# STRUCTURAL DAMAGE IDENTIFICATION IN BEAMS BY FLEXIBILITY MATRIX.

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***Abstract.*** *The structural damage identification in beams from the vibration characteristics is addressed in this work. The method has been developed to identify and quantify damage scenarios in slender beams. The main idea is that modal parameters are a function of the physical properties of the structure. The damage identification is assessed based on the flexibility matrix that presents some advantages. For instance, the reduced flexibility matrix of the structure is related only to the measured degrees of freedom and may be accurately estimated from a few of the lower frequency modes in a modal test. The finite elements method (FEM) is used to model an Euler-Bernouilli beam and the damage field. The Luus-Jaakola and the Newton optimization methods are applied to minimize the difference between the flexibility matrix obtained from modal testing and the analytical flexibility matrix. The Luus-Jaakola optimization method presented good results, however with a high computational cost. The Newton optimization method is fast but, in several cases, diverges from the solution. The hybridization of the two optimization methods has been used and presented excellent results in the damage identification. The influence of the position of sensors was studied and numerical results are presented for several damage scenarios.*

***Keywords:*** *Damage identification in beams, Matrix flexibility, Luus-Jaakola method, Newton method.*

## 1. INTRODUCTION

Several phenomena can cause damage to a structure and thus compromise its proper functioning and often endanger people's lives, like impacts, actions of wind, people and vehicle's moving etc.

For the security of people and structures, many authors have used dynamic tests to determine characteristics of the structure vibration, such as frequencies, modes and modal damping, in order to evaluate possible structural failure (Alvandi & Cremona, 2006, Jauregui & Farrar, 1996, Chang et al., 2003, Carrilo & Laier, 2006). The basic thought for this methodology is that the modal parameters are functions of the physical properties of the structure (mass, stiffness and damping) and therefore changes in physical properties will be reflected in the modal properties, which in turn can be obtained in an experimental trial and used to deduce about the damage. The structural damage identification in its initial stage and the continuous monitoring of the structure contribute to the reduction of maintenance or repair costs, and increase its reliability and lifetime.

In this paper, the finite element method is used to model a Euler-Bernouilli beam and the damage field, where the damage in the structure is described through a parameter of cohesion (Stutz, 2005). The inverse problem of identifying the structural damage is defined as a minimization problem, where a set of parameters of cohesion is determined to minimize a functional based on the difference between the experimental matrix of flexibility and the one provided by the finite element model. The Luus-Jaakola and the Newton optimization methods are applied to minimize the difference between the flexibility matrix obtained from modal testing and the analytical flexibility matrix. The influence of the position of sensors was studied and numerical results are presented for several damage scenarios.

## 2. MATHEMATICAL MODEL

From an appropriate spatial discretization, using the finite element method, the equation of motion of the structure is given by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}, \qquad (1)$$

where **M**, **K** and **C** are, respectively, the matrices of mass, stiffness and damping, with dimension of $n \times n$, **u** is a $n \times 1$ vector of generalized displacements and **f** is a $n \times 1$ vector of generalized forces.

The undamped natural frequencies and mode shapes of the structure can be obtained from the eigenvalue-eigenvector problem

$$(\mathbf{K} - \boldsymbol{\omega}_i^2\mathbf{M})\boldsymbol{\phi}_i = 0, \qquad (2)$$

where $\boldsymbol{\omega}_i$ e $\boldsymbol{\phi}_i$ refers, respectively, to the *i-th* natural frequency and to the *i-th* mode shape of the structure.

