AN ANALYTICAL DYNAMIC MODEL FOR DAMAGED BEAMS INCLUDING BENDING, BUCKLING, ROTATIONAL INERTIA AND SHEAR DEFORMATION.

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Abstract
In this paper an analytical dynamic model is developed for damaged beams including bending, buckling, rotational inertia and shear deformation. The damage is modeled using rotational springs that simulate the crack based on fracture mechanics. The developed model is used to predict variations on natural frequencies for several crack positions along the beam and damage magnitude. The importance of this work lies on the developed analytical model that has no approximations and discretizations of the displacement field. This initial theoretical approach describes the expected behaviors for fluctuations of the natural frequencies for simple supported and cantilever beams with the precision that only analytical methods allow. The results provide a useful benchmark for comparisons with approximate numerical methods that can be used to model and analyze the problem. The inclusion of rotational inertia and shear deformation effects rendered improvements in the dynamic behavior mainly in the case of slender and short beams when compared with a simplified Euler-Bernoulli model.

Keywords*: damaged beams, natural frequencies, analytical dynamic model for beams.*

1. INTRODUCTION

The development of an analytical model for dynamic modeling of beams has a great importance since no approximations due to discretizations are included in the formulation. This result in unbiased values that can be used as benchmark for comparisons with new numerical models. Research on the experimental investigations of the effects of cracks and damages on the integrity of structures was performed by Owolabi *et al.* (2003), with a view to detect, quantify, and determine their extents and locations. The damage detection schemes used in this study depended on the measured changes in the first three natural frequencies and the corresponding amplitudes of the measured acceleration frequency response functions.

A method is proposed by Binici (2005), to obtain the eigenfrequencies and mode shapes of beams containing multiple cracks and subjected to axial force. Cracks were assumed to introduce local flexibility changes and they are modeled as rotational springs. The method uses one set of end conditions as initial parameters for determining the mode shape functions. The proposed method could efficiently be used in detecting crack locations, severities and axial forces in beam columns. Binici (2005) stated that the method can be used to predict the critical load of damaged structures based on eigenfrequency measurements.

Also Naniwadekar *et al.* (2007) use a technique based on measurement of change in natural frequencies and modeling of crack by rotational spring is employed to detect a crack with straight front in different orientations in a section of straight horizontal steel hollow pipe. In their paper the rotational spring stiffness with crack size and orientation has been obtained experimentally by deflection and vibration methods. They reported that the proposed method was found to be very robust, since the maximum variation in location was 2.68%, which is much less than the change in frequency difference introduced.

In this paper it is analyzed the changes of the first two frequencies of simply supported and cantilever beams. It is proposed two models for the vibration equations: (a) one model including bending and buckling effects (Simplified Model) and another (b) including bending, buckling, rotational inertia, shear deformation effects and combinations (Complete Model). These models are compared using the simple beam examples for long and short spans including one damaged site. The idea is to analyze the models regarding their accuracy in evaluating frequency changes in these situations.

2. VIBRATION OF CRACKED BEAMS INCLUDING BENDING, BUCKLING, SHEAR DEFORMATION AND ROTATIONAL INERTIA.

The general equation for beams including bending, shear deformation and rotational inertia can be improved, based on the equation proposed by Clough and Penzien (1975) including the buckling effect. This formulation is written as indicated by Equation(1).

$$\underbrace{EI\frac{\partial^4 v}{\partial x^4}+\bar{m}\frac{\partial^2 v}{\partial t^2}}_{bending}+\underbrace{N\frac{\partial^2 v}{\partial t^2}}_{buckling}-\underbrace{\bar{m}r^2\frac{\partial^4 v}{\partial x^2\partial t^2}}_{rotational\ inertia}$$
$$\underbrace{\frac{EI}{k'AG}\frac{\partial^2}{\partial x^2}\left(-\bar{m}\frac{\partial^2 v}{\partial t^2}\right)}_{shear\ deformation\ effect}+\underbrace{\frac{\bar{m}r^2}{k'AG}\frac{\partial^2}{\partial t^2}\left(\bar{m}\frac{\partial^2 v}{\partial t^2}\right)}_{rotational\ inertia\ and\ shear\ deformation\ coupling}=0 \tag{1}$$

where E is the Young modulus, G is the Shear modulus, I is the moment of inertia, A is the sectional area, N is the axial force acting in the beam axis, $\bar{m}$ is the mass per unit length, r is the radius of gyration, $k'A$ is the effective shear area of the section, x is the coordinate space along beam axis, v is the displacement in y direction and t is the time variable.

