\Phi}{\partial\theta} + \dfrac{1}{r}\dfrac{\partial^2\Phi}{\partial r\partial\theta} \end{Bmatrix} \tag{4}$$

By substituting the William stress function, Eq. (1) or Eq. (2), in Eq. (4), results in:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = r^{\lambda-1} \begin{Bmatrix} F''(\theta^*) + (\lambda+1)\, F(\theta^*) \\ \lambda\,(\lambda+1)\, F(\theta^*) \\ -\lambda\, F'(\theta^*) \end{Bmatrix} \tag{5}$$

in which $F'(\theta^*)$ is the derivative of *F* with respect to $\theta^*$. For the condition of a free surface, where there is no tractions on the surface, the following equation may be written:

$$\sigma_{\theta\theta}(0) = \sigma_{\theta\theta}(2\pi) = \sigma_{r\theta}(0) = \sigma_{r\theta}(2\pi) = 0 \tag{6}$$

Therefore:

$$F(0) = F(2\pi) = F'(0) = F'(2\pi) = 0 \tag{7}$$

If Eq. (1) constants are not null, the boundary conditions described in Eq. (7) are satisfied if:

$$\sin(2\pi\lambda) = 0 \tag{8}$$

$$\lambda = \frac{n}{2},\ n = 1, 2, 3, \ldots \tag{9}$$

Substituting Eq. (7) and Eq. (9) in Eq. (1), two constants are eliminated, resulting in:

$$\Phi = r^{\frac{n}{2}+1}\begin{bmatrix} c_3\left[\sin\left(\frac{n}{2}-1\right)\theta^*\right] - \frac{n-2}{n+2}\sin\left(\frac{n}{2}+1\right)\theta^* + \\ c_4\left[\cos\left(\frac{n}{2}-1\right)\theta^*\right] - \cos\left(\frac{n}{2}+1\right)\theta^* \end{bmatrix} \tag{10}$$

Expressing $\theta = \theta^* - \pi$ in Eq. (10), results in:

$$\Phi = r^{\frac{3}{2}}\left[ s_1\left[-\cos\left(\frac{\theta}{2}\right) - \frac{1}{3}\cos\left(\frac{3\theta}{2}\right)\right] + t_1\left[-\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right)\right]\right] + s_2 r^2\left[1-\cos(2\theta)\right] + \cdots \tag{11}$$

in which $s_i$ and $t_i$ are constants that must be defined (i = 1, 2, 3, …). Substituting Eq. (11) in Eq. (4):

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{Bmatrix} = \frac{1}{4\sqrt{r}}\begin{Bmatrix} s_1\left[-5\cos\left(\frac{\theta}{2}\right)+\cos\left(\frac{3\theta}{2}\right)\right] + t_1\left[-5\sin\left(\frac{\theta}{2}\right)+3\sin\left(\frac{3\theta}{2}\right)\right] \\ s_1\left[-3\cos\left(\frac{\theta}{2}\right)-\cos\left(\frac{3\theta}{2}\right)\right] + t_1\left[-3\sin\left(\frac{\theta}{2}\right)-3\sin\left(\frac{3\theta}{2}\right)\right] \\ s_1\left[-\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{2}\right)\right] + t_1\left[\cos\left(\frac{\theta}{2}\right)+3\cos\left(\frac{3\theta}{2}\right)\right] \end{Bmatrix} + \begin{Bmatrix}\cdots\\ \cdots\\ \cdots\end{Bmatrix} \tag{12}$$

The constants $s_1$ and $t_1$ may be related to modes I and II. Therefore:

$$s_1 = -\frac{K_I}{\sqrt{2\pi}} \text{ and } t_1 = \frac{K_{II}}{\sqrt{2\pi}} \tag{13}$$

Considering the first term of the stress field generated from the William series, it is possible to obtain the stress field generated by SIF. This shows that the Williams series result, for points very close to the crack tip, in functions that have $K_I$ as a single parameter.

