The Characteristc Based Split (CBS) scheme for incompressible flow simulations over tube bundles

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Abstract. *In this work, the Characterisc Based Split (CBS) scheme is applied for two-dimensional incompressible flow simulations over tube bundles. Numerical simulations of unsteady flows have been carried out on unstructured meshes of linear triangles. With the CBS scheme is possible to use the standard Galerkin finite element method for convective dominant problems without oscillations in the solution. The results of some test cases show that the CBS formulation is accurate and efficient.*

Keywords*: Finite element, CBS, Incompressible flows, Tube bundle.*

1. INTRODUCTION

The Bubnov-Galerkin finite element method or the standard Galerkin finite element method (GFEM) gives the minimum error in the L_2 norm for self-adjoint problems and in this case the system of algebraic equations resulting is symmetric. However, if the GFEM is used to solve incompressible flow equations without any stabilization it may result in violent oscillations. The instability due to the non-linear convective acceleration terms, which make the flow equations non-self-adjoint, leads to a system of non-symmetric equations, Liu (2005). Also, the incompressible limit, Ladyshenskaya-Babuska-Brezzi (LBB) conditions, introduces instability if equal order interpolation functions are used for velocity and pressure fields. In this way, use of simple linear triangular elements results in highly oscillatory solutions when the viscous flows of incompressible fluids is solved using equal order interpolation. The violation of this condition often results in numerically unphysical oscillations in the pressure field.

There are several stable approximations available to deal with the steady-state situations which reduce/eliminate oscillations resulted from standard discretization of convective acceleration terms. These schemes of stabilization include the Streamline Upwind Petrov-Galerkin (SUPG), the Finite Increment Calculus (FIC) method and the Subgrid Scale (SGS). For stabilization via transient formulation there are, for example, the Characteristic-Galerkin (CG) method and the Taylor-Galerkin method (TG), Liu (2005).

The Characteristic Based Split (CBS) based on firstly removing all pressure gradient terms from the Navier-Stokes equations leads to a non-singular solution for any shape functions used for velocity and pressure. In the second step, the pressure is obtained from the continuity equation and finally the intermediate velocity variables obtained from the first step are corrected to get the final velocity values.

The CBS for both incompressible and compressible flows was initially presented by Zienkiewicz and Codina (1995) and has been extended to investigate other applications: solid dynamics, shallow water flows, thermal and porous medium flows, for example. The CBS scheme has been combined with the standard Artificial Compressibility (AC) method to obtain an efficient and accurate explicit matrix free procedure, Liu (2005). In this work a semi-implicit CBS scheme, in which a matrix solution procedure is required for the implicit solution of the pressure Poisson equation, has been applied for computational solution of the Navier-Stokes equations in two-dimensional domains, Pereira (2010).

2. THE GOVERNING EQUATIONS AND THE CBS SCHEME

The governing equations for incompressible viscous flows with heat transfer are the continuity, the momentum and the thermal energy equations, generally called the Navier-Stokes equations. These equations are now considered and the CBS scheme is described in the subsection 2.2.

2.1. Governing equations

The governing equations are presented in a non-dimensional form of the governing equations as presented in Pereira (2010) and this form is:

$$\frac{\partial U_i}{\partial X_i} = 0 \tag{1}$$

$$\frac{\partial U_i}{\partial t} + u_j \frac{\partial U_i}{\partial X_j} + \frac{\partial P}{\partial X_i} - \frac{1}{Re}\frac{\partial \tau_{ji}}{\partial X_j} = S_i \tag{2a}$$

$$\tau_{ji} = \mu\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{2b}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial X_j} - \frac{1}{RePr}\frac{\partial}{\partial X_j}\left(\alpha \frac{\partial T}{\partial X_j}\right) = Q \tag{3}$$

where $X_i = x_i / L$; $U_i = \rho u_i$; $P = p / \rho_\infty u_\infty^2$; $T = \left(T^* - T_\infty\right)/\left(T_w - T_\infty\right)$ and all physical properties in Eqs. (1) - (3) are non-dimensionalized in relation to reference properties at T_∞ . $\rho = \rho^* / \rho_\infty$; $\mu = \mu^* / \mu_\infty$; $\alpha = \alpha^* / \alpha_\infty$ are the density, the dynamic viscosity and the thermal diffusivity respectively. L is a reference length; $u_i = u_i^* / u_\infty$ and the Reynolds and the Prandtl numbers are defined as $Re = \rho_\infty u_\infty L / \mu_\infty$ and $Pr = \mu_\infty c_{p\infty} / k_\infty$ respectively. An asterisk indicates dimensional values.