Considering the mode shapes of the structure normalized with respect to the mass matrix, one has

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathbf{I}; \tag{3}$$

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \mathbf{\Lambda}, \tag{4}$$

where $\mathbf{\Phi}$ is the modal matrix and T means the transpose, $\mathbf{I}$ is the identity matrix and $\mathbf{\Lambda}$ is a diagonal matrix consisted of the squared values of natural frequencies. The stiffness matrix can be written as

$$\mathbf{K} = (\mathbf{\Phi}\mathbf{\Lambda}^{-1}\mathbf{\Phi}^{\mathrm{T}})^{-1}. \tag{5}$$

The flexibility matrix $\mathbf{G}$ of a structure is defined as the inverse of its stiffness matrix and, therefore, it can be written in terms of the modal parameters as

$$\mathbf{G} = (\mathbf{\Phi}\mathbf{\Lambda}^{-1}\mathbf{\Phi}^{\mathrm{T}}). \tag{6}$$

Considering Eq. (6), the flexibility matrix can be rewritten as

$$\mathbf{G} = \sum_{i=1}^{n} \frac{1}{{\boldsymbol{\omega}_i}^2}\boldsymbol{\phi}_i \otimes \boldsymbol{\phi}_i, \tag{7}$$

where $\otimes$ stands for the tensor product.

The Eq. (7) shows that the modal contribution to the flexibility matrix decreases with the increasing of the frequency, which characterizes a great advantage to the use of the flexibility matrix for damage identification purposes, since modes of higher frequencies are more difficult to obtain in a modal test. Due to pratical limitations, the following flexibility matrix can be obtained from an experimental test.

$$\bar{\mathbf{G}}_{\mathrm{exp}} = \sum_{i=1}^{n_{\mathrm{exp}}} \frac{1}{\boldsymbol{\omega}^2_{\mathrm{i,exp}}}\boldsymbol{\phi}_{\mathrm{i,exp}} \otimes \boldsymbol{\phi}_{\mathrm{i,exp}}, \tag{8}$$

where $n_{\mathrm{exp}}$ is the number of modes obtained in the modal test and $\boldsymbol{\phi}_{i,\mathrm{exp}}$ and $\omega_{i,\mathrm{exp}}$ are, respectively, the *i-th* mode shape and *i-th* natural frequencies determined only in a subset of $m$ degrees of freedom of the structure. From Eq.(8), it should be noted that the size of the matrix $\bar{\mathbf{G}}_{\mathrm{exp}}$ depends only on the number $m$ of degrees of freedom (DOF) which were measured in the experiment.

To obtain an inverse relation between the analytical flexibility matrix $\bar{\mathbf{G}}$, related only to the $m$ DOF measured in the vibration test, and a matrix of the same size, containing information about the stiffness properties of the structure, the original stiffness matrix must be partitioned as

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{\mathrm{mm}} & \mathbf{K}_{\mathrm{mo}} \\ \mathbf{K}^T_{\mathrm{mo}} & \mathbf{K}_{\mathrm{oo}} \end{bmatrix}, \tag{9}$$

where $T$ means transposed, and the subscripts $m$ and $o$ refer, respectively, to the DOF measured and omitted. It can be seen (Alvin et al., 1995) that a reduced flexibility matrix $\bar{\mathbf{G}}$ may be obtained as the inverse of the reduced stiffness matrix, obtained by Static Guyan Reduction (1965),

$$\bar{\mathbf{G}} = \bar{\mathbf{K}}^{-1} = [\mathbf{K}_{\mathrm{mm}} - \mathbf{K}_{\mathrm{mo}}\mathbf{K}^{-1}_{\mathrm{oo}}\mathbf{K}^T_{\mathrm{mo}}]. \tag{10}$$

In this paper, the measure of cohesion parameter $\beta$ is used to describe the behavior of the damage in the structure. This parameter is related to the connection between the material points and can be interpreted as a measure of the state of local material cohesion, where $0 \leq \beta \leq 1$. If $\beta = 1$, it is assumed that all connections between the material points have been preserved, there is no damage in the structure. If $\beta = 0$, a local rupture is considered, since all connections between the material points were removed.

