HIGH ORDER AND HIGHER REGULARITY FEM TO APPROACH THE FREE VIBRATION PROBLEMS IN BEAMS AND PLANE FRAMES

Tiago dos Santos, t.santos@brturbo.com.br
Department of Mechanical Engineering, Universidade Federal do Rio Grande do Sul, Rua Sarmento Leite 425, Porto Alegre, RS 90050-170, Brazil.

Oscar A. Garcia, oagsuarez@ucs.br
Leandro Corso, llcorso@ucs.br
Paulo R. Linzmaier, prlinzma@ucs.br
Department of Mechanical Engineering, Universidade de Caxias do Sul, Rua Francisco Getulio Vargas 1130, Caxias do Sul, 95070-560, Brazil.

Rodrigo Rossi, rrossi@ufrgs.br
Department of Mechanical Engineering, Universidade Federal do Rio Grande do Sul, Rua Sarmento Leite 425, Porto Alegre, RS 90050-170, Brazil.

Abstract. *One of the limitations of the low regularity approximation spaces constructed using the Finite Element Method (FEM) is the low accuracy in the determination of natural frequencies and modes corresponding to the relatively high frequencies of a elliptic eigenvalue problem in question. Usually the number of natural frequencies obtained by FEM with certain accuracy corresponds to a small percentage of the total natural frequencies and modes that can be approximated by the numerical model. This limitation is more pronounced when using higher order elements, which are denoted by the appearance of acoustic and optical branch of the frequency spectrum corresponding to the problem. This approximation spaces limitation of low regularity significantly increases the computational cost in problems involving high frequencies due to the necessity of using a large number of degrees of freedom in order to ensure certain accuracy with a reduced natural frequencies number of all the approximate eigenvalues numerically. These limitations are evident in problems of propagation of waves produced by impulsive forces on solid media and others. This article presents an alternative, using high regularity approximation spaces of the FEM, to improve the accuracy in obtaining relatively high natural frequencies of elliptic eigenvalue problems corresponding to Bernoulli-Euler beams and plane frames. The approximation space is constructed from a partition of unity, obtained from Hermite functions family, and enrichment functions that are associated with the edge of the element with the same order and regularity of Hermite PU. Results are shown for beams and plane frames, using normalized values of the natural frequencies obtained by numerical method with respect to the analytical solution of elliptic eigenvalue problem for thin beams.*

***Keywords**: natural frequency; high regularity; FEM.*

1. INTRODUCTION

In elliptic eigenvalue problems the possibility of obtaining a high percentage of numerical approximate eigenvalues with satisfactory accuracy is still an open research topic and there are numerous proposals for its approach.

Inside the context of Finite Element Method (FEM), a major factor in low efficiency in the approximation of elliptic eigenvalue problems is directly related to the low regularity and high order of approximation spaces. This fact can be verified by the error estimation of eigenvalues for the "*h*" version of the FEM (see: Hughes, 1987 and Givoli, 2008).

The use of *novel* numerical methods as well as: Component Mode Method (CMM, Weaver and Loh, 1985), *p*-Fourier Finite Element Method (p-FEM Fourier, Leung and Chan, 1998); recently Generalized Finite Element Method (*p*-GFEM, Arndt et all, 2009) show more accurate results than the version *h* and *p*-FEM for the free vibrations approach of bars and beams modeled with Bernoulli-Euler theory. In general, the construction of approximation spaces of the cited methods uses $C^0(\Sigma)$ FEM functions, linear functions, along with the enrichment functions constructed from information solution of the boundary value problem in question. The spaces obtained in this way do not have a uniform regularity, may even not approximate spaces with regularity $C^0(\Sigma)$, when the enrichment functions are multiplied by linear functions of the FEM as in the GFEM case. Low regularity spaces present compatible results with high regularity results for low-order frequencies, but for higher frequencies (beyond approximately ten percent of the frequencies obtained numerically) begins to experience loss of accuracy in the *p*-FEM case what can be aggravated by higher order polynomial interpolation space (see: Garcia et ali. 2010).

Recent research involving the approximation spaces construction based on high regularity isogeometric analysis methodologies (see: Cottrel et al. 2007a-b), are being used to capture more accurately the natural frequencies of higher order elliptic eigenvalue problems. The *k*-method, proposed by the just mentioned authors, employs the isogeometric analysis to build approximation spaces of desired order and regularity. The results obtained are more accurate than those from FEM analysis for low regularity in the natural frequencies of an elastic bar connected at the ends. The curves for normalized values of natural frequency are smooth and accuracy increases as the order of the polynomial approximation space is increased (see: Cottrel el al., 2007b). On the other hand, when used the low regularity FEM with high-order elements occurs a significant accuracy loss in the curve which is denoted as a jump in the diagram defined the so-called acoustic and optical branches of the spectrum.