2.2. The CBS Scheme

Details of the derivation of the CBS scheme, for the Navier-Stokes equations, can be found in Lewis, Nithiarasu and Seetharamu (2004) and it's an extension of the CG method of Zienkiewicz and Taylor (2000). It's based on evaluation of the time derivative along the characteristic that eliminate the convective term in the transport equation. So, a pure diffusion equation is obtained and after an expansion in Taylor series, a time discretized equation to a scalar variable similar to Eq. (3) has the form, see details also in Pereira (2010):

$$\begin{aligned}\Delta\phi = \phi(x_i,t_{n+1}) - \phi(x_i,t_n) = \Delta t\left[-u_j(x_j,t_n)\frac{\partial\phi(x_j,t_n)}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\alpha\frac{\partial\phi(x_i,t_n)}{\partial x_i}\right) - Q(x_i,t_n)\right] + \\ + \frac{(\Delta t)^2}{2}\left\{u_i(x_i,t_n)\frac{\partial}{\partial x_i}\left(u_j(x_j,t_n)\frac{\partial\phi(x_j,t_n)}{\partial x_j}\right)\right\} + \\ \frac{(\Delta t)^2}{2}\left\{-u_j(x_j,t_n)\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_i}\left(\alpha\frac{\partial\phi(x_i,t_n)}{\partial x_i}\right)\right] + u_j(x_j,t_n)\frac{\partial Q(x_j,t_n)}{\partial x_j}\right\}\end{aligned} \tag{4}$$

Note that the convection terms reappear in the time discretized equation and also higher order terms appeared. These high order terms work as stabilizing of the solution.

2.2.1 Time discretization of the Navier-Stokes equations

The time discretization of the continuity and the momentum equations is done in three steps. In the first step the pressure terms are dropped from the momentum equations and an intermediate velocity is obtained as

Step 1: Intermediate velocity

$$\Delta U_j^* = U_j^* - U_j^n = \Delta t\left[-\frac{\partial}{\partial x_k}\left(u_k U_j\right) + \frac{1}{\mathrm{Re}}\frac{\partial \tau_{ij}}{\partial x_i}\right]^n + \frac{(\Delta t)^2}{2}\left\{u_m\frac{\partial}{\partial x_m}\left[\frac{\partial}{\partial x_k}\left(u_k U_j\right) - \frac{1}{\mathrm{Re}}\frac{\partial \tau_{ij}}{\partial x_i}\right]\right\}^n \tag{5}$$

In the second step, a Poisson equation is solved for the pressure field considering the continuity equation. This Poisson equation is of the form

Step 2: Pressure calculation

$$\frac{\partial^2 p^n}{\partial x_j \partial x_j} = \frac{1}{\Delta t}\frac{\partial U_j^*}{\partial x_j} \tag{6}$$

In the third step the final velocity is obtained as
Step 3: Corrected velocity

$$U_j^{n+1} = U_j^* - \Delta t \frac{\partial p^n}{\partial x_j} \tag{7}$$

In incompressible flows, if there is heat transfer, the time discretization of the temperature field can be obtained in a fourth step, after applying the CBS scheme to the Eq. (3). The resulting equation is:
Step 4: Temperature transport equation

$$T^{n+1} = T^n + \Delta t\left[-u_j\frac{\partial T}{\partial x_j} + \frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{\partial}{\partial x_i}\left(\alpha\frac{\partial T}{\partial x_i}\right) - Q\right]^n + \frac{(\Delta t)^2}{2}\left\{u_i\frac{\partial}{\partial x_i}\left(u_j\frac{\partial T}{\partial x_j}\right)\right\}^n + \frac{(\Delta t)^2}{2}\left\{-u_j\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_i}\left(\alpha\frac{\partial T}{\partial x_i}\right)\right] + u_j\frac{\partial Q}{\partial x_j}\right\}^n \tag{8}$$

2.2.2 Spatial discretization and matrix form

Now the standard Galerkin approximation with the divergence theorem is applied to the time discretized Eqs. (5) to (8). In the GFEM the following set of interpolations are used for the variables:

$$U_j = N_u\tilde{U}_j\,;\ \Delta U_j = N_u\Delta\tilde{U}_j\,;\ \Delta U_j^* = N_u\Delta\tilde{U}_j^*\,;\ u_j = N_u\tilde{u}_j\,;\ p = N_p\tilde{p}\,;\ T = N_T\tilde{T} \tag{9}$$

Where $\tilde{U}_j = \begin{bmatrix} U_j^1 & U_j^2 & \cdots & U_j^k & \cdots & U_j^l \end{bmatrix}^T$ are the nodal variables and $N = \begin{bmatrix} N^1 & N^2 & \cdots & N^k & \cdots & N^l \end{bmatrix}$ are the interpolation functions. The integration of the equations and the use of the divergence theorem result in the weak formulation in the form