In the present model, it is considered that the damage affects only the elastic properties of the structure, so that the stiffness matrix obtained by the finite element method is given by

$$\mathbf{K}(\beta_h) = \int_0^l \beta_h(x)E_0I_0H^T(x)H(x)dx, \tag{11}$$

where $l$ corresponds to the beam length, $x$ is the space domain of the structure, $\beta_h$ is the approximation of the field $\beta$ obtained from its spatial discretization, $H$ is the differential discrete operator, and $E_0$ and $I_0$ are, respectively, the nominal elastic modulus and the area moment of inertia. In this paper, the discretization of the field $\beta$ used the same mesh of the displacement field. It should be noted that the discretization of the field $\beta(x)$ does not depend on the discretization of the displacement field, so that different meshes can be adopted.

Considering Eq. (11), the damage can be interpreted as a variation of the elastic modulus $E(x)$, of the area moment of inertia $I(x)$ or of both, $E(x)I(x)$. It is usually considered that the elastic modulus is uniform along the beam, $E(x) = E_0$, and the damage only alters its geometric properties. Therefore, the moment of inertia of area of the damaged structure is given as

$$I(x) = \beta(x) I_0 \tag{12}$$

and, therefore, the cohesion field may be written as

$$\beta(x) = \left(\frac{h(x)}{h_0}\right)^3, \tag{13}$$

where $h_0$ and $h(x)$ represents, respectively, the thickness of undamaged and damaged beam. The Fig. 1 shows this relation.

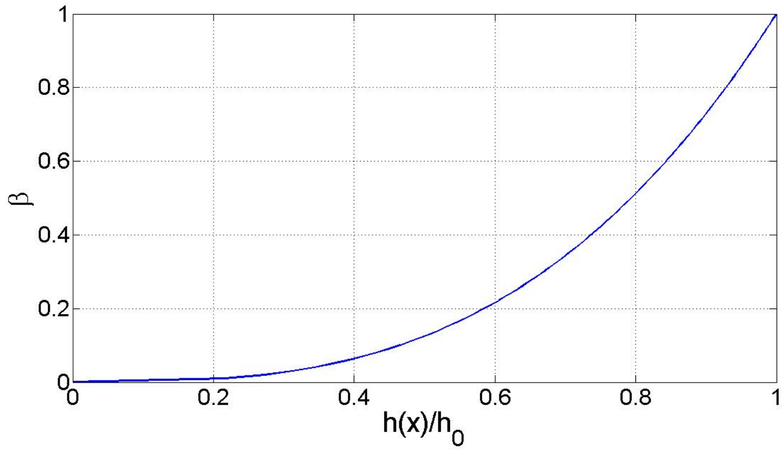


Figure 1. Cohesion parameter as a function of the relative thickness of the beam.

In order to present a more direct interpretation of the damage described by the cohesion parameter $\beta$, the damage parameter $D$ is defined, as

$$D = 1 - \beta. \tag{14}$$

It was considered that the field $\beta(x)$ for each finite element of the beam is interpolated by two nodal values. The stiffness matrix of a bidimensional Euler-Bernoulli is given by

$$\mathbf{K^e} = \left(\frac{2L}{E_0}\right) \begin{pmatrix} \frac{12I_0}{L^2}(\beta_i+\beta_j) & \frac{4I_0}{L}(2\beta_i+\beta_j) & -\frac{12I_0}{L^2}(\beta_i+\beta_j) & \frac{4I_0}{L}(\beta_i+2\beta_j) \\ \frac{4I_0}{L}(2\beta_i+\beta_j) & 2I_0(3\beta_i+\beta_j) & -\frac{4I_0}{L}(2\beta_i+\beta_j) & 2I_0(\beta_i+\beta_j) \\ -\frac{12I_0}{L^2}(\beta_i+\beta_j) & -\frac{4I_0}{L}(2\beta_i+\beta_j) & \frac{12I_0}{L^2}(\beta_i+\beta_j) & -\frac{4I_0}{L}(\beta_i+2\beta_j) \\ \frac{4I_0}{L}(\beta_i+2\beta_j) & 2I_0(\beta_i+\beta_j) & \frac{4I_0}{L}(\beta_i+2\beta_j) & 2I_0(\beta_i+\beta_j) \end{pmatrix}, \tag{15}$$

where $L$ indicate the length of the element, and $\beta_i$ and $\beta_j$ represent the cohesion parameter in each end of the element.

