A COMPARISON OF ANGULAR DISCRETIZATION SCHEMES OF THE HYBRID FINITE VOLUME/FINITE ELEMENT METHOD FOR THE SOLUTION OF THE RADIATIVE TRANSFER EQUATION

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Abstract. *A hybrid finite volume / finite element method was developed a few years ago to solve the radiative transfer equation. In this method, the radiation intensity is approximated as a linear combination of basis functions, dependent only on the angular direction. The coefficients of the approximation are unknown functions of the spatial coordinates. This Galerkin-like approximation is introduced into the radiative transfer equation. Then, this is multiplied by the nth basis function and integrated over all directions, i.e., over a solid angle of* 4π*, yielding a set of differential equations. The spatial discretization is carried out using the finite volume method, like in the discrete ordinates and finite volume methods, transforming the differential equations into algebraic equations. The angular discretization is accomplished using a methodology similar to that employed in the finite element method. In previous works, the basis functions were taken as the bilinear basis functions used in the finite element method. In the present work, spherical triangular basis functions are employed, and the results are compared with those computed using bilinear basis functions, as well as with results reported in the literature using other methods. Two-dimensional enclosures containing an emitting-absorbing, non-scattering, grey medium with prescribed temperature or in radiative equilibrium are considered. It is shown that the spherical triangular elements yield results slightly better than those calculated using bilinear elements, except in the case of optically thick media, where both elements perform similarly. The results of the hybrid method are less sensitive to the angular discretization than those obtained using the discrete ordinates method.*

Keywords*: Thermal Radiation, Radiative Transfer Equation, Finite Volume Method, Finite Element Method, Angular Discretization*

1. INTRODUCTION

Thermal radiation is an important heat transfer mode in many physical and engineering processes. The radiative transfer equation (RTE) describes the propagation of radiation in a participating medium that emits, absorbs and scatters radiation. This equation is difficult to solve, as it is an integro-differential equation, and involves seven independent variables in the most general case, since the radiation intensity may depend on the spatial coordinates, angular direction, wavelength and time. Many numerical methods have been developed to solve the RTE, as described in Modest (2003).

A hybrid finite volume / finite element method, referred to as HYDRA (HYbrid Discretization for RAdiation) was developed in Coelho (2005a, 2005b) to solve the RTE. The fundamentals of the method for one-dimensional problems are reported in Coelho (2005a) while the extension to multidimensional problems in Cartesian coordinates is addressed in Coelho (2005b). In this method, the radiation intensity is approximated as a linear combination of basis functions, dependent only on the angular direction, while the coefficients of the approximation are unknown functions of the spatial coordinates. The key difference between this method and the well known discrete ordinates (DOM) and finite volume methods is that while in these methods the radiation intensity is constant over a solid angle, in the present method the radiation intensity is a continuously varying function, because the basis functions vary continuously within the control angle elements. Further developments were reported in Coelho and Aelenei (2008), who used high resolution discretization schemes for the spatial discretization, and in Coelho (2006), who compared the accuracy of bilinear and biquadratic elements for the angular discretization, and of the STEP and CLAM schemes for the spatial discretization. Recently, Coelho (2008) extended the method to non-grey media, while Coelho (2009) applied the method to enclosures of complex geometry.

Two other works based on a finite element approach for the angular discretization were recently presented. Becker *et al*. (2010) solved the even-parity form of the RTE using a finite element approximation for both the spatial and the angular discretizations. The angular domain was discretized using triangular linear basis functions. Widmer *et al*. (2008) approximated the radiation intensity by a tensor product where the coefficients (unknowns) are multiplied by piecewise linear basis functions dependent on the spatial location and piecewise constant basis functions dependent on the direction. They employed a least-squares approach using weight functions that depend also on the spatial location and direction, and proposed a sparse tensor product approximation to reduce the number of degrees of freedom and so decrease the computational requirements. The angular domain was discretized into triangular elements. Another distinct feature of their work is the use of wavelet finite element functions rather than standard finite elements.

In the present work, triangular spherical elements are used along with the HYDRA method, and compared with bilinear elements formerly employed. Radiative transfer is calculated for two-dimensional enclosures containing an emitting-absorbing, non-scattering, grey medium.

2. MATHEMATICAL FORMULATION

The RTE in a non-grey medium may be written as (Modest, 2003):

$$\mathbf{s}\cdot\nabla I_\nu(\mathbf{r},\mathbf{s}) = -\beta_\nu\, I_\nu(\mathbf{r},\mathbf{s}) + \kappa_\nu\, I_{b\nu}(\mathbf{r}) + \frac{\sigma_{s\nu}}{4\pi}\int_{4\pi} I_\nu(\mathbf{r},\mathbf{s}')\,\Phi_\nu(\mathbf{s}',\mathbf{s})\,d\Omega' \tag{1}$$

where I is the radiation intensity, $I_{b\nu}$ the Planck function, Φ the phase function, Ω the solid angle, **r** the position vector, **s** the unit vector along the direction of propagation of radiation, κ the absorption coefficient, σ_s the scattering coefficient, $\beta = \kappa + \sigma_s$ the extinction coefficient, and subscript ν the wavenumber. This subscript will be dropped from now on, as well as the last term of the right hand side, since a non-scattering grey medium is considered in the present work. The direction of propagation **s** does not depend on the spatial coordinates. Therefore, the RTE may be written as

$$\nabla\cdot(\mathbf{s}\, I(\mathbf{r},\mathbf{s})) = -\kappa I(\mathbf{r},\mathbf{s}) + \kappa I_b(\mathbf{r}) \tag{2}$$