In this article the authors use the high-order and high regularity FEM to approach free vibrations problems of Bernoulli-Euler beams and frames. The high order and high regularity approximation spaces construction occurs via polynomial approximation functions with regularity $C_0^k(\Sigma)$, $k = 0,1,2,3,...,\infty$.

2. APPROXIMATION SPACES $C_0^k(\Sigma)$, $k = 0,1,2,3,...,\infty$

The construction of the polynomial approximation space with high order and high regularity functions is performed in two steps: first will be built the partition of unit functions (PU) with the desired regularity and defined on the natural element domain. In the second step the enrichment functions of the approximation space will be constructed. As shown in the sequence, the enrichment functions and the PU functions constitute the Hermite function family with regularity $k = 1,2,...,\infty$.

2.1. PU $C_0^k(\Sigma)$, $k = 0,1,2,3,...,\infty$

The PU functions with regularity $C_0^k(\Sigma)$, $k = 0,1,2,3,...,\infty$, defined on element natural domain, $\Sigma = [-1,1]$, are obtained by normalization process of the weight functions according to indicate in Eqs. (1)-(2).

$$\varphi_1 = \frac{W_1(\xi)}{W_1(\xi)+W_2(\xi)}; \tag{1}$$

$$\varphi_2 = \frac{W_2(\xi)}{W_1(\xi)+W_2(\xi)}. \tag{2}$$

The functions φ_1 and φ_2 obtained by polynomial weight functions have the following properties:

$$\begin{aligned} &\text{i. } \operatorname{supp}(\varphi_i) \subset \Sigma,\ i = 1,2 \\ &\text{ii. } \sum \varphi_i = 1;\ \ \forall \xi \in \Sigma \\ &\text{iii. } \|\varphi_i\|_{L^\infty(R)} \le C_\infty \\ &\text{iv. } \|\nabla \varphi_i\|_{L^\infty(R)} \le \frac{C_G}{2} \end{aligned} \tag{3}$$

Table 1. Boundary conditions for $W_i \in C_0^k(\Sigma)$, $k = 1,2,3,...,\infty$.

i	$\int_\Sigma W_i(\xi)\,d\xi = 1$	$W_i(-1) = 1$	$W_i(1) = 0$	$\frac{dW_i}{d\xi}(-1) = 0$	$\frac{dW_i}{d\xi}(1) = 0$	$\cdots$	$\frac{dW_i^k}{d\xi^k}(-1) = 0$	$\frac{dW_i^k}{d\xi^k}(1) = 0$
1	1	1	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0

To build the partition of unit functions defined in Eqs. (1)-(2) it is needed to define the functions to be used and what are the boundary conditions they must satisfy, ensuring the desired regularity.

To this end, we propose the following problem: *Determine the weight function* $W_i(\xi)$ *with regularity* $C_0^k(\Sigma)$, $k = 1,2,3,...,\infty$ *, which must satisfy the boundary conditions indicate in Tab. 1.*

From the boundary conditions defined in Tab.1, it is concluded that the weight function could be a polynomial of order:

$$n = 2(1+k). \tag{4}$$

On the other hand, it is possible to show by mathematical induction that the order of the polynomial weight function that satisfies the boundary conditions of Tab.1 is $m = 2k + 1$. The prove of this statement is part of the following problem: *Determine the weight functions* $\{W_i\}_{i=1}^{2}$ *obtained from the polynomial functions defined on the element natural domain, with regularity* $C_0^k(\Sigma)$, $k = 0,1,2$.

For the function with regularity $C_0^0(\Sigma)$, we have $k = 0$, so the order of the polynomial $W_i(\xi)$ is $n = 2$, in other words, a second order polynomial defined on Σ, as indicated in Eq. (5).

$$W_i(\xi) = a_2\xi^2 + a_1\xi + a_0, \quad i = 1,2; \tag{5}$$

The boundary conditions for this specific problem are indicated in Tab. 2.

Table 2. Boundary conditions for $W_i \in C_0^0(\Sigma)$.

i	$\int_\Sigma W_i(\xi)\,d\xi = 1$	$W_i(-1) = 1$	$W_i(1) = 0$
1	1	1	0
2	1	0	1

Substituting the boundary conditions of Tab. 2 in the Eq. (5) we get the following expression for the weight functions associate to element nodes:

$$W_1(\xi) = \frac{1}{2}(1 - \xi); \tag{6}$$

$$W_1(\xi) = \frac{1}{2}(1 + \xi). \tag{7}$$

Note that in Eq. (7), $a_2 = 0$, obtaining a first order polynomial, so with $m = 1$.