The problem of damage identification can be defined as an optimization problem where the following functional must be minimized

$$\min_{\beta} \ \mathcal{F}, \ \text{satisfying} \ 0 \le \beta \le 1 \tag{16}$$

with,

$$\mathcal{F}(\beta) = \frac{\|\bar{\mathbf{G}}_{\text{exp}} - \bar{\mathbf{G}}\|^2}{2\|\bar{\mathbf{G}}_{\text{exp}} - \bar{\mathbf{G}}_0\|}, \tag{17}$$

where $\bar{\mathbf{G}}_\mathbf{0}$ is the reduced flexibility matrix computed with the inicial estimate of the field $\beta$.

## 3. OPTIMIZATION METHOD USED

In the damage identification problem proposed, the aim is the minimization over the cohesion parameter $\beta$ of an error based on the difference between the experimental flexibility matrix and the original flexibility matrix of the FEM. For this, it will be used the optimization method described in this Section.

### 3.1 Luus-Jaakola Method

The basic idea of the stochastic algorithm of Luus & Jaakola (1973), is to select random solutions in a region that decreases in size over the course of iterations, as the pseudocode below.
Choose the size of initial search $r^0$;
Choose the number of internal $n_{\text{int}}$ and external loops, $n_{\text{ext}}$;
Choose the contraction coefficient $c$;
Choose the initial solution $\beta^* = \beta_0$

For $i = 1 : n_{\text{ext}}$
  For $j = 1 : n_{\text{int}}$
    $\beta^j = \beta^* + R^j r^{(j-1)}$, where $R$ is a diagonal matrix consisted by random numbers
    between -0,5 and 0,5
      If $\mathcal{F}(\beta^j) < \mathcal{F}(\beta^*)$
        $\beta^* = \beta^j$
      end (If)
  end (For)
  $r^i = (1-c)r^{(i-1)}$
end (For)

### 3.2 Newton's Method

Newton's method is a deterministic procedure that will be used to minimize the functional described in Eq. (17) in relation to the parameter $\beta$. Thus, the solution is provided by the cohesion parameters $\beta$ that satisfy the following equation,

$$\mathbf{F} = \frac{\partial \mathcal{F}}{\partial \beta} = 0, \;\; \text{satisfying} \;\; 0 \leq \beta \leq 1 \tag{18}$$

By Eq. (17), we have the derivative of the scalar function $F$ in relation to the *i-th* cohesion parameter $\beta_i$, given by

$$F_i = -\frac{1}{N}(\bar{\mathbf{G}}_{\text{exp}} - \bar{\mathbf{G}})\frac{\partial \bar{\mathbf{G}}}{\partial \beta_i}. \tag{19}$$

The derivative of the flexibility matrix regarding $\beta_i$ parameter can be calculated as in,

$$\frac{\partial \bar{\mathbf{G}}}{\partial \beta_i} = -\bar{\mathbf{G}}\frac{\partial \bar{\mathbf{K}}}{\partial \beta_i}\bar{\mathbf{G}}. \tag{20}$$

The increment $\triangle\boldsymbol{\beta}^{\text{p}}$ to the vector of parameters, from the *p-th* iteration is given by the equation

$$\triangle\boldsymbol{\beta}^{\text{p}} = (\mathbf{S}^{(\text{p}-1)})^{+}\mathbf{F}^{(\text{p}-1)}, \tag{21}$$

where (+) represents the pseudo-inverse of Moore-Penrose (Golub and Van Loan, 1983), since the S matrix shown below, can be singular and not owning a classic inverse. The $S_{\text{ij}}$ element of the **S** matrix is obtained through the derivative of the *i-th* element of vector **F**, Eq. (18), in relation to the *i-th* parameter of cohesion $\beta_j$,

$$S_{\text{ij}} = \frac{\partial F_i}{\partial \beta_j} = \frac{\partial \bar{\mathbf{G}}}{\partial \beta_j}\frac{\partial \bar{\mathbf{G}}}{\partial \beta_i} - (\bar{\mathbf{G}}_{\text{exp}} - \bar{\mathbf{G}})\frac{\partial^2 \bar{\mathbf{G}}}{\partial \beta_i \partial \beta_j}. \tag{22}$$