Step 1: Weak form of intermediate momentum

$$\int_\Omega N_u^T\Delta U_j^* d\Omega = \Delta t\left[-\int_\Omega N_u^T\frac{\partial}{\partial x_k}\left(u_k U_j\right)d\Omega - \frac{1}{\mathrm{Re}}\int_\Omega\frac{\partial N_u^T}{\partial x_i}\tau_{ij}d\Omega\right]^n + \frac{\Delta t^2}{2}\left[\int_\Omega\frac{\partial}{\partial x_m}\left(u_m N_u^T\right)\left(-\frac{\partial}{\partial x_k}\left(u_k U_j\right)\right)d\Omega\right]^n + \Delta t\left[\int_\Gamma N_u^T t_d d\Gamma\right]^n \tag{10}$$

Step 2: Weak form of pressure equation (semi-implicit CBS)

$$\int_\Gamma N_p^T\frac{\partial p^{n+1}}{\partial x_j}n_j d\Gamma - \int_\Gamma\frac{\partial N_p^T}{\partial x_j}\frac{\partial p^{n+1}}{\partial x_j}d\Gamma = \frac{1}{\Delta t}\int_\Omega N_p^T\frac{\partial U_j^*}{\partial x_j}d\Omega \tag{11}$$

Step 3: Weak form of momentum correction

$$\int_\Omega N_u^T\Delta U_j d\Omega = \int_\Omega N_u^T\Delta U_j^* d\Omega + \Delta t\int_\Omega\frac{\partial N_u^T}{\partial x_j}p^{n+1}d\Omega - \Delta t\int_\Gamma N_u^T t_p d\Gamma \tag{12}$$

Step 4: Weak form of temperature transport equation

$$\int_\Omega N_T^T \Delta T d\Omega = \Delta t \left[-\int_\Omega N_T^T \frac{\partial}{\partial x_k}\left(u_k T\right) d\Omega - \frac{1}{\mathrm{Re}\,\mathrm{Pr}} \int_\Omega \frac{\partial N_T^T}{\partial x_i} \alpha \frac{\partial T}{\partial x_i} d\Omega \right]^n + \\ + \frac{\Delta t^2}{2} \left[\int_\Omega \frac{\partial}{\partial x_m}\left(u_m N_T^T\right)\left(-\frac{\partial}{\partial x_k}\left(u_k T\right)\right) d\Omega \right]^n + \Delta t \frac{1}{\mathrm{Re}\,\mathrm{Pr}} \left[\int_\Gamma N_T^T \alpha \frac{\partial T}{\partial x_i} n_i d\Gamma \right]^n \tag{13}$$

The final matrix form of the weak formulation (10)-(13) is
Step 1: Intermediate momentum

$$\Delta \tilde{U}^* = -M_u^{-1} \Delta t \left[\left(C_u \tilde{U} + K_\tau \tilde{u} - f_u \right) - \Delta t \left(K_u \tilde{U} \right) \right]^n \tag{14}$$

Step 2: Pressure

$$H \Delta \tilde{p} = -\frac{1}{\Delta t} G \tilde{U}^* + f_p \tag{15}$$

Step 3: Momentum correction

$$\Delta \tilde{U} = \Delta \tilde{U}^* - M_u^{-1} \Delta t \left[G^T \left(\tilde{p}^n + \Delta \tilde{p} \right) \right] \tag{16}$$

Step 4: Temperature transport equation

$$\Delta \tilde{T} = -M_T^{-1} \Delta t \left[C_T \tilde{T} + K_t \tilde{T} + K_s \tilde{T} - f_T \right]^n \tag{17}$$

In the Equations (14) – (17) the matrices and vectors are defined as

$$M_u = \int_\Omega N_u^T N_u d\Omega;\; C_u = \int_\Omega N_u^T \left(\nabla^T \left(u N_u \right) \right) d\Omega;\; K_\tau = \int_\Omega B^T \frac{1}{\mathrm{Re}} \left(I_o - \frac{2}{3} m m^T \right) B d\Omega;\; f_u = \int_\Gamma N_u^T t_d d\Gamma$$

$$K_u = -\frac{1}{2} \int_\Omega \left(\nabla^T \left(u N_u \right) \right)^T \left(\nabla^T \left(u N_u \right) \right) d\Omega;\; H = \int_\Omega \left(\nabla N_p \right)^T \nabla N_p d\Omega;\; G = \int_\Omega \left(\nabla N_p \right)^T N_u d\Omega;\; f_p = \int_\Gamma N_p^T \nabla p^{n+1} n^T d\Gamma;$$

$$M_T = \int_\Omega N_T^T N_T d\Omega;\; C_T = \int_\Omega N_T^T \left(\nabla^T \left(u N_T \right) \right) d\Omega;\; K_T = -\frac{1}{2} \int_\Omega \left(\nabla^T \left(u N_u \right) \right)^T \left(\nabla^T \left(u N_u \right) \right) d\Omega;$$

$$K_s = \frac{1}{\mathrm{Re}\,\mathrm{Pr}} \int_\Omega \left(\nabla N_T \right)^T \alpha \nabla N_T d\Omega;\; f_T = \frac{1}{\mathrm{Re}\,\mathrm{Pr}} \int_\Gamma N_u^T q d\Gamma \tag{18}$$

Where *B* is a matrix defined by

$$B = S N_u \tag{19}$$

and *S* is a differential operator that for two-dimensional problems is of the form:

$$S^T = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \tag{20}$$

And the vectors m and I_o are

$$m = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T; \quad I_o = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix} \tag{21}$$

The CBS scheme as has been proved by Liu (2005) doesn't have the LBB restriction for interpolations functions. Velocity and pressure fields can be interpolated by functions of the same order without any spurious oscillation in the pressure field. For the semi-implicit form used in this work, the mass matrices in Eqs. (14), (16) and (17) are the lumped matrices.