For a polynomial weight function defined in Σ with regularity $C_0^1(\Sigma)$, we need in principle a polynomial of order $n = 4$, defined by expression:

$$W_i(\xi) = a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0, \; i = 1,2; \tag{8}$$

which must satisfy de boundary conditions indicate in Tab. 3:

Table 3. Boundary conditions for $W_i \in C_0^1(\Sigma)$.

i	$\int_\Sigma W_i(\xi)\,d\xi = 1$	$W_i(-1) = 1$	$W_i(1) = 0$	$\frac{dW_i}{d\xi}(-1) = 0$	$\frac{dW_i}{d\xi}(1) = 0$
1	1	1	0	0	0
2	1	0	1	0	0

Substituting the Eq. (8) in the boundary conditions of Tab. 3, we obtain the following expressions for the weight functions:

$$W_1(\xi) = \frac{1}{4}(1 - \xi)^2(2 + \xi); \tag{9}$$

$$W_2(\xi) = \frac{1}{4}(1 + \xi)^2(2 - \xi). \tag{10}$$

In Eqs. (9)-(10), again it appears that $a_4 = 0$, resulting in a polynomial of order $m = 3$.

For a $C_0^2(\Sigma)$ polynomial function defined on domain Σ, at first glance it is necessary $n = 6$, as shown in Eq. (11).

$$W_i(\xi) = a_6\xi^6 + a_5\xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0, \quad i = 1,2 . \tag{11}$$

Table 4. Boundary conditions for $W_i \in C_0^2(\Sigma)$.

i	$\int_\Sigma W_i(\xi)\,d\xi = 1$	$W_i(-1) = 1$	$W_i(1) = 0$	$\frac{dW_i}{d\xi}(-1) = 0$	$\frac{dW_i}{d\xi}(1) = 0$	$\frac{dW_i^2}{d\xi^2}(-1) = 0$	$\frac{dW_i^2}{d\xi^2}(1) = 0$
1	1	1	0	0	0	0	0
2	1	0	1	0	0	0	0

But, substituting the boundary conditions of Tab.4 in Eq. (11), we obtain the following weight functions:

$$W_1(\xi) = \frac{1}{16}(1-\xi)^3\left(8 + 9\xi + 3\xi^2\right); \tag{12}$$

$$W_2(\xi) = \frac{1}{16}(1+\xi)^3\left(8 - 9\xi + 3\xi^2\right). \tag{13}$$

Observing Eqs. (12)-(13) we note that the resulting polynomial from the weight functions have order $n = 5$, in general, following the procedure described above, to obtain a function with regularity $C_0^k(\Sigma)$, $k = 0,1,2,...,\infty$, the resulting weight function will have order $m = 1 + 2k$.

From the weight functions W_1 and W_2 the associated partition of unit functions $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are obtained using the Eqs. (1)-(2). For all cases studied, $W_1 + W_2 = 1$, so we can conclude that for the function types used so far we have:

$$\varphi_i(\xi) = W_i(\xi), \; W_i \in C_0^k(\Sigma), \; i = 1,2 \;\; k = 0,1,2,...,\infty . \tag{14}$$

The curves of the partition of unit functions with regularity $C_0^k(\Sigma)$, $k = 0,1,2$ are shown in Fig. 1.

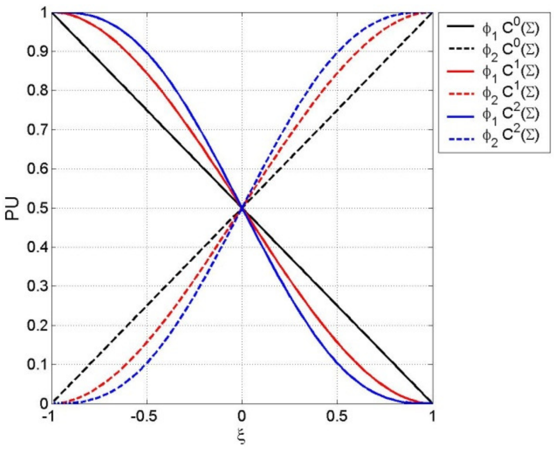


Figure 1. Partition of unity with regularity $C_0^k(\Sigma)$, $k = 0,1,2$.

