

POSSIBILITY TO DESCRIBE HYPERBOLIC HEAT CONDUCTION USING FOURIER HEAT CONDUCTION ASSUMPTION

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Abstract. *The present paper proves that an adequate formulation of the thermomechanical couplings may lead to a thermodynamically admissible hyperbolic heat equation even if the Fourier Law is taken into account. The main assumption is that the intrinsic dissipation is not only a function of the state variables, but also of the temperature time rate. Various alternative models have been proposed to lead to a finite thermal wave speed. Generally they try to replace the classical Fourier heat conduction assumption. Nevertheless, most of these alternative heat conduction assumptions clearly violate the principle of objectivity. Besides, it is very difficult to assure that the resulting governing equations are thermodynamically admissible.*

Keywords: *hyperbolic heat conduction, finite thermal wave speed, fourier equation, second law of thermodynamics...*

1. Introduction

The classical linear equation for heat conduction based on Fourier's law is a parabolic equation in terms of the temperature field. Although this parabolic equation leads to an adequate description of heat conduction in most engineering applications, it predicts an infinite wave speed of heat conduction what is physically unrealistic. Consequently, any thermal disturbance exerted on a body is instantaneously felt through the whole body. It is now accepted that in situations involving very short times, extreme thermal gradients or temperatures near absolute zero may lead to a finite thermal wave speed. Finite thermal wave speed must be considered in the study of microelectronic devices such as IC chips (Xu, 1995), in the use of heat sources such as lasers and microwaves and scientific research (e.g., measuring physical properties of thin films, exhibiting microscopic heat transport dynamics). Lasers and microwaves are used in numerous applications related to material processing (e.g., surface annealing, welding and drilling of metals, and sintering of ceramics, see (Liu, 2009), for instance). In applications involving high heating rates induced by a short-pulse laser, the typical response time is in the order of picoseconds (Al-Nimr, 1997, 1999; Qiu, 1992). Lasers are also routinely used in medicine.

Various alternative models have been proposed to lead to a finite thermal wave speed. Generally they try to replace the classical Fourier heat conduction assumption. Some of them clearly violate a notion of the second law of thermodynamics since the heat may flow from cold to hot regions during finite time periods. For some of them it is very difficult to assure the resulting governing equations are thermodynamically admissible or the principle of objectivity is satisfied.

Vernotte (Vernotte, 1958), and Cattaneo (Cattaneo, 1958), based on the concept of heat transmission by waves, independently introduced an alternative equation aiming at describing problems involving high rates of temperature change, heat flow in an extremely short period of time or very low temperatures near absolute zero. After the pioneer works of Vernotte and Cattaneo, a number of research contributions have been dedicated to the study of problems involving hyperbolic heat conduction (Jackson, 1971; Narayanamurti, 1972; Joseph, 1990b,a; Kaminski, 1990; Rubin, 1992; Tien, 1993; Özisik, 1994; Guillemet, P., 1997; Barletta, 1997; Christov, 2005; Tibullo, 2011).

This study proves that an adequate thermomechanical formulation can lead to a hyperbolic heat equation even if we use the Fourier law. The main idea is that the Helmholtz postulate is not necessarily valid outside the states of equilibrium. Note that the idea here is not to say that other models are wrong and that the model proposed here is the only physically realistic that provides heat waves of finite speed. The main objective is to show an alternative approach that cannot be neglected in future experimental studies.

2. Some remarks about the first and second laws of thermodynamics

The basic thermodynamic framework and the main definitions necessary to the analysis are summarized on this section. In order to present the formal theoretical arguments, it is considered as a system an arbitrary part P of a body B that occupies a region $P_t \subset R^3$ at each time instant is taken as mechanical system. By definition, the boundary of the region P_t will be called ∂P_t .