3. SOLUTION AND RESULTS

The only system of equations solved in this scheme is Eq. (15), all others solutions are obtained explicitly. The solution of Eq. (15) is obtained by applying the pre-conditioned conjugate gradient method. In the explicit solution the time step can be calculated locally for stability considerations as demonstrated by Liu (2005). In this work, however, a small fixed time step of the order 10^{-5} has been specified. The original source program can be obtained from Lewis, Nithiarasu and Seetharamu (2004). This program has been validated in several problems. Some modifications of the original program were done and the modified code was tested in the case of the classical benchmark problem of the lid-driven cavity, in the domain $0 \le X \le 1;\ 0 \le Y \le 1$, presenting good agreement with results from the literature as shown in Fig. 1. Here is also presented an application for the case of simulation of flow over a tube bundle. The problem considered has application in many heat transfer processes. In the case it is considered the staggered arrangement of tubes.

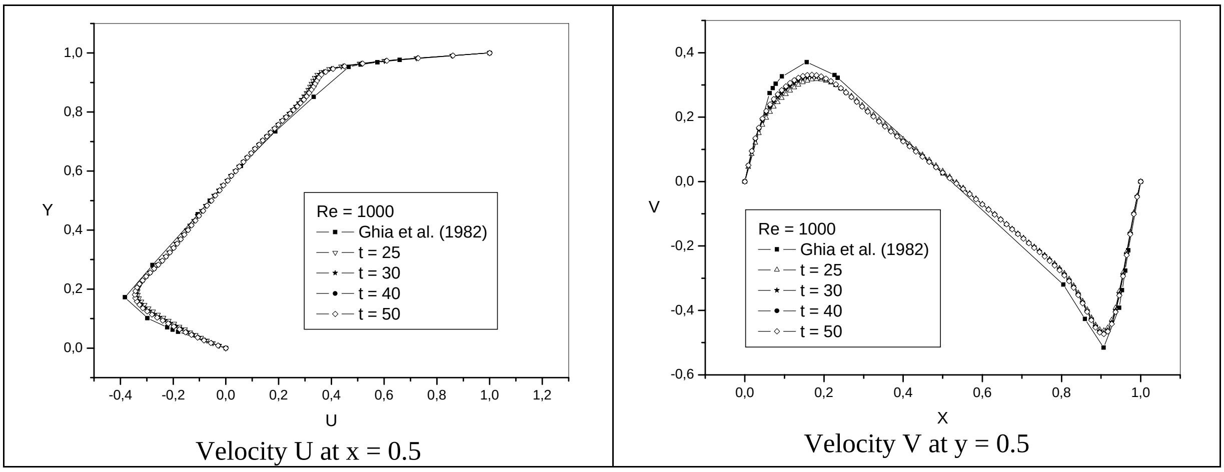
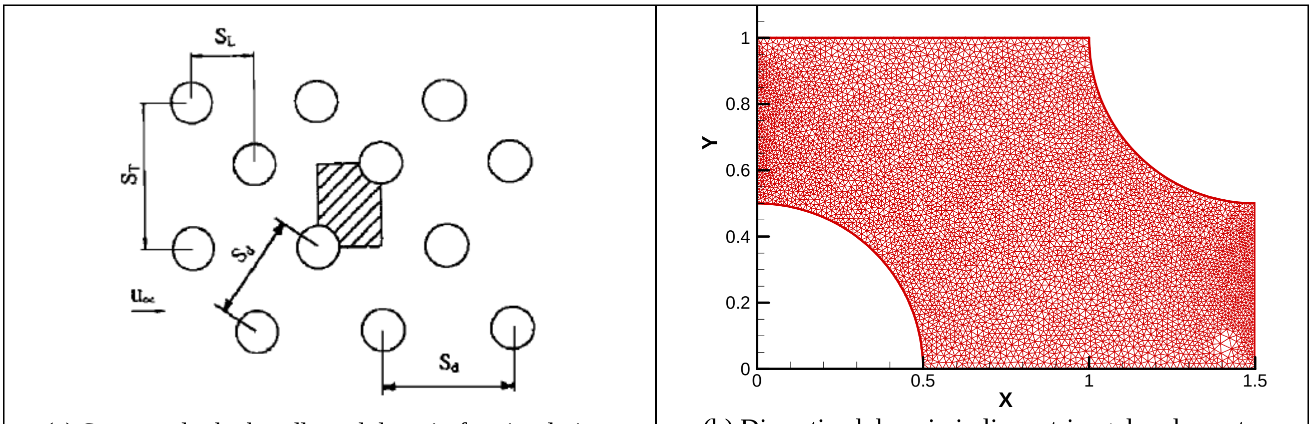


Figure 1. Flow at different times in a lid-driven cavity for Re = 1000.