Until now, we have commented only about the PU functions associated with element nodes. This sort of functions has the delta of kronecker delta property and satisfies the requisites defined in Eq. (3), i-iv. Even being high order functions these functions alone do not ensure the space completeness. Thus, urges the necessity of using enrichment functions, which ensure the continuity of the functions derivatives with unitary value at the element ends, essential condition for thin beam analysis. The number of enrichment functions to be used is equal to twice the desired regularity.

The order and regularity of the enrichment functions in this work are the same of the PU functions. However, the enrichment functions are null in the element nodes, but its derivatives with order $k = 1,2,...,\infty$ are not null on all nodes.

To build the enrichment functions with regularity $C_0^k(\Sigma)$, $k = 1,2,...,n$ it is necessary a polynomial with order $m = 2k + 1$. Its generic form is defined in Eq. (15). The number of functions to be constructed is equal to $n = 2k$.

$$\psi_i(\xi) = a_m\xi^m + a_{m-1}\xi^{m-1} +,....,+ a_1\xi + a_0, \quad i = 1,...,2k . \tag{15}$$

The polynomial functions defined in Eq. (15) must satisfy de boundary conditions informed in Tab. 5.

Table 5. Boundary conditions for enrichment functions with regularity $C_0^{\alpha}(\Sigma)$, $\alpha = 1,2,3,...,k$.

i	$\psi_i(-1)=1$	$\psi_i(1)=0$	$\frac{d\psi_i}{d\xi}(-1)=0$	$\frac{d\psi_i}{d\xi}(1)=0$	$\cdots$	$\frac{d\psi_i^k}{d\xi^k}(-1)=0$	$\frac{d\psi_i^k}{d\xi^k}(1)=0$
1	0	0	1	0	0	0	0
2	0	0	0	1	0	0	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2k-1$	0	0	0	0	0	1	0
$2k$	0	0	0	0	0	0	1

Enrichment function examples with regularity $C_0^k(\Sigma)$, $k = 1,2,3$:

i. Enrichment functions with regularity $C_0^1(\Sigma)$:

$$\psi_1(\xi) = \frac{1}{4}\left(1-\xi^2\right)\left(1-\xi\right); \tag{16}$$

$$\psi_2(\xi) = \frac{1}{4}\left(-1+\xi^2\right)\left(1+\xi\right). \tag{17}$$

ii. Enrichment functions with regularity $C_0^2(\Sigma)$:

$$\psi_1(\xi) = \frac{1}{16}(1-\xi)^3(1+\xi)(5+3\xi); \tag{18}$$

$$\psi_2(\xi) = \frac{1}{16}(1+\xi)^3(-1+\xi)(5-3\xi); \tag{19}$$

$$\psi_3(\xi) = \frac{1}{16}(1-\xi)^3(1+\xi)^2; \tag{20}$$

$$\psi_4(\xi) = \frac{1}{16}(1+\xi)^3(1-\xi)^2. \tag{21}$$

iii. Enrichment functions with regularity $C_0^3(\Sigma)$:

$$\psi_1(\xi) = \frac{1}{32}(1+\xi)\left(11+14\xi+5\xi^2\right)(-1+\xi)^4; \tag{22}$$

$$\psi_2(\xi) = \frac{1}{32}(-1+\xi)\left(11-14\xi+5\xi^2\right)(1+\xi)^4; \tag{23}$$

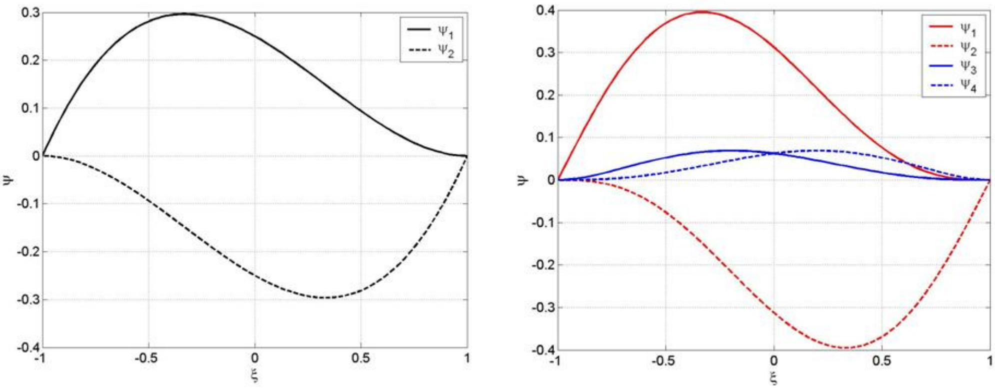
$$\psi_3(\xi) = \frac{1}{32}(3+2\xi)(1+\xi)^2(-1+\xi)^4; \tag{24}$$