The boundary condition for an opaque, diffuse and grey surface is given by (Modest, 2003)

$$I(\mathbf{r}_w,\mathbf{s}) = \varepsilon I_b(\mathbf{r}_w) + \frac{\rho}{\pi}\int_{\mathbf{n}\cdot\mathbf{s}'<0} I(\mathbf{r}_w,\mathbf{s}')\,|\mathbf{n}\cdot\mathbf{s}'|\,d\Omega' \tag{3}$$

where ε and ρ are the emissivity and the reflectivity of the surface, respectively, **n** is the outward unit vector normal to the wall, and subscript w refers to a wall. Equation (2) is independent of the coordinate system. Hence, integrating both left and right-hand side terms over an arbitrary control volume centred at grid node P, whose volume is V, and applying the Gauss divergence theorem to the term on the left hand side, the following equation is obtained

$$\int_A \mathbf{s}\cdot\mathbf{n}\, I(\mathbf{r},\mathbf{s})\, dA = -\kappa V\, I_P(\mathbf{s}) + \kappa V\, I_{b,P} \tag{4}$$

Since the outer unit vector normal to a cell face, **n**, does not change along that face, Eq. (4) may be written as

$$\sum_{f=1}^{F} \mathbf{s}\cdot\mathbf{n}_f\; I_f(\mathbf{s})\, A_f = -\kappa V\, I_P(\mathbf{s}) + \kappa V\, I_{b,P} \tag{5}$$

where subscript f denotes a cell face, whose area is A_f, and F is the total number of cell faces of the control volume under consideration. The mean radiation intensity at cell face f along direction **s** was denoted by $I_f(\mathbf{s})$. The symbol $\mathbf{n}_f$ represents the outer unit vector normal to cell face f.

In the present method, the spatial and angular dependence of the radiation intensity is split according to the following approximation:

$$I(\mathbf{r},\mathbf{s}) = \sum_{m=1}^{N} I^m(\mathbf{r})\,\phi_m(\mathbf{s}) \tag{6}$$

In this approximation, $\phi_m(\mathbf{s})$ are linearly independent functions that constitute the basis of a space of dimension N. These basis functions depend on the direction of propagation of radiation, **s**, i.e., on the polar and azimuthal angles that define direction **s.** The functions $I^m(\mathbf{r})$, which depend only on the spatial coordinates, are unknowns. Substituting Eq. (6) into Eq. (5), multiplying both sides of by $\phi_n(\mathbf{s})$, with $1 \le n \le N$, and integrating over all directions, the following set of N simultaneous equations is obtained:

$$\begin{aligned}\sum_{f=1}^{F} A_f\;\mathbf{n}_f\cdot\sum_{m=1}^{N} I_f^m \int_{4\pi} \mathbf{s}\,\phi_m(\mathbf{s})\,\phi_n(\mathbf{s})\,d\Omega &= -\kappa V \sum_{m=1}^{N} I_P^m \int_{4\pi} \phi_m(\mathbf{s})\,\phi_n(\mathbf{s})\,d\Omega + \\ &+ \kappa V\, I_{b,P}\int_{4\pi} \phi_n(\mathbf{s})\,d\Omega \qquad n = 1,2,\cdots,N\end{aligned} \tag{7}$$

where I_f^m is the mean value of function $I^m(\mathbf{r})$ at cell face *f*. A spatial discretization scheme is required to compute the radiation intensity at the faces of a control volume as a function of the radiation intensity at the spatial grid nodes, i.e, I_f^m must be expressed in terms of I_P^m. The step scheme, which takes the radiation intensity at a cell face equal to the radiation intensity at the upstream grid node, was employed in the present work.

The basis functions, $\phi_n(\mathbf{s})$, are defined according to criteria employed in the finite element method. The *n*th basis function is equal to 1 at the *n*th angular node and equal to 0 at all other angular nodes. The restriction of function $\phi(\mathbf{s})$ to an angular element is referred to as shape function and denoted by ψ. The basis functions are defined element by element and have been assumed in past work as bilinear within every angular element (see Fig. 1a). The use of biquadratic basis functions was addressed in Coelho (2006).

In the present work, spherical triangular elements were used, as shown in Fig. 1b. The basis functions were defined following the method described in Becker *et al*. (2010). First, the sphere is divided into solid angles in the same way as in the T_N quadrature (Turghood *et al*., 1995). In this method, the equilateral triangle whose vertices are (1, 0, 0), (0, 1, 0) and (0, 0, 1) is mapped onto the first octant of a unit radius sphere using the relation $\mathbf{s} = \mathbf{r}/|\mathbf{r}|$, where $\mathbf{r}$ stands for a point on the triangle and $\mathbf{s}$ denotes the position vector of a mapped point on the sphere. The triangle is tessellated into smaller identical triangles by dividing each side into *N* equally spaced segments and connecting the points that result from that division by lines parallel to the sides of the original triangle. The projection of the vertices of every small triangle onto the surface of the sphere defines spherical triangles on the surface of the sphere. Second, the basis functions are defined by parameterization of the planar triangles (see details in Becker *et al*., 2010).