## 4. Numerical Results

In this paper, it was considered an aluminum clamped beam with $0.9$ m in length, $0.005$ m thick, $0.05$ m wide, nominal elasticity modulus $E_0 = 7.2582 \times 10^{10}$ Pa and moment of inertia of nominal area $I_0 = 5.2083 \times 10^{-10}$ kgm$^2$. All damages have a triangular profile, as illustrated in Fig. 2.

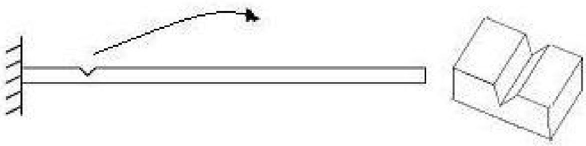

Figure 2. Clamped beam with a triangular damage profile.

The beam was discretized by the finite element method in 20 elements, where each element has two nodes and each nodal point has two degrees of freedom: one transverse (v) and one of rotation $(\theta)$, see Fig. 3. The structure has a total of 21 nodes and 40 DOF, because one node of the structure is clamped.

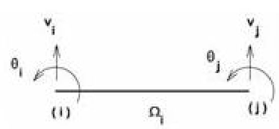


Figure 3. Degrees of freedom for each element

The imposition of the damage to the beam is accomplished through the cohesion parameter through the definition of the vector $\beta$. In the damaged nodes $\beta \neq 1$ (or $D \neq 0$), and the nodes where there is no damage $\beta = 1$ (or D = 0). The FEM with the values of the cohesion parameter required to describe a damage scenario are used to generate the frequencies and vibration modes of the damaged structure, thus representing the "experimental" data.

The Luus-Jaakola method, as a method of random search, may find an approximated solution to the problem proposed but with the disadvantage of being very time consuming. The deterministic Newton's method, shows speed to obtain the results, but in many cases it stucks in points of local minimum and it may even diverge for an unsuitable initial guess. Therefore, in order to outperform these methods, an hybridization of both is considered here. In the hybrid method, the Luus-Jaakola method is used with a few number of iterations to generate an initial estimate of values of the parameters and hence the location of the damage. The Newton method was then used starting from the estimate of the previous method and updating only the parameters within the indicated region.

As in practice, the goal is to use less sensors and obtain accurate information of the structure by measuring a small number of vibration modes. It is important that the few sensors used are positioned in strategic points, making the attainment of the modal characteristics of the structure easier. The curvature of the vibration modes were analyzed to avoid the sensors to stay positioned on the nodes. The Fig. 4 show the shape of the first four vibration modes of a clamped beam and the position of sensors.

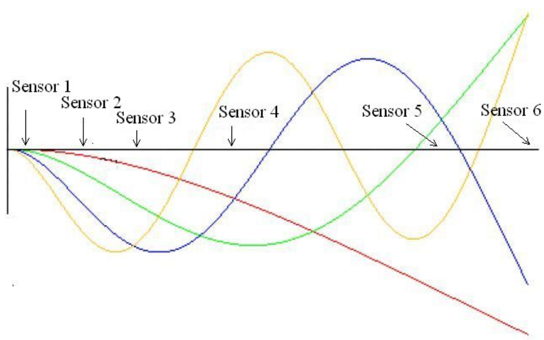


Figure 4. Position of sensors and first four vibration modes of a clamped beam, red, green, blue and yellow respectively.

According to Meirovitch, 2000, the first four vibration modes has a node located as table below.

Table 1. Localization of nodes

| Vibration Mode | Node Position (m) |
|---|---|
| First mode | $-$ |
| Second mode | $0.7047$ |
| Third mode | $0.4536$ and $0.7812$ |
| Fourth mode | $0.3222, 0.5796$ and $0.8154$ |

Three different scenarios of damage were considered: only one damaged region, two distant regions damaged and two adjacent regions damaged. Table 2 shows the cases considered.