$$\psi_4(\xi) = -\frac{1}{32}(-3+2\xi)(-1+\xi)^2(1+\xi)^4; \tag{25}$$

$$\psi_5(\xi) = \frac{1}{96}(1+\xi)^3(-1+\xi)^4; \tag{26}$$

$$\psi_6(\xi) = \frac{1}{96}(-1+\xi)^3(1+\xi)^4. \tag{27}$$

The enrichment functions with regularity $C_0^1(\Sigma)$ and $C_0^2(\Sigma)$ defined on the natural domain element are shown in Fig. 2(a) and (b), respectively.



(a) (b)

Figure 2. a) Enrichment function for $C_0^1(\Sigma)$ regularity; b) enrichment function for $C_0^2(\Sigma)$ regularity.

3. FREE VIBRATION OF BERNOULLI – EULER BEAM

The weak formulation corresponding to the Bernoulli-Euler (B-E) beam model on the local system of coordinates, see Fig. 3, consists in determine $u_o(x), w(x) \in \delta$ such that:

$$EA\int_0^L \frac{du_o}{dx}\frac{d\tilde{u}_o}{dx}dx + EI\int_0^L \frac{d^2w}{dx^2}\frac{d^2\tilde{w}}{dx^2}dx - \omega^2\left[\rho A\left(\int_0^L u_o\tilde{u}_o dx + \rho A\int_0^L w\tilde{w}dx\right)\right] = 0,\ \forall \tilde{u}_o, \tilde{w} \in Var\,. \quad (28)$$

The sets of trial functions and weight functions are defined by:

$$\delta = \left\{u_o, w \in H^3\left[0, L\right] / u_o = w = 0, x = 0, x = L\right\}; \quad (29)$$

$$Var = \left\{\tilde{u}_o, \tilde{w} \in H^3\left[0, L\right] / \tilde{u}_o = \tilde{w} = 0, x = 0, x = L\right\}. \quad (30)$$

In Eq. (28) E, A, I and ρ are the material Young modulus, the beam cross-section, the inertia moment with relation to the "z" axis and the material specific mass, respectively.

3.1. Discretization

The discretized B-E problem will be approximated using an approximation space with regularity $C_0^1(\Sigma)$, as indicated in Eqs. (31)-(33) corresponding to an element in the local coordinate system.

$$\begin{Bmatrix} u \\ w \end{Bmatrix}_h = \boldsymbol{N}(\xi)\boldsymbol{U}\,; \quad (31)$$

$$\boldsymbol{N}(\xi) = \begin{bmatrix} \varphi_1(\xi) & \psi_1(\xi) & yJ^{-1}\frac{d\varphi_1}{d\xi} & yJ^{-1}\frac{d\psi_1}{d\xi} & \varphi_2(\xi) & \psi_2(\xi) & yJ^{-1}\frac{d\varphi_1}{d\xi} & yJ^{-1}\frac{d\psi_1}{d\xi} \\ 0 & 0 & \varphi_1(\xi) & \psi_1(\xi) & 0 & 0 & \varphi_2(\xi) & \psi_2(\xi) \end{bmatrix}; \quad (32)$$

$$\boldsymbol{U}^T = \{u_1 \quad \alpha_1 \quad w_1 \quad \theta_1 \quad u_2 \quad \alpha_2 \quad w_2 \quad \theta_2\}. \quad (33)$$

The shape matrix, Eq. (32), can be rewritten in terms of displacement components, as:

$$u_h = \boldsymbol{N}_u\boldsymbol{U}\,; \quad (34)$$

$$\boldsymbol{N}_u(\xi)=\left[\varphi_1 \quad \psi_1 \quad yJ^{-1}\frac{d\varphi_1}{d\xi} \quad yJ^{-1}\frac{d\psi_1}{d\xi} \quad \varphi_2 \quad \psi_2 \quad yJ^{-1}\frac{d\varphi_2}{d\xi} \quad yJ^{-1}\frac{d\psi_2}{d\xi}\right]; \tag{35}$$

$$w_h=\boldsymbol{N}_w\boldsymbol{U}\ ; \tag{36}$$

$$\boldsymbol{N}_w(\xi)=[0 \quad 0 \quad \varphi_1 \quad \psi_1 \quad 0 \quad 0 \quad \varphi_2 \quad \psi_2]. \tag{37}$$