## 3. THE WESTERGAARD STRESS FUNCTION

The general expression for obtaining stress components from the $Z(z)$ Westergaard stress function is (Castro and Meggiolaro, 2009):

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix} = \begin{Bmatrix}\mathrm{Re}(Z(z)) - y\,\mathrm{Im}(Z'(z))\\ \mathrm{Re}(Z(z)) + y\,\mathrm{Im}(Z'(z))\\ -y\,\mathrm{Re}(Z'(z))\end{Bmatrix} \tag{14}$$

in which $z = x + iy$ and $i = \sqrt{-1}$. In the case of Griffith plate, the Westergaard stress function is:

$$Z = \frac{\sigma_n z}{\sqrt{z^2 - a^2}} \tag{15}$$

When this infinite plate (Griffith plate) is loaded in the direction parallel to the crack plane, it is necessary to add $-\sigma_n$ in component $\sigma_{xx}$ to force $\sigma_{xx}(\infty) = 0$. This addition is an adequate mathematical trick since a constant stress in the $x$ direction does not affect the stress field near the crack tip.

Using Eq. (15), defining an axis translation $x_p = x - a$, in which the origin of the coordinate system is at the crack tip, and interpreting $\sigma_n/\sqrt{\pi a}$ constant as $K_I$, the component $\sigma_{yy}(x_p, 0)$ very close of the crack tip ($x_p << a$) is given by:

$$\sigma_{yy} = \frac{\sigma_n (x_p + a)}{\sqrt{(x_p + a)^2 - a^2}} \cong \frac{\sigma_n a}{\sqrt{2 a x_p}} = \frac{\sigma_n \sqrt{\pi a}}{\sqrt{2\pi x_p}} = \frac{K_I}{\sqrt{2\pi x_p}} \tag{16}$$

It is noted that the Irwin solution reproduces the singularity $\sigma_{ij} = f\left(1/\sqrt{x_p}\right)$ of the Williams series. Using relations $r^2 = (x-a)^2 + y^2$ and $\theta = \arctan[y/(x-a)]$, it is possible to transform the original function $Z(z)$ in an equivalent function with the origin at the crack tip, resulting in:

$$Z_p(z_p) = \frac{K_I}{\sqrt{2\pi z_p}} = \frac{K_I}{\sqrt{2\pi r e^{i\theta}}} \tag{17}$$

Substituting Eq. (17) in the stress field represented by Eq. (14), one has:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \begin{Bmatrix} 1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \\ 1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right) \end{Bmatrix} \tag{18}$$

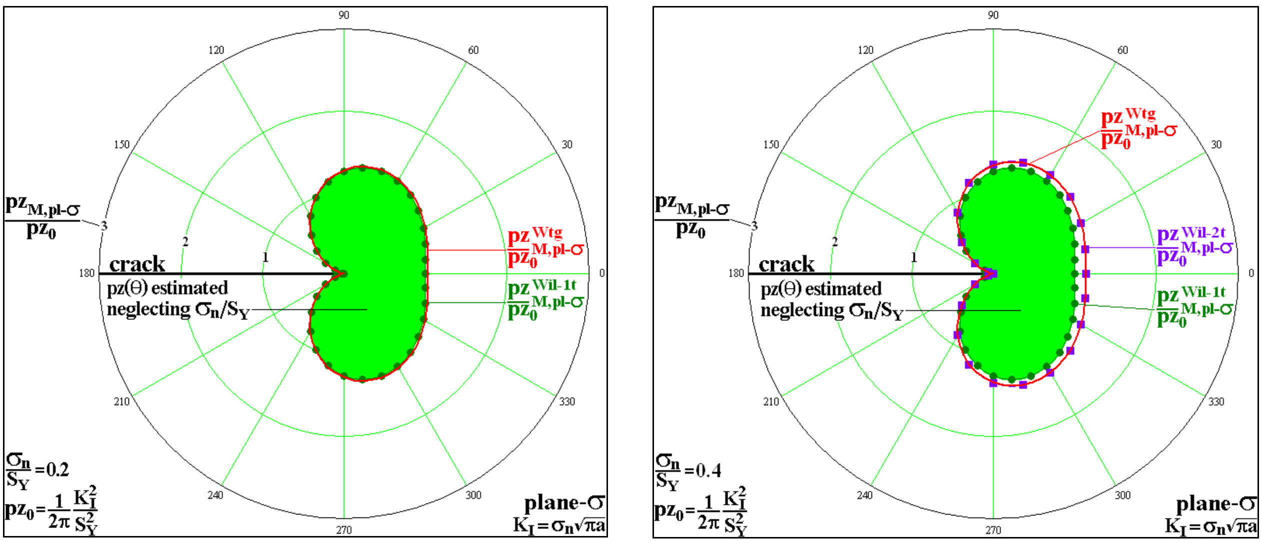
Equation (18) shows that the Westergaard stress function also results in functions that have $K_I$ as a single parameter at locations close to the crack tip. Although it is not demonstrated in this work, it can be shown (Lopes, 2002) that the stress field obtained from the William series converges to the Westergaard stress field, if an adequate number of terms are considered. The number of terms depends on the problem. This convergence problem is illustrated by the *pz* estimates developed in the next section.