The domain of the problem is presented in Fig. 2. For this case is considered that the flow characteristics repeat periodically, so only the dashed domain in Fig. 2 is discretized. Following Roychowdhury, Das and Sundararajan (2002), the domain for simulation is the range: $0 \le x \le 1{,}5;\quad 0 \le y \le 1$. In the case the longitudinal pitch is $S_L = 2$ and the transversal pitch is $S_T = 1.5$. In this simulation the reference length is the cylinder diameter chosen to be $D = 1$. There is some difficult to specify the boundary condition at the vertical inlet and outlet of the domain because they are not known before the solution. These boundary conditions can be specified during the solution to satisfy the mass conservation. However, in the present work a fictitious unitary axial velocity is applied at inlet and null pressure is applied at outlet. Non-slip boundary conditions are applied at tube wall. At the horizontal contours is imposed condition of symmetry as boundary conditions. It considered that the flow is heated by the walls of the tube. The non-dimensional temperature is zero at inlet and unitary at the tube walls.



(a) Staggered tube bundle and domain for simulation.

(b) Discretized domain in linear triangular elements.

Figure 2. Tube bundle in staggered arrangement and discretized domain (5864 nodes and 11420 elements)

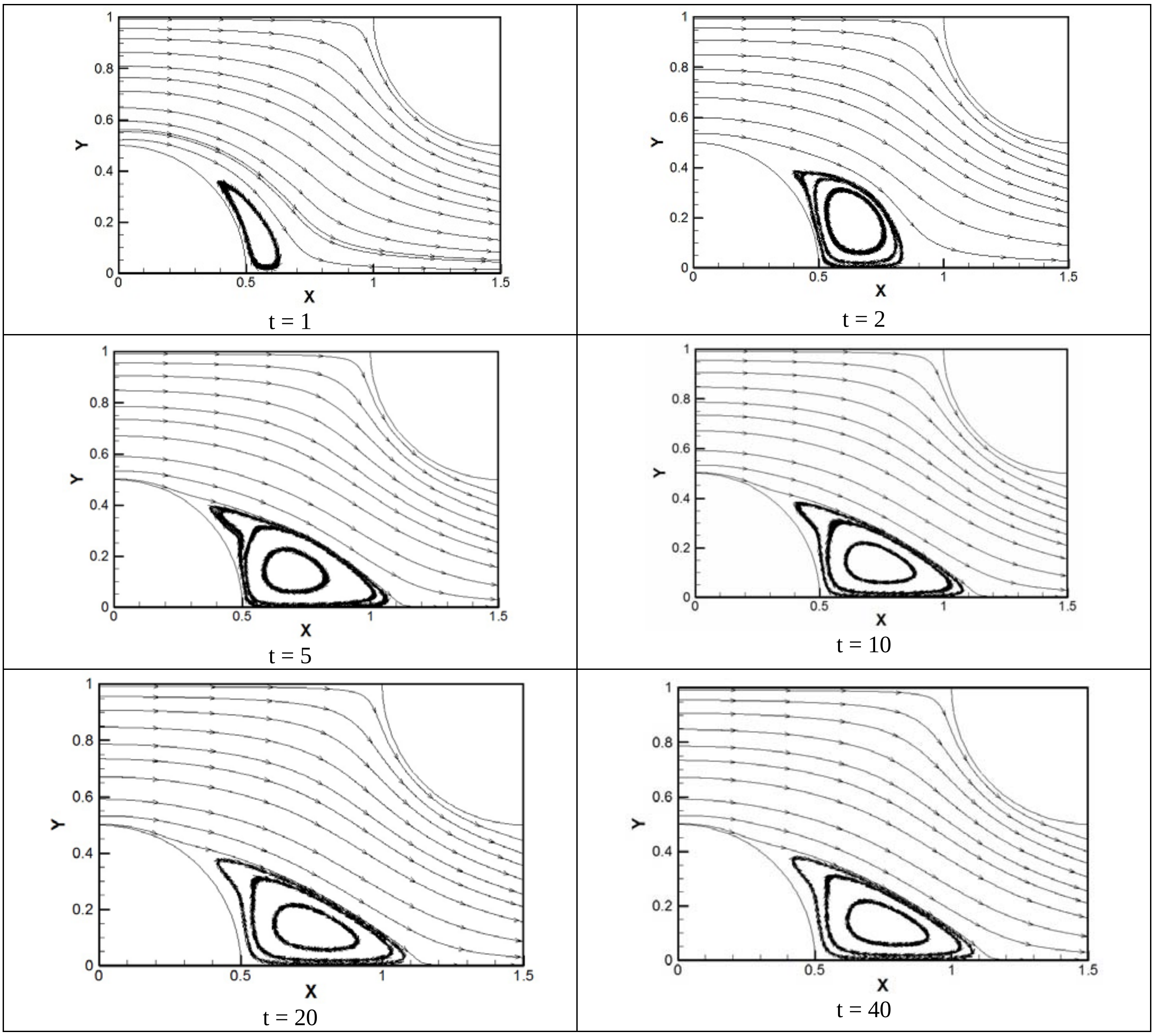


Figure 3. Flow development along the time for Re = 1000.

Now, some results of simulations are presented for the velocity and temperature fields. In Figure 3 is shown the evolution of the velocity field considering the streamtraces of the flow. It can be observed that the growing of the recirculation zone behind the lower tube presents the behavior expected. In front of the upper cylinder none recirculation is formed. To a better analysis of the flow, a simulation of the complete domain should be more appropriated. Also, others combinations of longitudinal and transversal pitches have to be considered to analyze the influence of the distance between cylinders on the flow. In this work none turbulence model has been implemented. For higher Reynolds numbers flows may be necessary some turbulence model for the corrected simulations of the flow field. With a time step of 10^{-5} it takes about forty minutes to run one unity of time, in the case of this simulation on a personal computer of 2 Gb de RAM and Intel processor core i5, 750 of 2.67 GHz.

In Figure 4 is shown the temperature field at different times for a Prandtl number equal to 0.71. The maximum temperature must be 1. Notice that the expected values have been simulated. By inspection of the Fig. 4, it can be observed that the fluid is more heated in the wake of the lower cylinder. The region of heated fluid in front of the upper cylinder is thin. The characteristic of boundary layer is simulated in front of the upper cylinder.