In the case of triangular elements, different angular node numbers have been assigned to solid angles in different octants. It was found that this option yields more accurate results than using the same node numbers for the angular nodes lying on the $x = 0$ and $y = 0$ planes, which share different octants.

The governing equations are solved iteratively. The *N* simultaneous equations for a spatial grid node are solved using the Gauss elimination method adapted to band matrices, and the spatial control volumes are visited sequentially in every iteration.

Two important quantities in thermal radiation are the heat flux, which allows the calculation of the radiative energy transferred to a boundary of the domain, and the incident radiative, which allows the evaluation of the local radiative energy source. The incident radiation at a grid node is evaluated as follows

$$G = \int_{4\pi} I(\mathbf{r},\mathbf{s})\, d\Omega = \sum_{m=1}^{N} I^m(\mathbf{r}) \int_{4\pi} \phi_m(\mathbf{s})\, d\Omega \tag{8}$$

The radiative heat flux at a surface normal to direction *k*, denoted by the subscripts *w*,*k*, is given by

$$q_{w,k} = \int_{\mathbf{s}\cdot\mathbf{n}_f<0} \left|\mathbf{s}\cdot\mathbf{n}_f\right| I(\mathbf{r_w},\mathbf{s})\, d\Omega = \sum_{m=1}^{N} I^m(\mathbf{r_w}) \int_{\mathbf{s}\cdot\mathbf{n}_f<0} \left|\mathbf{s}\cdot\mathbf{n}_f\right| \phi_m(\mathbf{s})\, d\Omega \tag{9}$$

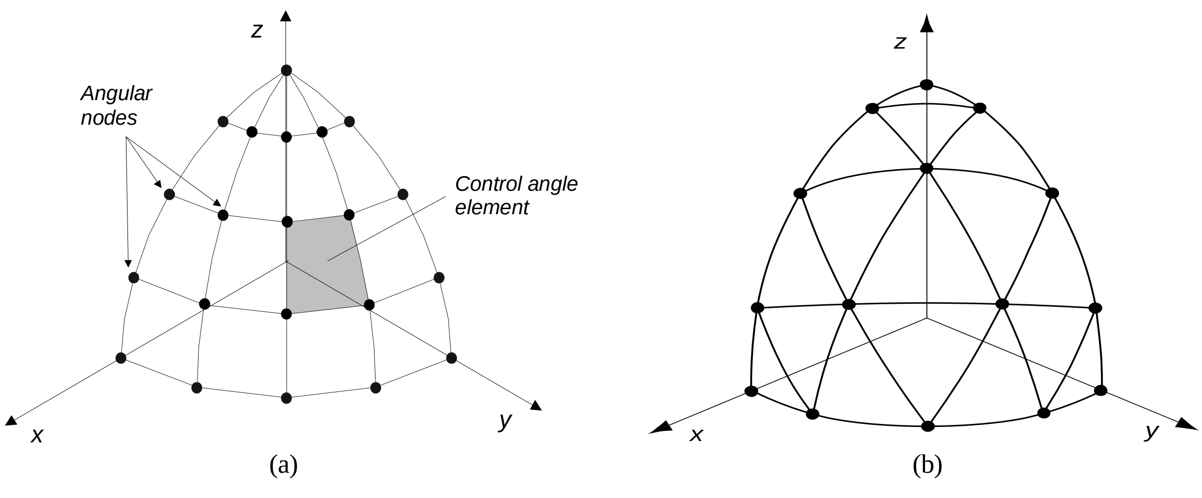


Figure 1. Angular discretization by means of bilinear (a) and triangular (b) elements.

3. RESULTS AND DISCUSSION

Two test cases have been selected to compare the different angular discretization schemes described in the previous section, and to assess their accuracy. Analytical solutions and other numerical predictions are also presented for comparison purposes. The angular discretization was carried out using spherical triangular elements defined according to the T_N quadrature (Thurgood *et al*., 1995) when the HYDRA method and triangular elements were used. When the HYDRA method and bilinear elements were employed, the angular discretization was defined by spherical lines of constant latitude and spherical lines of constant longitude, yielding $N_\theta \times N_\phi$ solid angles per octant. The results of calculations performed using the discrete ordinates method (DOM), Fiveland (1984), along with the step scheme and the S_N level symmetric quadrature satisfying sequential odd moments (Fiveland, 1991) are presented for comparison