## 4. PLASTIC ZONES ESTIMATED FROM LINEAR ELASTIC ANALYSIS WITH THE WILLIAMS SERIES AND THE WESTERGAARD STRESS FUNCTION

The first example is the biaxially loaded Griffith plate. Two levels of $\sigma_n/S_Y$ are evaluated: 0.2 and 0.8, for both plane stress and plane strain situations. The second example is the uniaxially loaded Griffith plate, in which it is shown the convergence of the estimates of plastic zones obtained by the series of Williams in plane stress, $pz(\theta)^{Wil-N\,t}_{M,pl-\sigma}$, and in plane strain, $pz(\theta)^{Wil-N\,t}_{M,pl-\varepsilon}$, with the increase of the number $N$ of terms in the series. In the latter example, six levels of $\sigma_n/S_Y$ are evaluated: 0.2, 0.4, 0.5, 0.6, 0.7 and 0.8.

### 4.1. Griffith plate (biaxially loaded)

This example has an analytic solution, given by Eq. (14). Figure 1 presents the estimates of the plastic zone under plane stress and Fig. 3 shows the estimates of the plastic zone under plane strain.




(a) (b)

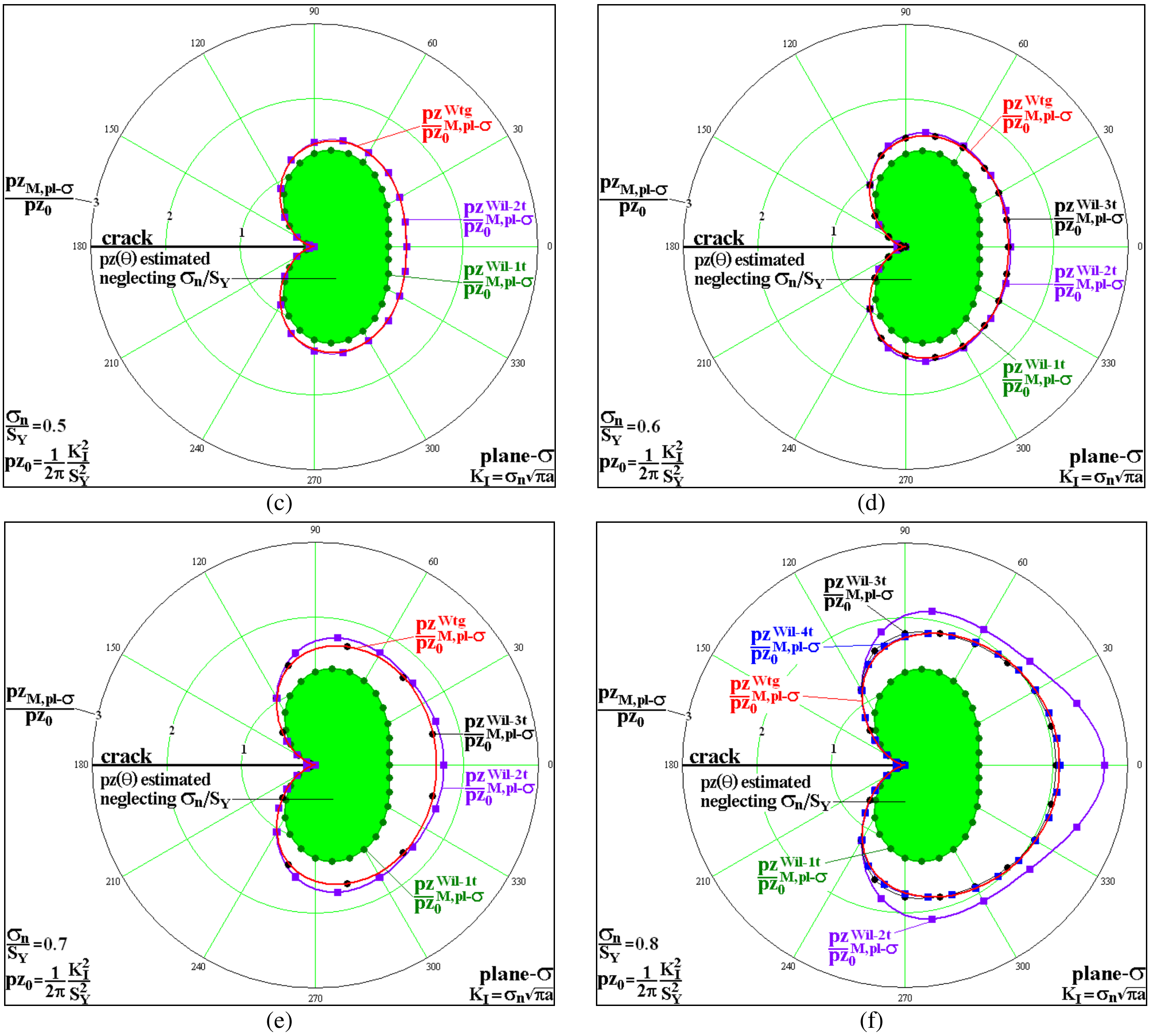
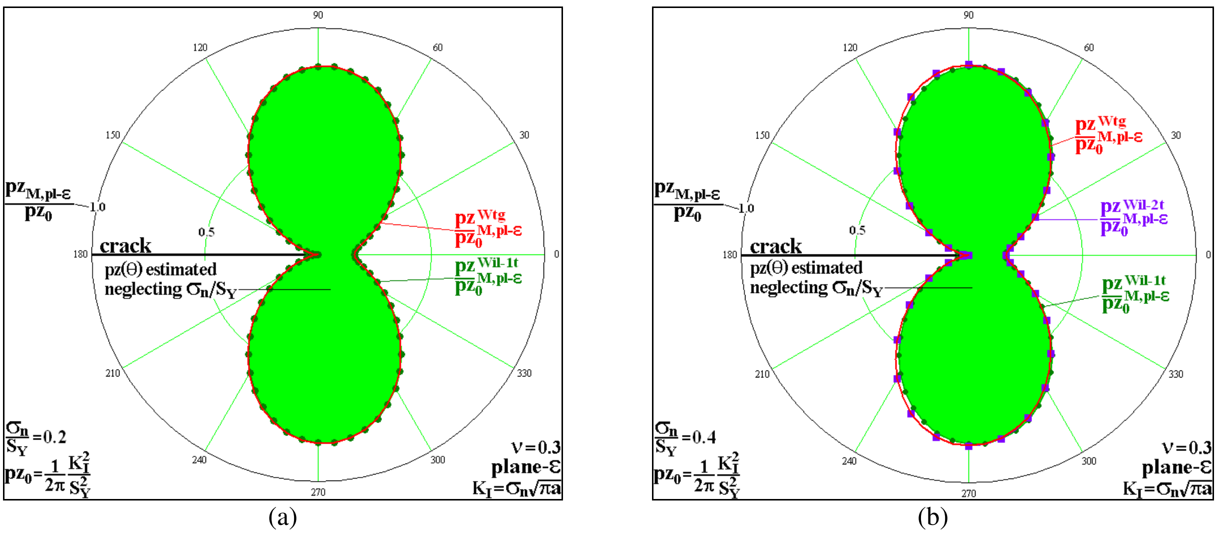



Figure 1. Convergence of estimates of $pz(\theta)_M$ obtained from Williams series to estimates from Westergaard stress function under (a) $\sigma_n/S_Y = 0.2$, (b) $\sigma_n/S_Y = 0.4$, (c) $\sigma_n/S_Y = 0.5$, (d) $\sigma_n/S_Y = 0.6$, (e) $\sigma_n/S_Y = 0.7$ and (f) $\sigma_n/S_Y = 0.8$, for the Griffith plate biaxially loaded in plane stress.