Table 2. Damage Scenarios

| CASE | POSITION | DAMAGE (D) |
|---|---|---|
| 1 | $9$ | $0.2$ |
| 2 | $3; 16$ | $0.2; 0.2$ |
| 3 | $9; 11$ | $0.4; 0.1$ |

In the simulations presented in this work it were considered as measured only the first six vibration modes and six DOF. The measured DOF's (and the distance of the clamp to this) were: DOF 3 (0.045m) DOF 7 (0.135m), DOF 11 (0.225m), DOF 19 (0.405m), DOF 33 (0.72m) and DOF 41 (0.9m). The table below shows the natural frequencies of the first six vibration modes, where can it be seen that the range of frequency of interest goes up to 450Hz.

Table 3. Natural Frequencies

| Vibration mode | Natural Frequencies (Hz) |
|---|---|
| First mode | $5.1605$ |
| Second mode | $32.3405$ |
| Third mode | $90.5557$ |
| Fourth mode | $177.4611$ |
| Fifth mode | $293.3873$ |
| Sixth mode | $438.3598$ |

For the first scenario of simulated damage, it was adopted as external and internal loops for the Luus-Jaakola method the number of, 10 and 10 000 iterations, taking about 10 minutes to complete the implementation, see Fig. 5a. It can be noticed that the method was not able to provide the precise location and the severity of the damage. Running it again but with 15 external loop and 25000 internal loop, it can be seen an improvement in the results, but still not satisfactory as can be seen in Fig. 5b. In this case, the execution time reached one hour. Newton method, in this case, get the correct values for the location and severity of the damage, taking around 3 minutes to generate the result shown in Fig. 5c. Another method used was the hybridization between the two previous ones (see Fig. 5d). It was used the simulation results presented in Fig. 5a as initial guess for the Newton method and it was updated only the parameters of cohesion in the region indicated with $\beta \neq 1$ (or $D \neq 0$) by the Luus-Jaakola method. The hybridization took 13 minutes.

The results were computed in a laptop with the following configuration: Intel dual core,1.73GHz, 2 GB of RAM and 160 GB HD.