Noting $U(P_t)$, $K(P_t)$, $P_{ext}(P_t)$ and $P_h(P_t)$, respectively, the internal energy, the kinetic energy, the power of the external forces and the non mechanical power (heat) of the system P at instant t , the first law of thermodynamics (FLT)

for the system can be expressed as an energy balance:

$$U(P_t) = \int_{P_t} \rho e \, dV \quad (1)$$

$$K(P_t) = \frac{1}{2} \int_{P_t} \rho (\mathbf{v} \cdot \mathbf{v}) \, dV \quad (2)$$

$$P_{ext}(P_t) = \int_{\partial P_t} (\mathbf{f} \cdot \mathbf{v}) \, dS + \int_{P_t} \mathbf{b} \cdot \mathbf{v} \, dV \quad (3)$$

$$P_h(P_t) = \int_{\partial P_t} (\mathbf{q} \cdot \mathbf{n}) \, dS + \int_{P_t} \rho \mathbf{r} \, dV \quad (4)$$

$$FLT \implies \frac{dU(P_t)}{dt} + \frac{dK(P_t)}{dt} = P_{ext}(P_t) + P_h(P_t) \quad (5)$$

Where ρ is the mass density, e the internal energy per unit mass, \mathbf{v} the velocity, \mathbf{f} the contact force applied on the boundary on ∂P_t , \mathbf{b} the body force applied on P_t , \mathbf{q} the heat flux vector, \mathbf{n} the unit outward normal to the surface ∂P_t and \mathbf{r} a heat supply or source per unit mass and unit time. If the principle of virtual power is taken into account, the following relation holds:

$$\frac{K(P_t)}{dt} = P_{int}(P_t) + P_{ext}(P_t) \quad (6)$$

$$P_{int}(P_t) = \int_{P_t} (\mathbf{T} : \nabla \mathbf{v}) \, dV \quad (7)$$

With \mathbf{T} being the Cauchy stress tensor. From equations (1) and (7) it comes that the FLT can be expressed as:

$$FLT \implies \frac{dU(P_t)}{dt} = P_{int}(P_t) + P_h(P_t) \quad (8)$$

Under suitable regularity assumption, using the balance of mass equation, equation (8) and the symmetry of the Cauchy stress tensor, it is possible to obtain the following classical local version of the FLT:

$$\rho \dot{e} = -\nabla \cdot \mathbf{q} + \mathbf{T} : \mathbf{D} + \rho \mathbf{r} \quad (9)$$

Where $\dot{\phi}$ denotes the material derivative of (ϕ) . The tensor \mathbf{D} is the symmetrical part of $\nabla \mathbf{v}$ and is usually called the deformation rate tensor. Another alternative way to express the first law of thermodynamics is by introducing directly the concept of entropy balance. In this case, the entropy and the absolute temperature are introduced as primitive quantities. Noting $S(P_t)$, $H_1(P_t)$, $H_2(P_t)$ and $H_3(P_t)$, respectively, the entropy of the system, the variation of entropy due to internal dissipative mechanisms, the entropy flux at the boundaries and an entropy source, it is possible to write:

$$S(P_t) = \int_{P_t} \rho s \, dV \quad (10)$$

$$H_1(P_t) = \int_{P_t} \frac{d}{\theta} \, dV \quad (11)$$

$$H_2(P_t) = - \int_{\partial P_t} \frac{(\mathbf{q} \cdot \mathbf{n})}{\theta} \, dS \quad (12)$$

$$H_3(P_t) = \int_{P_t} \frac{\rho \mathbf{r}}{\theta} \, dV \quad (13)$$

$$SLT \implies \frac{dS(P_t)}{dt} = H_1(P, t) + H_2(P, t) + H_3(P, t) \quad (14)$$

Where s is the entropy per unit mass, θ the absolute temperature and d the rate of energy dissipation per unit volume. Using the balance of mass equation and the symmetry of the Cauchy stress tensor, it is possible to prove that, in order to have equivalence between (9) and (14), must have the following form:

$$d = \mathbf{T} : \mathbf{D} - \rho (\dot{e} - \theta \dot{s}) - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \quad (15)$$

The second law of thermodynamics (SLT) can then be stated simply as:

$$SLT \implies H_1(P, t) \geq 0 \quad (16)$$

$$SLT \implies d \geq 0 \quad (17)$$

The second law of thermodynamics makes a distinction between possible ($d \geq 0$) and impossible ($d < 0$) processes. The possible processes may be reversible (the rate of energy dissipation is always equal to zero) or not. This local version of the SLT does not exclude the possibility of unusual behaviors such as a decreasing temperature if heat is added to the medium. To exclude the possibility of such kind of unusual behavior, here we only consider fluids that always satisfy a further restrictive constraint:

$$d_1 = \mathbf{T} : \mathbf{D} - \rho (\dot{e} - \theta \dot{s}) \geq 0 \quad (18)$$

$$d_2 = -\frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \geq 0 \quad (19)$$