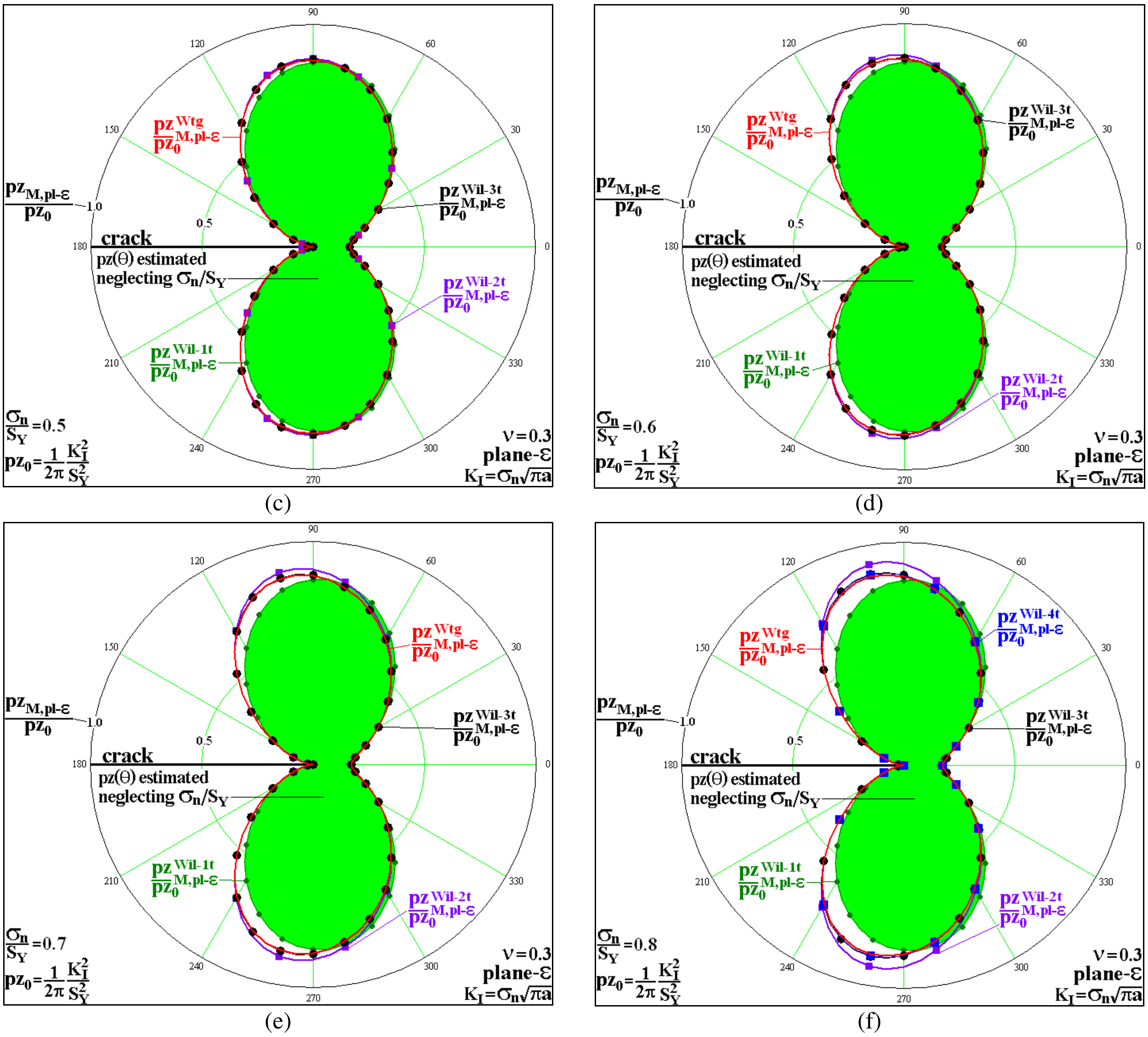



Figure 2. Convergence of estimates of $pz(\theta)_M$ obtained from Williams series to estimates from Westergaard stress function under (a) $\sigma_n/S_Y = 0.2$, (b) $\sigma_n/S_Y = 0.4$, (c) $\sigma_n/S_Y = 0.5$, (d) $\sigma_n/S_Y = 0.6$, (e) $\sigma_n/S_Y = 0.7$ and (f) $\sigma_n/S_Y = 0.8$, for the Griffith plate biaxially loaded in plane strain.