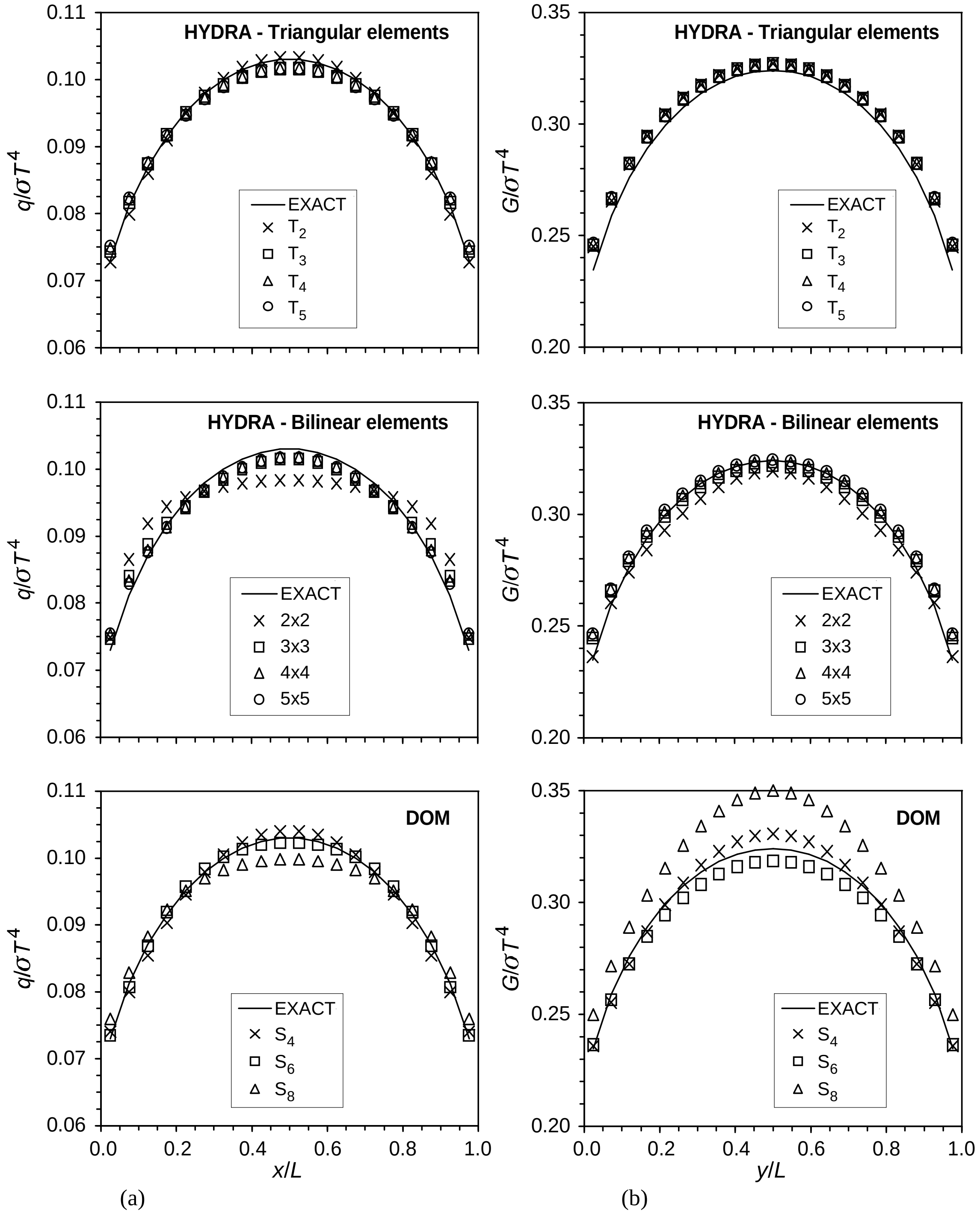
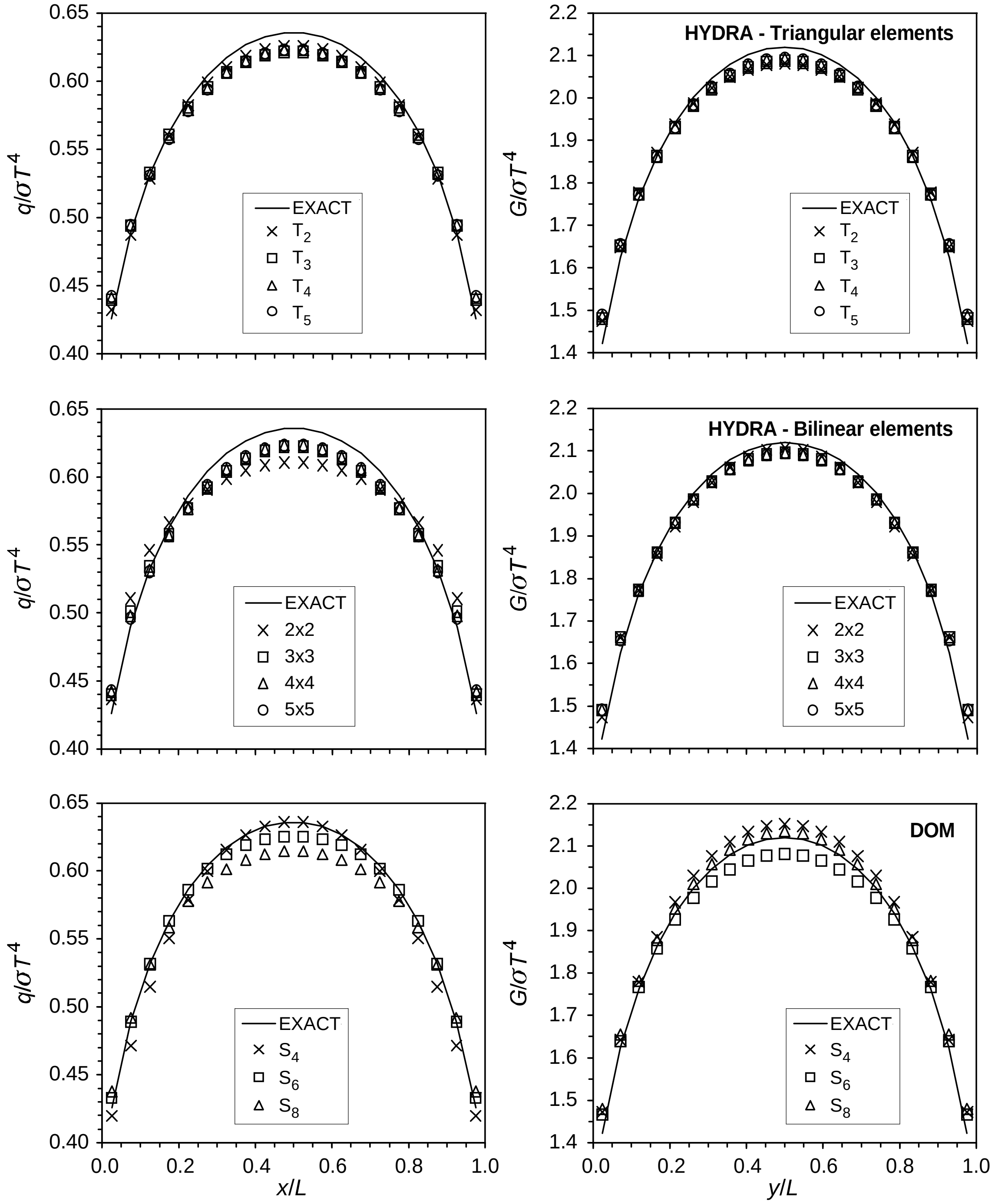