Obviously, if the above relations are satisfied, then the local version of the SLT presented in (17) will also be satisfied. It is also simple to verify that the expression in (19) leads to the classical heat conduction inequality $-\mathbf{q} \cdot \nabla \theta \geq 0$ since the absolute temperature θ is a positive quantity. This relation implies that heat flows in the direction of decreasing temperature when \mathbf{q} is parallel to the temperature gradient. The quantity d_1 , defined in (18), is usually called the intrinsic dissipation and the quantity d_2 , the thermal dissipation.

A convenient local form for the first law of thermodynamics which will be useful in the next sections can be obtained introducing the definition of the intrinsic dissipation d_1 in (9):

$$-\nabla \cdot \mathbf{q} = \rho \dot{s} \theta - d_1 - \rho \mathbf{r} \quad (20)$$

It will be also convenient to use the Helmholtz free energy per unit mass ψ in the place of the internal energy per unit mass e :

$$\psi = e - \theta s \quad (21)$$

$$d_1 = \mathbf{T} : \mathbf{D} - \rho (\dot{\psi} + s \dot{\theta}) \quad (22)$$

3. Basic constitutive assumptions

The main constitutive assumption considered in this paper is:

1. The classical Fourier heat conduction assumption is valid

$$\mathbf{q} = -k \nabla \theta \quad (23)$$

Where k is a positive function of θ , usually called the thermal conductivity. Using (23) it is simple to verify that:

$$d_2 = -\frac{1}{\theta} \mathbf{q} \cdot \nabla \theta = \frac{k}{\theta} \nabla \theta \cdot \nabla \theta \geq 0 \quad (24)$$

Since the absolute temperature θ is a positive quantity and k is a positive function of θ . Therefore, in order to assure the equations are thermodynamically admissible, it is only necessary to propose sufficient conditions to assure the inequality ($d_1 \geq 0$) is automatically satisfied in any process. The additional information about the material behaviour is obtained from two thermodynamic potentials: the Helmholtz free energy ψ and the dissipation potential $\hat{\phi}$. To simplify the discussion, the demonstration is restricted to an incompressible material ($\nabla \cdot \mathbf{v} = 0$) such that:

$$\mathbf{T} = -p \mathbf{I} \quad (25)$$

Where p is the hydrostatic pressure, which is a Lagrange multiplier related to the incompressibility constraint ($\nabla \cdot \mathbf{v} = \text{tr}(\mathbf{D}) = 0$) and \mathbf{I} is the second order unity tensor. It is important to emphasize that such a simplifying assumption is adopted only to reduce the paper size and that it is not necessary in order to prove that an adequate formulation of the thermomechanical couplings may lead to a thermodynamically admissible hyperbolic heat equation even if the Fourier Law is taken into account. Besides the hypothesis H1, the following additional constitutive assumptions are taken into account:

2. The Helmholtz Free Energy is a differentiable function of absolute temperature:

$$\psi = \hat{\psi}(\theta) \quad (26)$$

3. The rate of energy dissipation d_1 has the following form:

$$d_1 = \frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} \dot{\theta} \quad (27)$$

It is simple to verify that, for an incompressible material, the product $\mathbf{T} : \mathbf{D} = 0$ since:

$$\mathbf{T} : \mathbf{D} = -p\mathbf{I} : \mathbf{D} = -p \cdot \text{tr}(\mathbf{D}) = 0 \quad (28)$$

Hence, introducing (26) and, (27) into the definition of the intrinsic dissipation in (22), it is possible to obtain:

$$\rho \left(s + \frac{\partial \hat{\psi}(\theta)}{\partial \theta} + \frac{1}{\rho} \frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} \right) \dot{\theta} = 0; \quad \forall \dot{\theta} \quad (29)$$

Consequently, the following constitutive relation must hold:

$$s = - \left(\frac{\partial \hat{\psi}(\theta)}{\partial \theta} + \frac{1}{\rho} \frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} \right) \quad (30)$$

In order to assure that equations (23), (25), (30) are thermodynamically admissible, the following additional condition is considered:

4. The potential $\hat{\phi}$ is a differentiable, positive and strictly convex function of the variable $\dot{\theta}$ and θ such that $\hat{\phi}(\dot{\theta} = 0, \theta) = 0, \forall \theta$.

$$\hat{\phi} = \hat{\phi}(\dot{\theta}, \theta) \quad (31)$$

If these conditions hold, a classical result of Convex Analysis (Ekeland, 1976; Rockafellar, 1970) is that:

$$\frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} \dot{\theta} \geq 0 \quad (32)$$

In this case, these conditions are sufficient to assure that equations (23), (25), (30), form a complete set of thermodynamically admissible constitutive equations, i.e., for any particular set of constitutive equations obtained within the context proposed on this paper, the inequalities ($d_1 \geq 0$) and ($d_2 \geq 0$) (and hence the Clausius-Dühen inequality $d \geq 0$) are automatically satisfied in any process.

4. Heat Equation

The heat equation is a version of the first law of thermodynamics for the materials described by the constitutive theory presented in the previous section. From (30) it is possible to obtain:

$$\dot{s} = - \left(\frac{\partial^2 \hat{\psi}(\theta)}{\partial \theta^2} \dot{\theta} + \frac{1}{\rho} \frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}^2} \ddot{\theta} + \frac{1}{\rho} \frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \theta \partial \dot{\theta}} \dot{\theta} \right) \quad (33)$$

Introducing the previous equation into the local version (20) of the FLT, it comes that

$$-\nabla \cdot \mathbf{q} = \rho \left(-\frac{\partial^2 \hat{\psi}(\theta)}{\partial \theta^2} \dot{\theta} - \frac{1}{\rho} \frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}^2} \ddot{\theta} - \frac{1}{\rho} \frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \theta \partial \dot{\theta}} \dot{\theta} \right) \theta - d_1 - \rho r \quad (34)$$

Finally, combining equations (23), (27) with (34) it is possible to conclude that:

$$\nabla \cdot (k \nabla \theta) = \left(-\rho \frac{\partial^2 \hat{\psi}(\theta)}{\partial \theta^2} \dot{\theta} - \frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \theta \partial \dot{\theta}} \dot{\theta} - \frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} \right) \dot{\theta} + \left(-\frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}^2} \dot{\theta} \right) \ddot{\theta} - \rho r \quad (35)$$

Equation (35) allows a hyperbolic description of heat transfer, depending on the choice of the potentials $\hat{\psi}$ and $\hat{\phi}$. For instance, it is possible to the following particular form for the dissipation potential $\hat{\phi}$.

$$\hat{\phi}(\dot{\theta}, \theta) = \frac{\tau}{\theta} f(\dot{\theta}) \quad (36)$$

With τ being a nonnegative constant and f a convex function that assures H4 is verified. In this case, a strong simplification can be made in the heat equation (35) since

$$\frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} = \frac{\tau}{\theta} f'(\dot{\theta}) \quad (37)$$

$$\frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \theta \partial \dot{\theta}} \theta = -\frac{\tau}{\theta} f'(\dot{\theta}) \quad (38)$$

Thus form equations (37) and (38):

$$\frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} + \frac{\partial^2 \hat{\phi}(\dot{\theta}, \theta)}{\partial \theta \partial \dot{\theta}} \theta = 0 \quad (39)$$

Thus, from (35), (36) and (39), the following simpler expression for the heat transfer equation is obtained

$$\nabla \cdot (k \nabla \theta) = \rho \hat{c} \dot{\theta} - \hat{\tau} \ddot{\theta} - \rho \mathbf{r} \quad (40)$$

Where:

$$\hat{c} = -\theta \frac{\partial^2 \psi}{\partial \theta^2} \quad (41)$$

$$\hat{\tau} = \theta \frac{\partial^2 \phi(\dot{\theta}, \theta)}{\partial \dot{\theta}^2} = \tau f''(\dot{\theta}) \quad (42)$$

If equations (41) and (42) holds, \hat{c} and $\hat{\tau}$ can be related to the reversible and irreversible parts (s^r, s^{ir}) of the entropy s (given in (30)) trough the following relations:

$$s = s^r + s^{ir} \quad (43)$$

Where:

$$s^r = -\frac{\partial \hat{\psi}(\theta)}{\partial \theta} \quad (44)$$

$$s^{ir} = -\frac{1}{\rho} \frac{\partial \hat{\phi}(\dot{\theta}, \theta)}{\partial \dot{\theta}} \quad (45)$$