In order to solve Eq. (1), assuming that time and position variables can be separated, one can write the following general solution in time and space:

$$v(x,t)=Y(x)e^{i\omega t} \tag{2}$$

$Y(x)$ is a modal shape, ω is the circular frequency, i *is* the imaginary number and x is a space position on the x-axis. If Eq.(2) is substituted into Eq.(1), and assuming a total length of L, this yields a well-known general solution in the following form:

$$Y(x)=C_1\sinh(\alpha x/L)+C_2\cosh(\alpha x/L)+C_3\sin(\beta x/L)+C_4\cos(\beta x/L) \tag{3}$$

where the constants α and β are defined depending on the retained terms in the Eq. (1). If bending and buckling effect are retained, the parameters α and β are defined as indicated by equations (4) and (5) as proposed by Binici (2005). This model will be referred in this paper as the simplified model.

$$\alpha=\sqrt{-(\frac{NL^2}{2EI})+\sqrt{(\frac{NL^2}{2EI})^2+(\frac{\rho A}{EI})(\omega L^2)^2}} \tag{4}$$

$$\beta=\sqrt{(\frac{NL^2}{2EI})+\sqrt{(\frac{NL^2}{2EI})^2+(\frac{\rho A}{EI})(\omega L^2)^2}} \tag{5}$$

If all the terms are retained, the parameters α and β are defined as indicated by equations (6) and (7).

$$\alpha=\sqrt{-\frac{1}{2}\left(\frac{\bar{m}r^2L^2\omega^2}{EI}+\frac{\bar{m}L^2\omega^2}{k'AG}+\frac{NL^2}{EI}\right)+\frac{1}{2}\sqrt{\left(\frac{\bar{m}r^2L^2\omega^2}{EI}+\frac{\bar{m}L^2\omega^2}{k'AG}+\frac{NL^2}{EI}\right)^2-4\left(\frac{\bar{m}r^2L^2\omega^2}{EIk'AG}-\frac{\bar{m}L^2\omega^2}{EI}\right)}} \tag{6}$$

$$\beta=\sqrt{\frac{1}{2}\left(\frac{\bar{m}r^2L^2\omega^2}{EI}+\frac{\bar{m}L^2\omega^2}{k'AG}+\frac{NL^2}{EI}\right)+\frac{1}{2}\sqrt{\left(\frac{\bar{m}r^2L^2\omega^2}{EI}+\frac{\bar{m}L^2\omega^2}{k'AG}+\frac{NL^2}{EI}\right)^2-4\left(\frac{\bar{m}r^2L^2\omega^2}{EIk'AG}-\frac{\bar{m}L^2\omega^2}{EI}\right)}} \tag{7}$$

The constants C_1, C_2, C_3 and C_4 from Eq. (3) depend on the beam's boundary conditions. In order to find them, the appropriate boundary condition should be applied. As stated by Binici (2005), if one defines $\chi=x/L$, where χ represents a dimensionless crack position, then the general solution simplifies to:

$$Y(\chi)=Y(0)A(\chi)+Y'(0)B(\chi)+Y''(0)C(\chi)+Y'''(0)D(\chi) \tag{8}$$

where $Y(0)$, $Y'(0)$, $Y''(0)$ and $Y'''(0)$ mean the modal vertical displacement, first, second and third derivative at x=0 respectively. These parameters are related to forces, moments, and displacements by the following expressions:

$$Y'(\chi) = \theta(\chi)L \qquad Y''(\chi) = M(\chi)L^2 / EI \qquad Y'''(\chi) = V(\chi)L^3 / EI \tag{9}$$

The other constants $A(\chi), B(\chi), C(\chi)$ and $D(\chi)$ show up when the boundaries conditions are applied. These functions are selected to be linearly independent in such way that the following equation is satisfied:

$$\begin{bmatrix} A(0) & A'(0) & A''(0) & A'''(0) \\ B(0) & B'(0) & B''(0) & B'''(0) \\ C(0) & C'(0) & C''(0) & C'''(0) \\ D(0) & D'(0) & D''(0) & D'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

At the vicinity of a crack, some compatibilities conditions should be enforced to assure the continuity of the force and displacement field across the crack. Figure 1 shows a sketch for the required compatibilities.