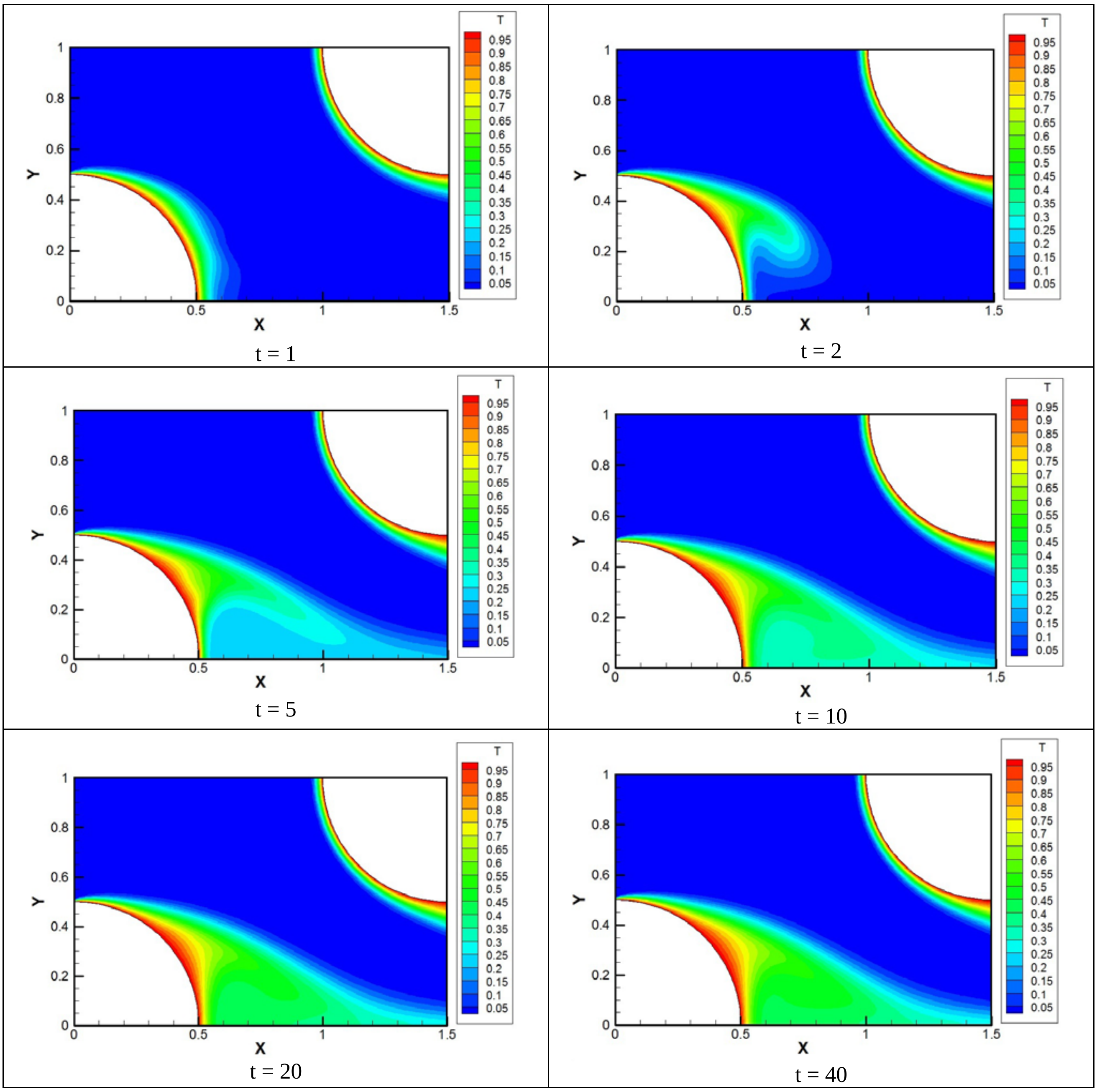


Figure 4. Temperature field in different times for Re = 1000.