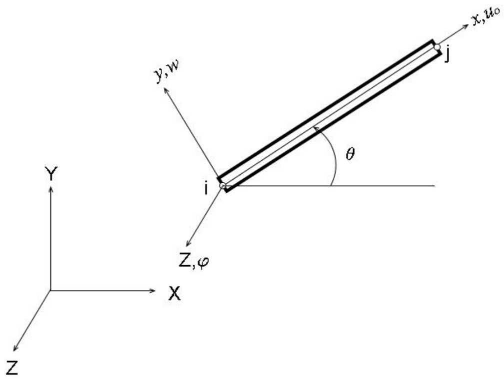


Figure 3. Beam according to a local coordinate system.

In the B-E beam model the membrane and bending deformations are uncoupled for isotropic and homogeneous beams. Then, the displacement component u can be written as the sum of the membrane displacement and the bending displacement, as follow:

$$u=u_m+u_b\ ; \tag{38}$$

where:

$$u_m=\boldsymbol{N}_m\boldsymbol{U}\ ; \tag{39}$$

$$N_m=[\varphi_1 \quad \psi_1 \quad 0 \quad 0 \quad \varphi_2 \quad \psi_2 \quad 0 \quad 0]; \tag{40}$$

$$u_b=\boldsymbol{N}_b\boldsymbol{U}\ ; \tag{41}$$

$$\boldsymbol{N}_b=\left[0 \quad 0 \quad yJ^{-1}\frac{d\varphi_1}{d\xi} \quad yJ^{-1}\frac{d\psi_1}{d\xi} \quad 0 \quad 0 \quad yJ^{-1}\frac{d\varphi_2}{d\xi} \quad yJ^{-1}\frac{d\psi_2}{d\xi}\right]. \tag{42}$$

The parameters α_i, $i=1,2$, of the displacement vector do not have physic interpretation. By substituting Eqs. (36), (39) and (41) in Eq. (28) one obtains the discretized form of the variational formulation for the B-E beam free vibration problem in Eq. (43).

$$\left\{EA\int_{-1}^{1} J^{-1}\boldsymbol{B}_m^T\boldsymbol{B}_m d\xi+EI\int_{-1}^{1} J^{-3}\boldsymbol{B}_b^T\boldsymbol{B}_b d\xi-\omega^2\left(\rho A\int_{-1}^{1}\boldsymbol{N}_m^T\boldsymbol{N}_m J d\xi+\rho A\int_{-1}^{1}\boldsymbol{N}_b^T\boldsymbol{N}_b J d\xi\right)\right\}\boldsymbol{U}=\boldsymbol{0}. \tag{43}$$

In Eq.(43), we have:

$$\boldsymbol{B}_m=\frac{d\boldsymbol{N}_m}{d\xi}\ ; \tag{44}$$

$$\boldsymbol{B}_b=\frac{d\boldsymbol{N}_b}{d\xi}\ . \tag{45}$$

The Eq. (43) can be expressed in terms of the stiffness and mass consistent matrices, as shown in Eq. (46)-(47).

$$\boldsymbol{K_e^L} = EA\int_{-1}^{1} J^{-1}\boldsymbol{B}_m^T\boldsymbol{B}_m d\xi + EI\int_{-1}^{1} J^{-3}\boldsymbol{B}_b^T\boldsymbol{B}_b d\xi\ ; \tag{46}$$

$$\boldsymbol{M_e^L} = \rho A\int_{-1}^{1} \boldsymbol{N_m^T}\boldsymbol{N}_m J d\xi + \rho A\int_{-1}^{1} \boldsymbol{N_b^T}\boldsymbol{N}_b J d\xi\ . \tag{47}$$

The kinematic problem is associated with the element local coordinate system. However, the element contribution must be constructed in the global system, so $\boldsymbol{K_e^L}$ and $\boldsymbol{M_e^L}$ must be transformed using a rotation operator according to:

$$\boldsymbol{K_e^G} = \boldsymbol{R}\boldsymbol{K}_e^L\boldsymbol{R}^T\ ; \tag{48}$$

$$\boldsymbol{M_e^G} = \boldsymbol{R}\boldsymbol{M_e^L}\boldsymbol{R}^T\ . \tag{49}$$

In Eq.(48) the rotation operator is given by:

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix}; \tag{50}$$

where:

$$\boldsymbol{Q} = \begin{bmatrix} cos(\theta) & 0 & -sen(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ sen(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

From Eqs. (48)-(49), the global matrix formulation for the beam element free vibration problem is obtained as follows:

$$\left(\boldsymbol{K_e^G} - \omega^2\boldsymbol{M_e^G}\right)\boldsymbol{U_e^G} = \boldsymbol{0}\ . \tag{52}$$

4. NUMERIC RESULTS

The aim of this section is to verify the influence of the higher regularity spaces, obtained by the Hermite family, in the approach of elliptic eigenvalue problems. The section has two case studies: the first consists in the analysis of the influence of the approximation space regularity on the relatively high order natural frequency accuracy for a simple supported beam. In the second example it is analyzed the effect of the higher regularity spaces on the eigenvalues accuracy for a plane frame problem.