Figure 2. Normalized incident heat flux along the bottom wall (a) and incident radiation along the vertical symmetry plane (b) of the enclosure of test case 1 for an optical thickness of 0.1.

purposes. The results reported by Becker *et al*. (2010) using two different methods based on the even-parity formulation of the RTE are also presented. In the first method, referred to as EP-DOM (Koch *et al*., 1995), the DOM was used. In the second one, referred to as EP-FEM-SAFE, the finite element method was employed for both spatial and angular discretization.

The total number of quadrature points, corresponding to directions along which the discretized RTE is solved, in the DOM and EP-DOM is 24, 48 and 80 for the S_4, S_6 and S_8 approximations, respectively. In the HYDRA method, the total number of angular elements is equal to 32, 72, 128 and 200 for $N_\theta = N_\phi = 2$, 3, 4 and 5, respectively, in the case of bilinear elements, and for T_2, T_3, T_4 and T_5, respectively, in the case of triangular elements. In the EP-FEM-SAFE method, the total number of angular elements is also 32, 72, 128 and 200 for SAFE 18, SAFE 38, SAFE 66 and SAFE 108, respectively. The number following SAFE denotes the number of angular nodes of the angular discretization. In

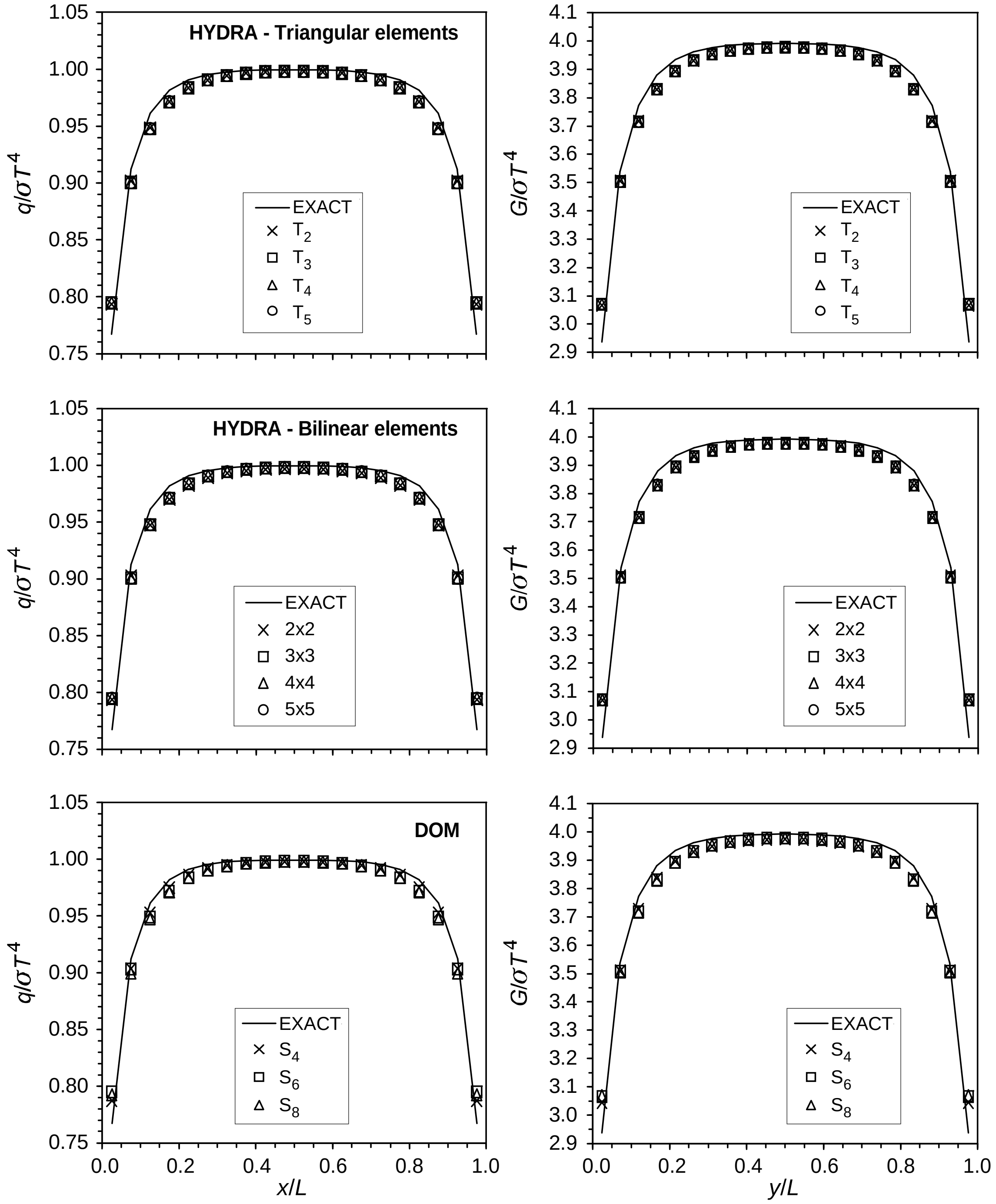


(a) (b)

Figure 3. Normalized incident heat flux along the bottom wall (a) and incident radiation along the vertical symmetry plane (b) of the enclosure of test case 1 for an optical thickness of 1.

practice, only directions and angular elements associated to positive (or negative) z directions need to be considered in two-dimensional problems.

In the first test case, a two-dimensional square enclosure of side length L is considered. The walls are black and cold, the medium is grey and emits and absorbs with a uniform unity emissive power. Three different values of the optical thickness of the medium, based on the side length, have been considered, namely 0.1, 1 and 10. The analytical solution is available in Fiveland (1984). A uniform grid with 20×20 control volumes was used, and different angular discretization schemes and refinements were employed. The predicted incident heat flux (q) on the bottom wall and the incident radiation (G) at the vertical symmetry plane of the enclosure are shown in Figs. 2, 3 and 4 for an optical thickness of 0.1, 1 and 10, respectively. The root mean square (rms) of these quantities is given in Table 1.



(a) (b)

Figure 4. Normalized incident heat flux along the bottom wall (a) and incident radiation along the vertical symmetry plane (b) of the enclosure of test case 1 for an optical thickness of 10.

The results show that, in all cases, the predictions are in good agreement with the analytical solution. The angular refinement often does not improve the solution accuracy of the HYDRA, DOM, EP-DOM and EP-FEM-SAFE methods. Although this may seem surprising, it should be noticed that the solution accuracy is influenced by both the spatial and the angular discretization, and that the errors associated to these two procedures tend to compensate each other (Raithby, 1999, Coelho, 2002). Therefore, refining the angular discretization while keeping the same spatial discretization, or refining the spatial discretization while maintaining the angular discretization, does not necessarily improve the solution accuracy. The accuracy is only expected to increase if both the angular and the spatial discretization are refined.