Analyzing Fig. 1 and Fig. 2, it is noted that in both cases of plane stress and plane strain, for higher values of $\sigma_n/S_Y$, more terms in the series of the Williams are necessary to adjust the estimates of plastic zones to the case of the plastic zones obtained from the Westergaard stress function.

For the plastic zones estimated from Williams series LE stress field in plane stress with $\sigma_n/S_Y$ equal to 0.2, only one term is necessary; for $\sigma_n/S_Y$ equal to 0.4 and 0.5, two terms are needed; for $\sigma_n/S_Y$ equal to 0.6 and 0.7, three terms are needed; and for $\sigma_n/S_Y$ equal to 0.8, four terms are needed for convergence.

For the plastic zones estimated from Williams series LE stress field in plane strain with $\sigma_n/S_Y$ equal to 0.2, only one term is necessary; for $\sigma_n/S_Y$ equal to 0.4, two terms are needed; for $\sigma_n/S_Y$ equal to 0.5, 0.6 and 0.7, three terms are needed; and for $\sigma_n/S_Y$ equal to 0.8, four terms are needed for convergence.

### 4.2 Griffith plate uniaxially loaded

This example also uses Eq. (14) as an analytical solution. However, it is necessary to add a constant stress value $-\sigma_n$ to the $\sigma_{xx}$ stress component to force the boundary condition $\sigma_{xx}(\infty) = 0$. Figure 3 presents estimates of plastic zones under plane stress and Fig. 5 shows estimates of plastic zones under plane strain.