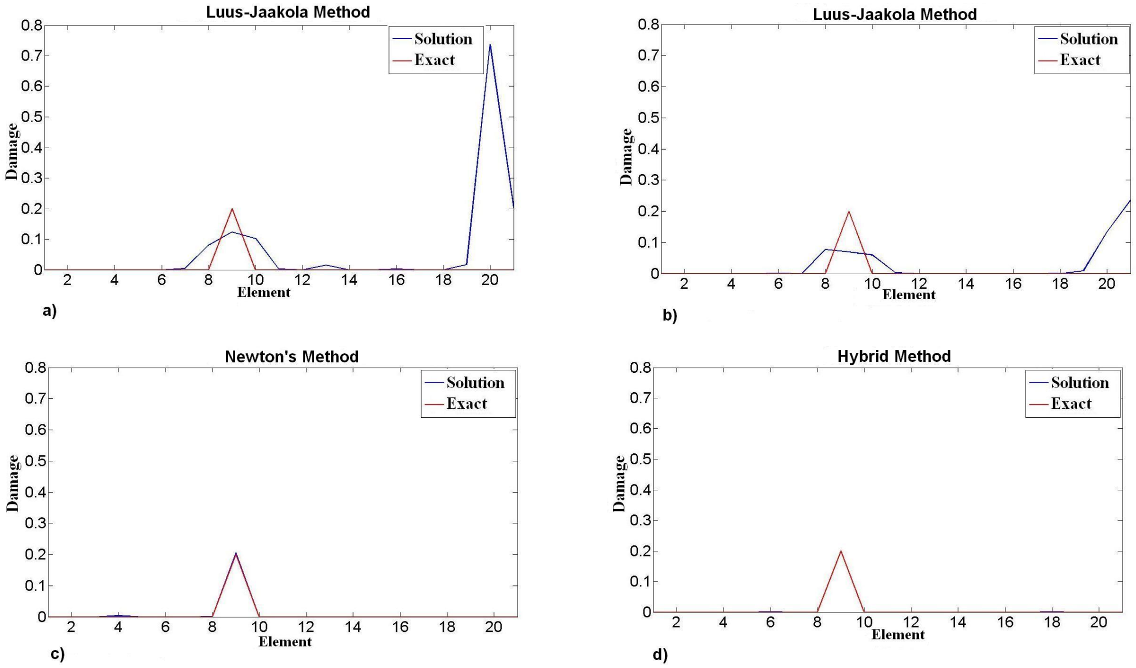


Figure 5. Damage identification for case 1.

The Luus-Jaakola method may take too much time to achieve to the exact result for the problem of identifying damage because it is a method of random search. For this reason, in the next simulations it was used for a period of 10 minutes, where the external loop was set at 10 iterations and in the internal loop 10000 iterations, enough to generate a good initial estimate for the hybrid method.

The simulation below shows the damage scenario where two distinct regions are damaged. The Newton method did it in five minutes, providing good results for the proposed problem, see Fig. 6b. In the hybrid method it was used as initial guess the results achieved by the Luus-Jaakola method and shown in Fig. 6a. It can be checked that hybridization between the two methods could accurately identify the location and the severity of the damage, see Fig. 6c. The whole process took 12 minutes.

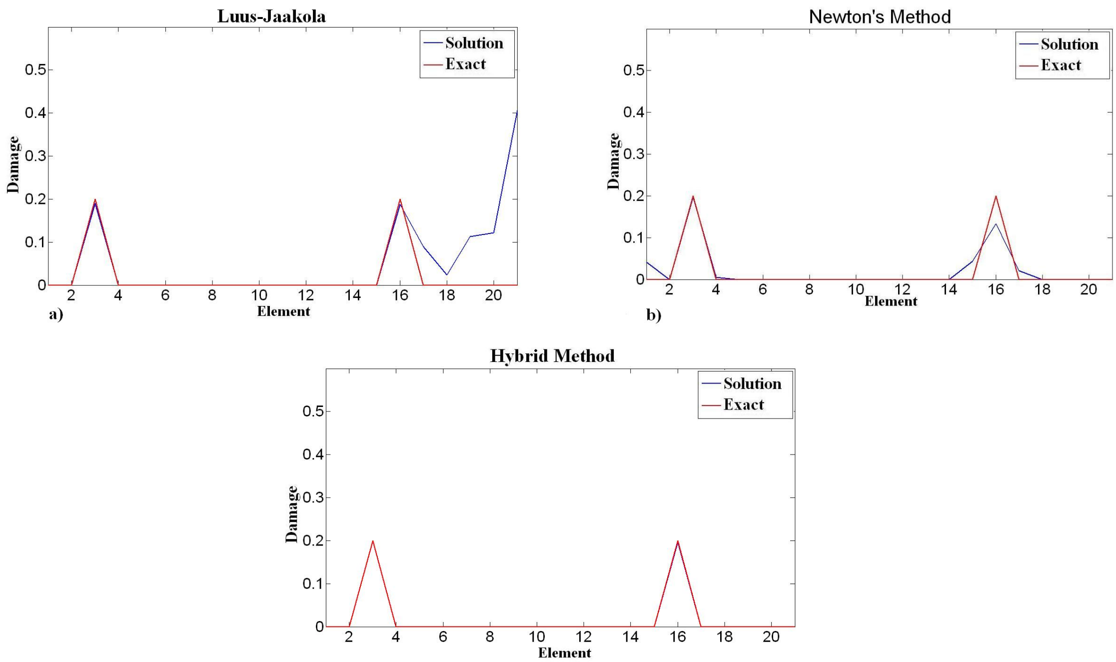


Figure 6. Damage identification for case 2.