In the case of an optical thickness equal to 0.1, the HYDRA solution using triangular elements provides the most accurate predictions of the incident heat flux, but the results obtained using bilinear elements are also satisfactory, except for the coarsest angular discretization, which yields a too flat q profile. There is no significant difference between the triangular and the bilinear elements as far as the incident radiation is concerned. In the case of an intermediate or optically thick medium, i.e., for an optical thickness of 1 or 10, respectively, the HYDRA predictions obtained using triangular or bilinear elements are similar. A marginally better accuracy was obtained for the incident heat flux in the case of triangular elements and an optical thickness of unity.

The DOM results are more influenced by the angular discretization than the HYDRA ones, except for the optically thick medium, where both the HYDRA and the DOM results are very similar. This is an expected result, since the HYDRA and the DOM differ only in the angular discretization. It turns out that in the case of an optically thick medium, the transmissivity of the medium is small, and the local radiation intensity tends to the blackbody radiation intensity, which is uniform in this test case, because the medium is at a uniform temperature. In the case of an optically intermediate or an optically thin medium, the transmissivity of the medium is higher, the influence of the distance travelled by a radiation medium becomes larger, and so does the angular discretization. This is more visible in the DOM than in the HYDRA, since the radiation from different directions is more coupled in the second case.

The results reported in Becker *et al*. [2010] using the EP-DOM and EP-FEM-SAFE are given in Table 1, and reveal that the HYDRA results are more accurate, except for the incident radiation and for an optically intermediate or thick medium.

The same two-dimensional square enclosure of side length L is considered in test case 2. The walls are black, except the top one, which is maintained at an emissive power of unity. The medium is grey, emits and absorbs, and is in radiative equilibrium. The optical thickness of the medium, based on the side length, is equal to 1. A quasi-analytical solution, referred to as exact, is reported in Crosbie and Schrenker (1984). A uniform grid with 20×20 control volumes was used, and different angular discretization schemes and refinements were employed. The predicted incident heat flux (q) on the bottom wall and the incident radiation (G) at the vertical symmetry plane of the enclosure are shown in Figs. 5 and 6, respectively. The root mean square (rms) of these quantities is given in Table 2.

Table 1. Root mean square of the incident heat flux on the bottom wall and incident radiation at the vertical symmetry plane of the square enclosure of test case 1.