Simple Supported Beam: The results of this case of study were obtained for a simple supported beam, where the boundary conditions, geometric and material properties are presented in Fig. 4.

The beam results are presented in terms of normalized values of the natural frequencies, see Dym and Shames (1973), obtained according to the strategies indicate in topics i-iii, with relation to the analytical solution, see Fig. 5.

i. Case 1: Space constructed with 29 Hermitian elements with regularity $C_0^1(\Sigma)$;
ii. Case 2: Space constructed with 19 Hermitian elements with regularity $C_0^2(\Sigma)$;
iii. Case 3: Space constructed with 14 Hermitian elements with regularity $C_0^3(\Sigma)$.

In all cases a discretization with 58 degree of freedom was used.

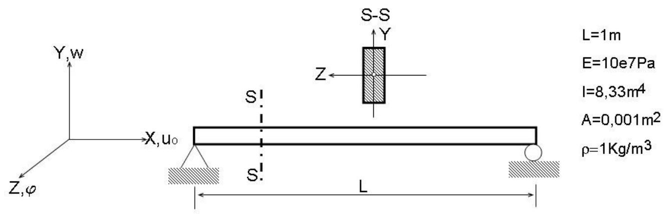


Figure 4. Simple supported beam model.

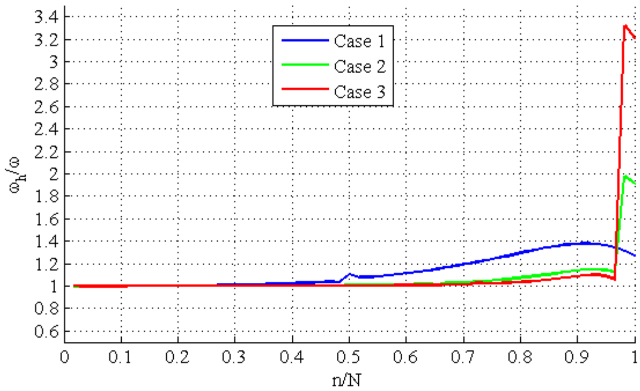


Figure 5. Normalized natural frequency values of the simple supported beam.

The results are very similar until n/N=0.5. Beyond this value the strategy **Case 1** shows a jump denoting the transition between the acoustic and optic branch of the frequency spectrum. The convergence loss observed in the optic region is significant, showing that it does not have a satisfactory relation between the regularity and the approximation space order. On the other hand, the **Case 2** and **Case 3** strategies, show results without jumps around n/N=0.5. A slight improvement in the **Case 3** convergence is noticed. Based on the results, the higher is the regularity the better is the accuracy. The simple supported beam problem show surprising results for solutions utilizing higher regularity spaces, however, we do not have shown the influence of the bending and membrane stiffness coupling. The next example, plane frame problem, deals with it.

Plane Frame: The boundary conditions, geometric and material properties of the plane frame are shown in Fig. 6. In this example, an analytical solution is not available to compare the results. In this case, for an elliptic eigenvalue problem with a symmetric positive defined stiffness matrix and mass consistent matrix, an upper convergence to the analytical solution takes place. So, the accuracy of the proposed method is analyzed by means of the upper convergence criterion with relation to the *a priori* error estimator for the h -FEM version, indicated in Eq. (53). In other words, for an eigenvalue λ_i obtained by different numerical strategies the more accurate is the lowest value.

$$\lambda_l \leq \lambda_l^h \leq \lambda_l + ch^{2(k+1-m)}\lambda_l^{(k+1)/m}. \tag{53}$$

In Eq. (53), λ_l and λ_l^h represent the analytical and approximated values, respectively, for an eigenvalue of order l. The terms k and m represent the polynomial order and the approximation space regularity, respectively. The term h corresponds to the element radius and c to a constant that not depends of h and λ_l.