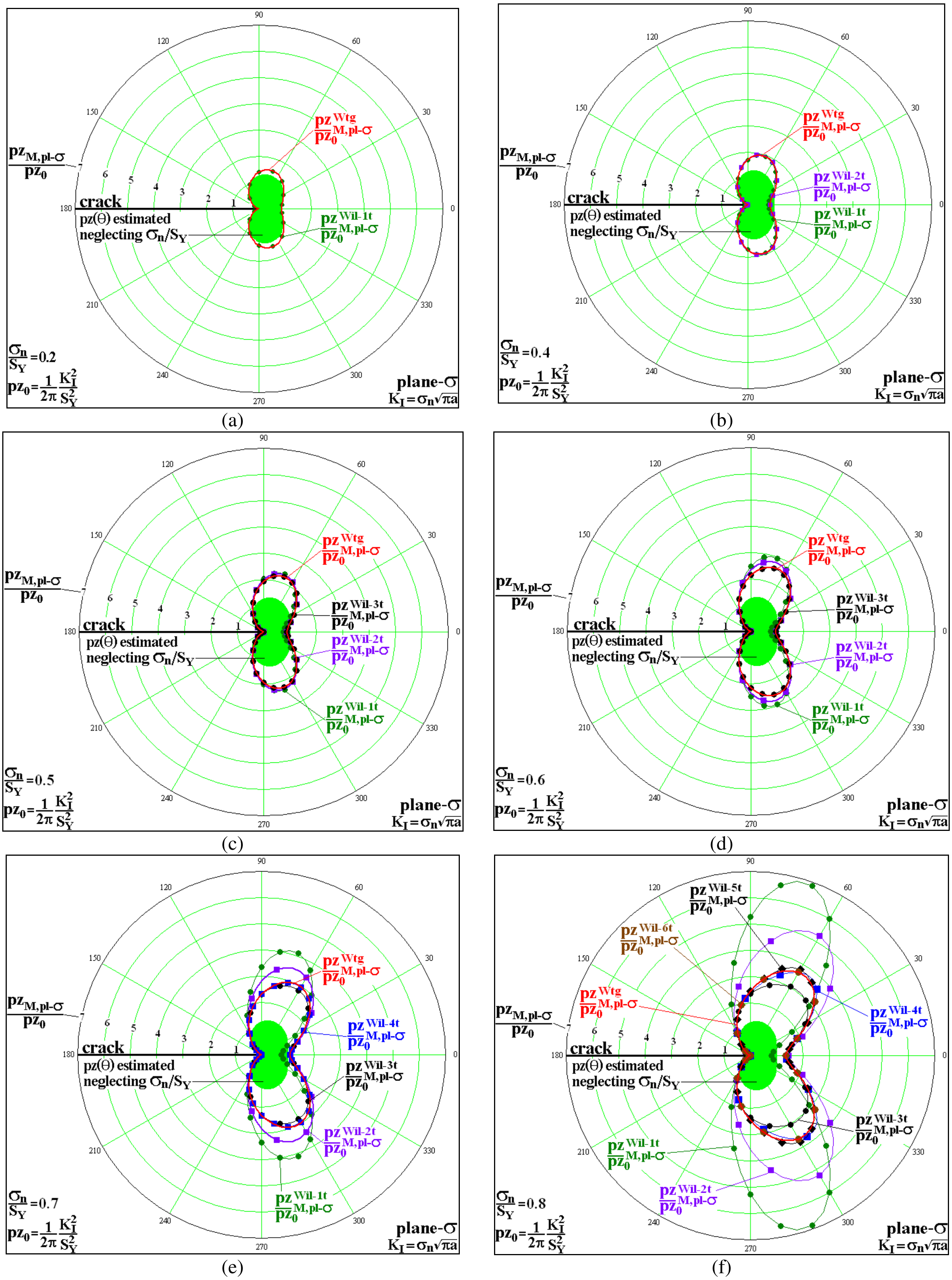



Figure 3. Convergence of estimates of $pz(\theta)_M$ obtained from Williams series to estimates from Westergaard stress function under (a) $\sigma_n/S_Y = 0.2$, (b) $\sigma_n/S_Y = 0.4$, (c) $\sigma_n/S_Y = 0.5$, (d) $\sigma_n/S_Y = 0.6$, (e) $\sigma_n/S_Y = 0.7$ and (f) $\sigma_n/S_Y = 0.8$, for the Griffith plate uniaxially loaded in plane stress.