Solution method	Angular discretization	$\tau = 0.1$		$\tau = 1$		$\tau = 10$	
		rms(q)	rms(G)	rms(q)	rms(G)	rms(q)	rms(G)
HYDRA – Triangular elements	T_2	0.00060	0.00556	0.00639	0.02841	0.01081	0.05110
	T_3	0.00084	0.00569	0.00986	0.02835	0.01127	0.05232
	T_4	0.00107	0.00572	0.01011	0.02796	0.01134	0.05251
	T_5	0.00114	0.00573	0.01011	0.02808	0.01133	0.05256
HYDRA – Bilinear elements	2×2	0.00353	0.00522	0.01735	0.02460	0.01147	0.05200
	3×3	0.00158	0.00412	0.01147	0.02833	0.01140	0.05250
	4×4	0.00137	0.00462	0.01082	0.02849	0.01137	0.05257
	5×5	0.00130	0.00496	0.01053	0.02837	0.01137	0.05258
DOM	S_4	0.00102	0.00385	0.00949	0.02960	0.00804	0.04485
	S_6	0.00044	0.00473	0.00578	0.02882	0.01091	0.05082
	S_8	0.00205	0.01912	0.01360	0.02306	0.01094	0.05206
EP-DOM (Koch et al., 1995)	S_4	0.01088	0.01644	0.05286	0.13579	0.10186	0.06440
	S_6	0.00965	0.02457	0.02238	0.03982	0.10607	0.03444
	S_8	0.00805	0.03724	0.02756	0.09387	0.10899	0.02012
EP-FEM-SAFE (Becker et al., 2010)	SAFE 18	0.00963	0.08116	0.03819	0.17569	0.09804	0.03412
	SAFE 38	0.00499	0.03071	0.01858	0.06889	0.11488	0.02728
	SAFE 66	0.00664	0.02834	0.01922	0.02429	0.12252	0.02453
	SAFE 102	0.00686	0.04319	0.02433	0.02030	0.12594	0.02449

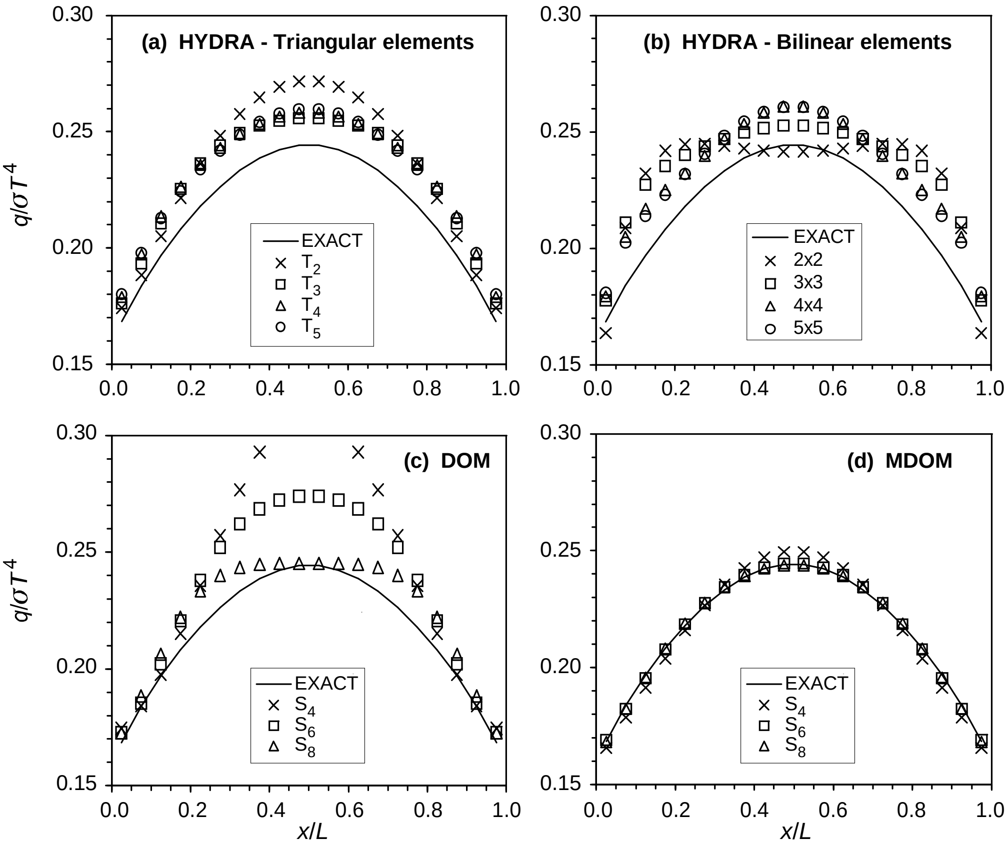


Figure 5. Normalized incident heat flux along the bottom wall of the enclosure of test case 2.