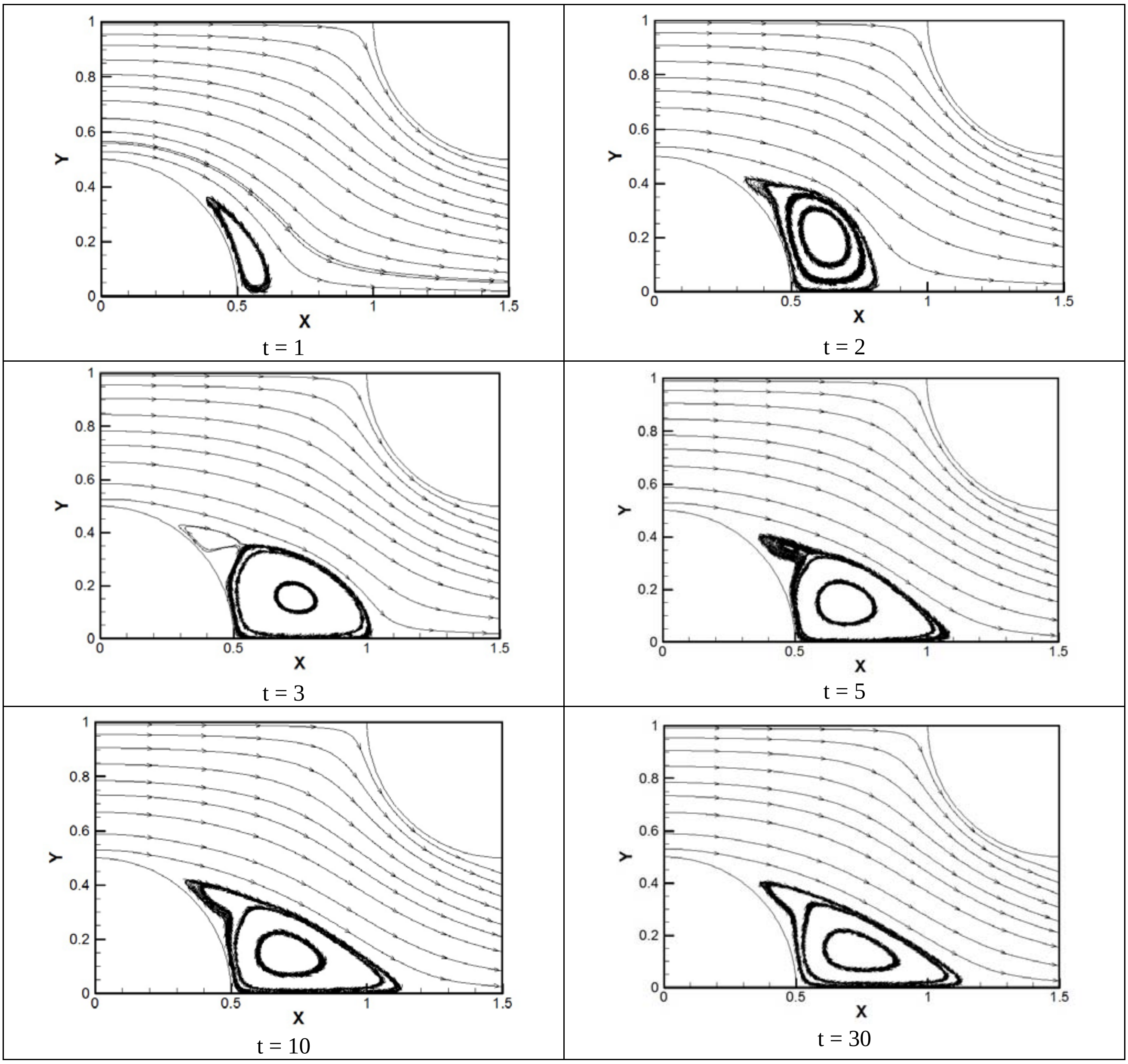


Figure 5. Flow development along the time for Re = 1500.