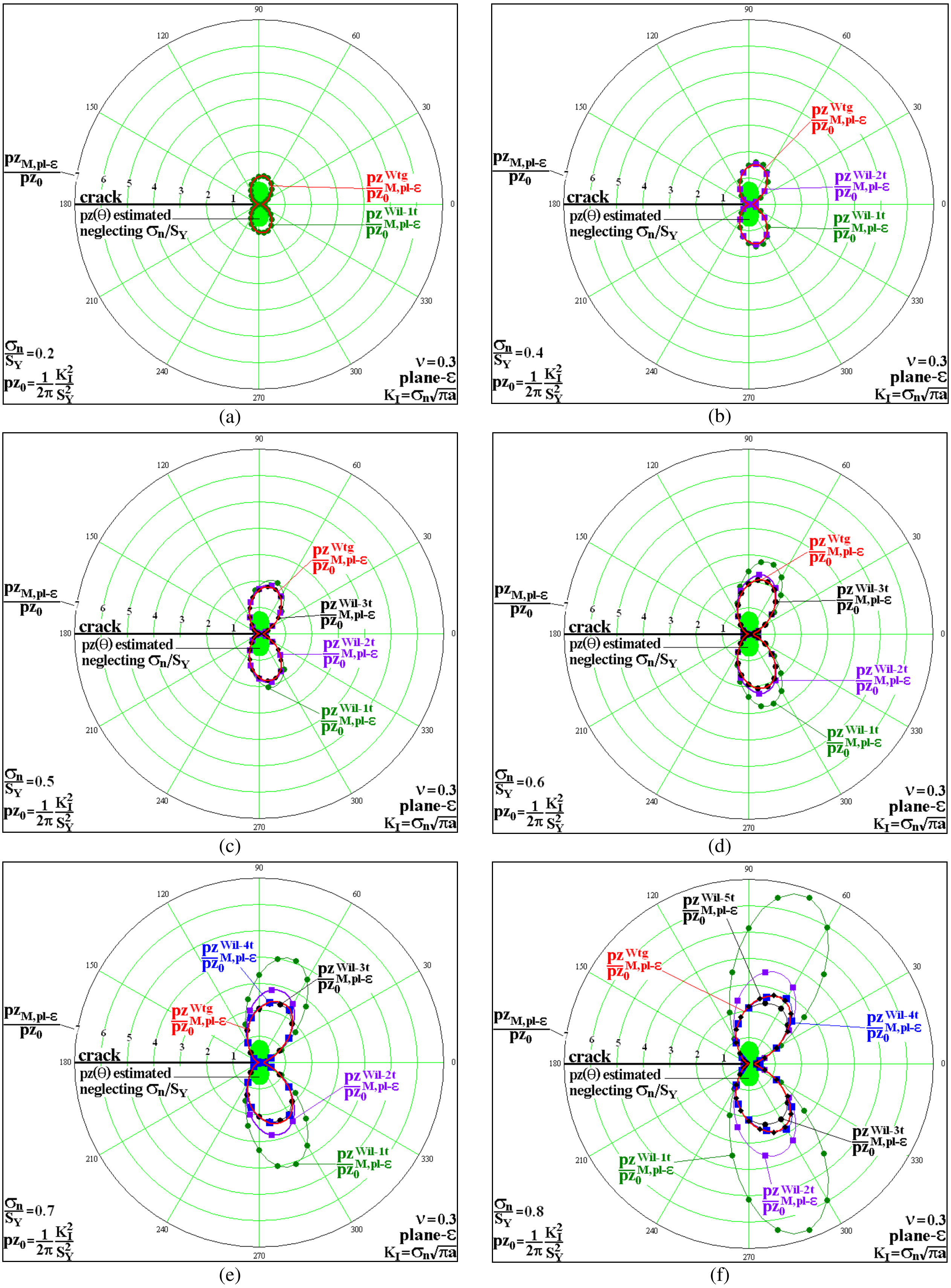


Figure 4. Convergence of estimates of $pz(\theta)_M$ obtained from Williams series to estimates from Westergaard stress function under (a) $\sigma_n/S_Y = 0.2$, (b) $\sigma_n/S_Y = 0.4$, (c) $\sigma_n/S_Y = 0.5$, (d) $\sigma_n/S_Y = 0.6$, (e) $\sigma_n/S_Y = 0.7$ and (f) $\sigma_n/S_Y = 0.8$, for the Griffith plate uniaxially loaded in plane strain.

Analyzing Fig. 3 and Fig. 4, it is noted that in both plane stress and plane strain cases, more terms are necessary in the William series to adjust their estimated plastic zones to the plastic zones estimated from the complete LE stress field generated from the Westergaard stress function for higher values of $\sigma_n/S_Y$.

For the plastic zones estimated from Williams series LE stress field in plane stress with $\sigma_n/S_Y$ equal to 0.2, only one term is necessary; for $\sigma_n/S_Y$ equal to 0.4, two terms are needed; for $\sigma_n/S_Y$ equal to 0.5 and 0.6, three terms are needed; for $\sigma_n/S_Y$ equal to 0.7, four terms are needed; and for $\sigma_n/S_Y$ equal to 0.8, six terms are needed for convergence.

For the estimative of plastic zones from Williams series in plane strain with $\sigma_n/S_Y$ equal to 0.2, only one term is necessary to adjust; for $\sigma_n/S_Y$ equal to 0.4, two terms are needed; for $\sigma_n/S_Y$ equal to 0.5 and 0.6, three terms are needed; for $\sigma_n/S_Y$ equal to 0.7, four terms are needed; and for $\sigma_n/S_Y$ equal to 0.8, five terms are needed.

## 5. CONCLUSION

Part 2 this work showed plastic zones estimates obtained from Westergaard stress function and from Williams series with more than one term that are sensitive to $\sigma_n/S_Y$. It was observed that, for locations close the crack tip, the stress fields obtained from these solutions could be simplified, resulting in the SIF stress field. Another important fact is that the estimates obtained from Westergaard stress function and from Williams series are coincident at all levels of $\sigma_n/S_Y$, if an adequate number of terms are used in the Williams series. The number of terms depends on the problem. This convergence is demonstrated in two examples of the Griffith plate.

Since Westergaard stress function satisfies all boundary conditions, it is expected that more consistent *pz* estimates can be obtained from the William series, although the singularity at the crack tip still persists in this LE solution.

## 6. ACKNOWLEDGEMENT

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