From the (44) and (45)

$$\hat{c} = -\theta \frac{\partial^2 \psi}{\partial \theta^2} = \theta \frac{\partial s^r}{\partial \theta} \quad (46)$$

$$\hat{\tau} = \theta \frac{\partial^2 \phi(\dot{\theta}, \theta)}{\partial \dot{\theta}^2} = \rho \theta \frac{\partial s^{ir}}{\partial \theta} \quad (47)$$

The particular choice of $\hat{\phi}$ made in (36) and condition H4 imply that $\hat{\tau}$ is always nonnegative. The expression for the variable \hat{c} in (41) is exactly the same found for the specific heat in the classical parabolic theory in which. Different expressions can be chosen for $f(\dot{\theta})$ and for $\hat{\psi}$. Possible choices of $f(\dot{\theta})$ and $\hat{\psi}$ are:

$$f(\dot{\theta}) = \frac{\dot{\theta}^2}{2} \quad (48)$$

$$\hat{\psi}(\theta) = \hat{\psi}_0 - \int_{\theta_0}^{\theta} \pi_1 \log(\xi) d\xi - \frac{1}{2} \pi_2 (\theta - \theta_0)^2 \quad (49)$$

With $\hat{\psi}_0$ being a positive material constant and c and π , nonnegative material constants. Hence

$$s^r = -\frac{\partial \hat{\psi}(\theta)}{\partial \theta} = c (\log(\theta) - \log(\theta_0)) + \pi (\theta - \theta_0) \quad (50)$$

And, therefore

$$\hat{c} = \theta \frac{\partial s^r}{\partial \theta} = c + \pi \theta \quad (51)$$

Besides, from the definition of $\hat{\tau}$, equation (42), and $f(\dot{\theta})$, equation (48):

$$\hat{\tau} = \tau f''(\dot{\theta}) = \tau \quad (52)$$

The constitutive equations proposed by Green and Naghdi, (Green, 1977), using a very different framework for hyperbolic heat conduction lead to a heat transfer equation which is exactly the one obtained from (35), using (36), (48) and (49), assuming a constant thermal conductivity k and taking $\pi = 0$ and $\mathbf{r} = 0$ (no heat supply or source).

$$k \nabla^2 \theta = \rho \left(c \dot{\theta} - \alpha \ddot{\theta} \right) \quad (53)$$

Where the symbol ∇^2 denotes the Laplacian operator with respect to the present position and $\alpha = \tau/\rho$.

5. Conclusion

This theoretical study proves that an adequate thermomechanical formulation can lead to a hyperbolic heat equation even if the Fourier law is taken into account. The main idea is that intrinsic dissipation d_1 is also a function of the temperature rate $\dot{\theta}$. Various alternative models have been proposed in the literature trying to obtain a finite thermal wave speed.

Generally the classical Fourier heat conduction assumption ($\mathbf{q} = -k \nabla \theta$) is replaced by alternative laws, such as the Cattaneo-Vernotte equation ($\mathbf{q} + \tau \dot{\mathbf{q}} = -k \nabla \theta$). Nevertheless, most of these alternative heat conduction assumptions clearly violate the principle of objectivity. Besides, it is very difficult to assure that the resulting governing equations are thermodynamically admissible, since heat may flow from cold to hot regions during finite time periods.

The main objective is to show an alternative approach that cannot be neglected in future experimental studies. The generalization of the demonstration to more complex material behaviours can be performed by considering a more sophisticated constitutive theory under the framework of thermodynamics of irreversible processes. All constitutive equations obtained in (Costa Mattos, 2009) in the case of inelastic damageable solids undergoing small transformations, and in (Costa Mattos, 1998) in the case of non Newtonian fluids, can be extended to account for hyperbolic heat transfer even if the classical Fourier heat conduction assumption is adopted. The key in all cases is to consider that the intrinsic dissipation is not only a function of the state variables, but also of the temperature rate.

6. ACKNOWLEDGEMENTS

This optional section must be placed before the list of references.

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