In the simulation of the third scenario of damage, it can be noticed the superiority of the hybrid method, which could provide information about the location and the severity of the damage accurately. The Luus-Jaakola method and the Newton method were not able to accurately identify the damage, the first one because of the high computational cost and the second one because still incorrect damage identification remains.

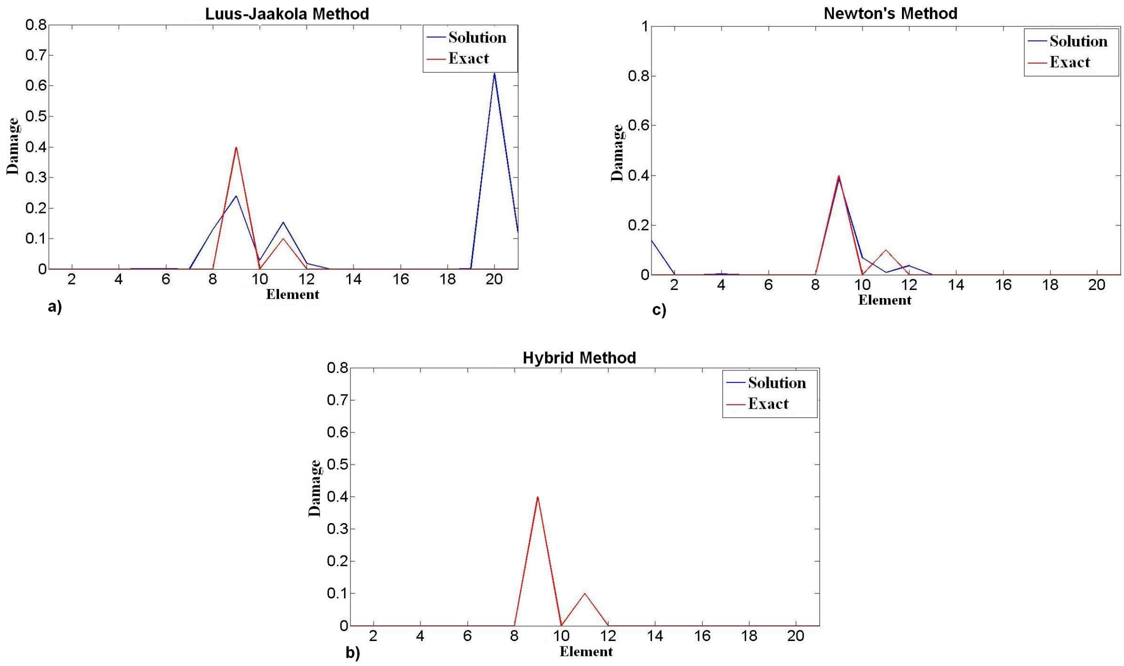


Figure 7. Damage identification for case 3.

In all cases presented, the hybrid method showed satisfactory results in the localization and quantification of the damage, which did not occur with the Luus-Jaakola and Newton method. It should be noted that few modes and DOFs were considered as measured, since, in practice, the number of available sensors is greatly reduced, thereby limiting the amount of information obtained from the experiment.

## 5. CONCLUSIONS

In this paper, it was presented the efficiency of the hybrid method for damage identification by flexibility matrix. Even considering only a very small number of DOF and vibration modes, the method proved to be very reliable for the detection of damage, but the location where the sensors were placed in the beam proved to be of great importance for good quality results. The Luus-Jaakola method when used alone and only or a few DOF's and a few measured modes, required a very high running time to generate good results.

Newton method, in these same conditions, when used alone, did not converge to the expected result in some damage scenarios. The hybridization between the two methods enabled to obtain great results, both in locating and identifying the severity of the damage, even when using a very small number of DOF and vibration modes. In hybridization, the Luus-Jaakola method updated every parameter and the sensors were placed along the beam to avoid the proximity of the nodes and the vibration modes. The Newton method was used to update only the parameters of the region indicated by the previous method, where the cohesion parameter $\beta$ shown value different of zero.

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