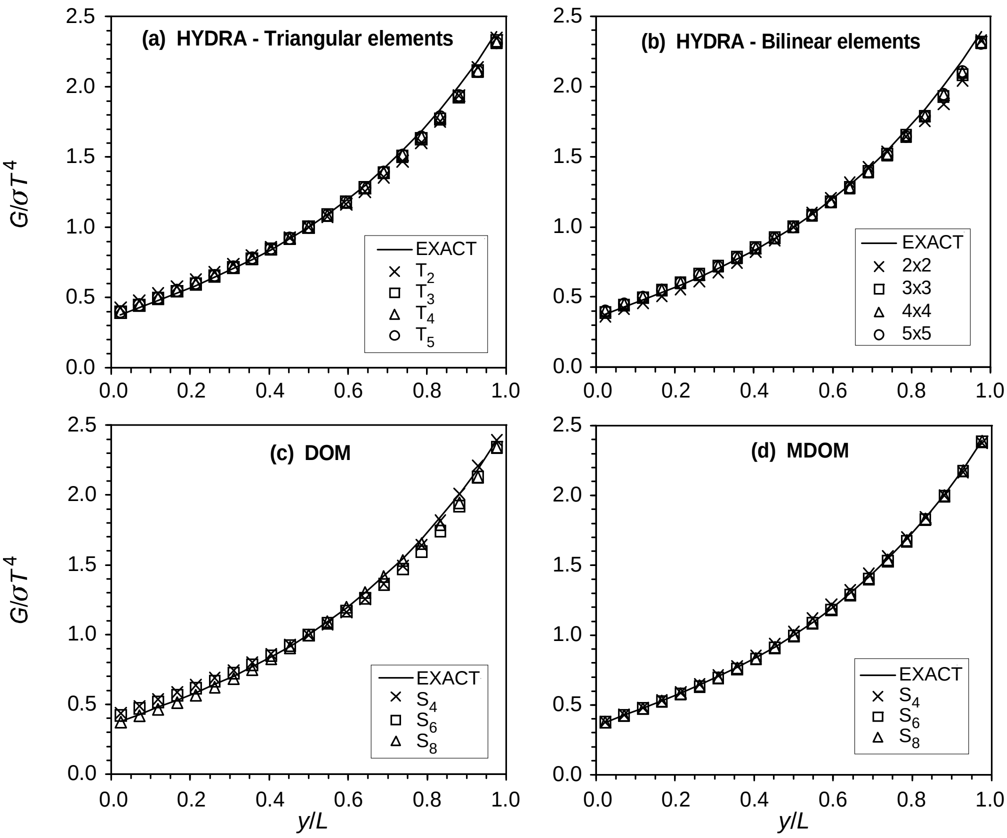


Figure 6. Normalized incident radiation along the vertical symmetry plane of the enclosure of test case 2.

The HYDRA predictions of the incident heat flux on the bottom wall of the enclosure exhibit a physically unrealistic solution for the bilinear elements and the coarser angular discretizations (T_2 and T_3), which is due to ray effects (Chai *et al*. 1993, Raithby, 1999, Coelho, 2002), while the spherical triangular elements show realistic solutions, but poor accuracy. The results are better for finer angular discretizations (T_4 and T_5), and the differences between the solutions obtained using the triangular and the bilinear elements become marginal. The corresponding DOM predictions are quite poor for S_4 and S_6, as a consequence of the ray effects. The S_8 solution is much better, but the profile is too flat in the middle of the bottom wall. The ray effects are significantly mitigated using the modified discrete ordinates method, MDOM (Ramankutty and Crosbie, 1999, Coelho, 2002). The predictions of the incident radiation, displayed in Fig. 6, are in good agreement with the exact one for all methods, but the MDOM outperforms again the other methods.

Table 2. Root mean square of the incident heat flux on the bottom wall and incident radiation at the vertical symmetry plane of the square enclosure of test case 2.

Solution method	Angular discretization	rms(q)	rms(G)
HYDRA – Triangular elements	T_2	0.01963	0.19479
	T_3	0.01423	0.14525
	T_4	0.01528	0.14070
	T_5	0.01516	0.14075
HYDRA – Bilinear elements	2×2	0.02051	0.20772
	3×3	0.01931	0.15299
	4×4	0.01599	0.14612
	5×5	0.01550	0.14274
DOM	S_4	0.03772	0.15978
	S_6	0.02183	0.19648
	S_8	0.00947	0.11275
MDOM	S_4	0.00397	0.05966
	S_6	0.00094	0.04254
	S_8	0.00091	0.02278
EP-DOM (Koch et al., 1995)	S_4	0.14479	0.25752
	S_6	0.06472	0.09945
	S_8	0.04343	0.11552
EP-FEM-SAFE (Becker et al., 2010)	SAFE 18	0.07181	0.18534
	SAFE 38	0.03732	0.12030
	SAFE 66	0.02665	0.11223
	SAFE 102	0.02234	0.10978

4. CONCLUSIONS

A hybrid finite volume / finite element method was used to solve the radiative transfer equation. The radiation intensity is approximated as a linear combination of basis functions, dependent only on the angular direction. In the present work, spherical triangular elements were used as basis functions for the angular discretization. The results were compared with those obtained using bilinear elements, which had already been employed in previous works, and with results determined using the standard and modified discrete ordinates method. Predictions reported in the literature based on the even parity formulation of the radiative transfer equation and using either the discrete ordinates method or a finite element method for angular discretization were also used for comparison purposes. Radiative transfer in two-dimensional enclosures containing a grey emitting-absorbing medium was considered to evaluate the accuracy of the methods, taking the analytical or quasi-analytical solution as reference. The influence of the optical thickness of the medium was also investigated.

The results show that the hybrid method using spherical triangular elements as basis functions for the angular discretization performs better than the same method using bilinear elements for optically thin or optically intermediate media. In the case of optically thick media, the two kinds of elements perform similarly. The discrete ordinates method is more sensitive to the angular refinement for the investigated test cases. When ray effects are present, the triangular elements perform better than the bilinear elements and the discrete ordinates method, in the case of relatively coarse angular discretization. However, the hybrid method is unable to achieve the high accuracy exhibited by the modified discrete ordinates method.

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