Some results of simulations for Reynolds number of 1500 are also presented in Figs. 5 and 6. In Figure 5 it is shown the development of the flow by the streamtraces at some initial times and in Fig. 6 it is shown results for the temperature field. In this case the results suggest the formation of a second small vortex after the separation of the flow from the cylinder surface, as can be seen at t= 5. Qualitatively the results seem to have the expected behavior.

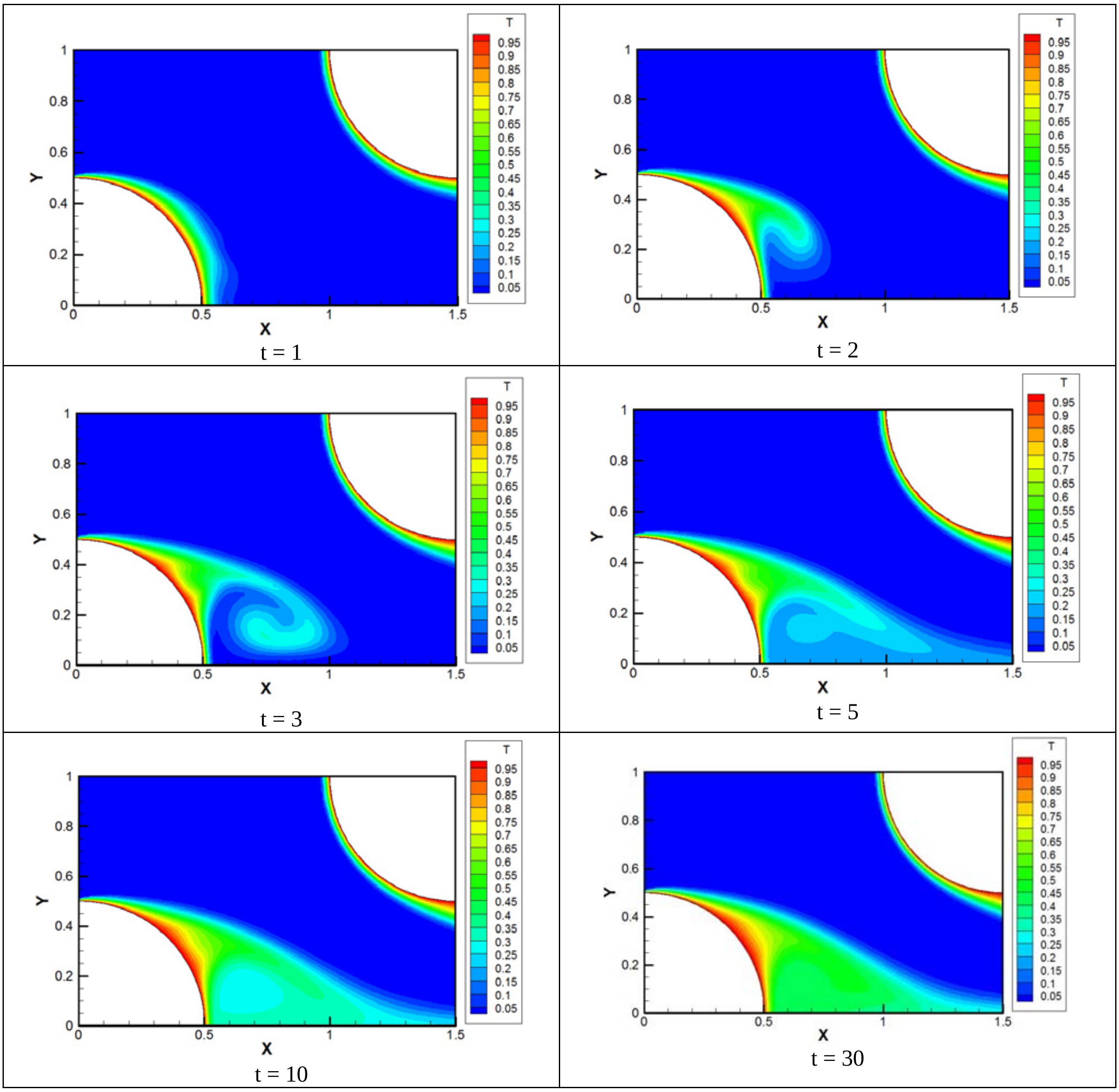


Figure 6. Temperature field in different times for Re = 1500.

3. CONCLUSIONS

The semi-implicit CBS scheme has been applied in this work to the heat transfer problem in a bank of staggered tubes. The corrected behaviors of the velocity and temperature fields were predicted. Only one configuration of longitudinal and transversal pitches was considered. The flows were simulated without any turbulence model for the Reynolds number adopted. The CBS was effective and presented good results for the cases analyzed. Simulations of others combinations of pitches are necessary for more insight in the evaluation of the influence of the distance between tubes.

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