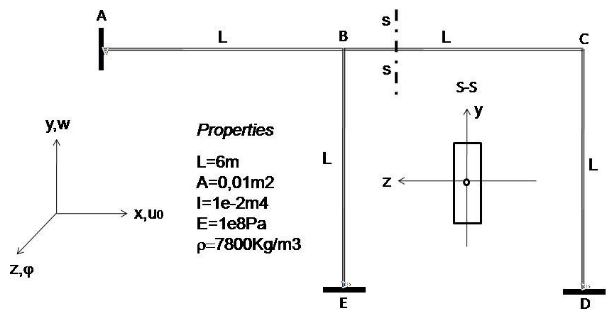


Figure 6. Physics and boundary conditions of the plane frame.

Four strategies are considered. They are:

i. Strategy 1: The frame was discretized in 48 elements, with 138 dofs. In this strategy were utilized linear functions with regularity $C_0^0(\Sigma)$ to approximate the membrane displacement u_o and cubic functions with regularity $C_0^1(\Sigma)$ to approximate the transverse displacement w, both in the local system.
ii. Strategy 2: The frame was discretized in 37 elements, with 139 dof. In this strategy were utilized cubic functions with regularity $C_0^1(\Sigma)$ to approximate the components u_o and w of the displacement field.

iii. Strategy 3: The frame was discretized in 21 elements, with 117 dof. In this strategy were utilized fifth-order functions with regularity $C_0^2(\Sigma)$ to approximate the components u_o and w of the displacement field.

iv. Strategy 4: The frame was discretized in 16 elements, with 127 dof. In this strategy were utilized seventh-order functions with regularity $C_0^3(\Sigma)$ to approximate the components u_o and w of the displacement field.

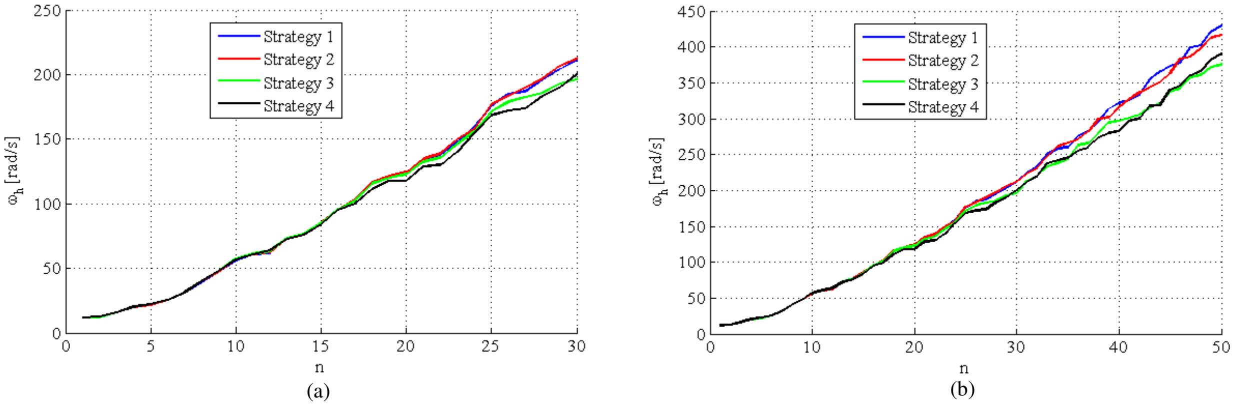


Figure 7. a) First thirty results for the natural frequencies; a) first fifty results for the natural frequencies.

The results obtained for the plane frame analysis are shown in Fig.7 for limits of $\omega_{30} = \sqrt{\lambda_{30}}$ and $\omega_{50} = \sqrt{\lambda_{50}}$. Applying the criteria shown in Eq. (53), we note in Fig.7a that until approximately 10% of natural calculated frequencies the strategies have a similar behavior, beyond this value we note an improvement in the convergence for Strategy 3 and Strategy 4. The influence of the approximation space regularity on the resulting convergence becomes apparent for greater frequencies, see Fig.7b. In this case, for frequency ω_{50}, we note a significant improvement in Strategy2-Strategy4, with relation to Strategy1.

5. CONCLUSIONS

For the examples analyzed the more accurate results were obtained with the increasing of the approximation space regularity. On the other hand, the approximation spaces based on Hermite family functions present regularity linked to the polynomial order, which turns this systematical construction of the approximation space less adequate for using in high computational demanding problems, as nonlinear problems or constructing approximation spaces in 2D and 3D. Despite the relationship between order-regularity has not been properly analyzed, no loss of convergence was found in the examples analyzed showing good results for the one-dimensional spaces used in the beam and plane frame.

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