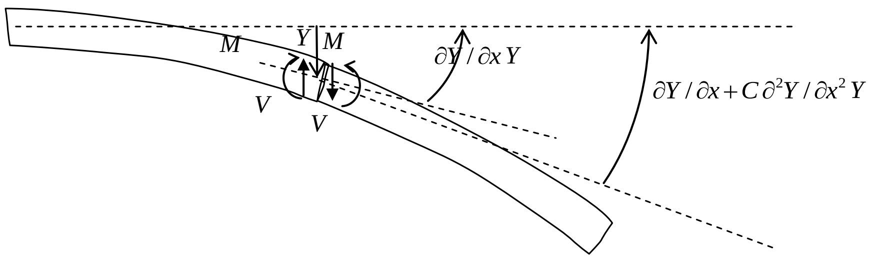

Figure 1. Differential "∂x" beam segment bending about *z* axis with axial load *P*, shear force *V*, bending moment *M* at crack site.

So, the following equations can be stated at the vicinity of the crack:

$$\begin{aligned} & Y_{bc}(\chi) = Y_{ac}(\chi) \\ & M_{bc}(\chi) = Y''_{bc}(\chi) EI / L^2 = M_{ac}(\chi) = Y''_{ac}(\chi) EI / L^2 \qquad Y''_{bc}(\chi) = Y''_{ac}(\chi) \\ & V_{bc}(\chi) = Y'''_{bc}(\chi) EI / L^3 = V_{ac}(\chi) = Y'''_{ac}(\chi) EI / L^3 \qquad Y'''_{bc}(\chi) = Y'''_{ac}(\chi) \\ & Y'_{bc}(\chi) = Y'_{ac}(\chi) + cY''_{ac}(\chi) \end{aligned} \tag{11}$$

where *c* represents the flexibility constant related to a rotational spring at the crack position. Using the fracture mechanics relations between strain energy release rate, the stress intensity factor and the Castigliano's Theorem, Liebowitz (1968) computed the cracked-region's local flexibility for plane strain as indicated bellow:

$$\begin{aligned} c = & 5.346(h/L)[1.8624(a/h)^2 - 3.95(a/h)^3 + 16.37(a/h)^4 - 37.226(a/h)^5 + \\ & + 76.81(a/h)^6 - 126.9(a/h)^7 + 172(a/h)^8 - 143.97(a/h)^9 + 66.56(a/h)^{10}] \end{aligned} \tag{12}$$

where *a* is the crack depth, *h* is the beam height and *L* is the beam length

It was found that this equation was misspelled for the last term in Binici's paper (2005), inducing to erroneous results along the text. This same equation appears in the Chondros (2001) paper typed correctly.

In the following, two cases of simple beams are analyzed using the complete formulation and the simplified one. The first is related to a simple supported beam and the second one is related to a cantilever beam.

3. Numerical Studies

3.1 Simply supported beam

A simple supported beam with only one crack, presents zero displacement and moments at the supports. Equation (3) may be simplified to Equation(13) and (14) for a beam segment before and after the crack, just applying the boundaries condition at $\chi = 0$ and $\chi = 1$ ($x = 0$ *and* $x = L$).

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$$Y_1(\chi) = Y_1'(0)B(\chi) + Y_1'''(0)D(\chi) \tag{13}$$

$$Y_2(\chi) = Y_1(\chi) + c_1 Y''(\chi_1) G(\chi - \chi_1) \tag{14}$$

where $B(\chi)$, $D(\chi)$ and $G(\chi)$ functions are defined as:

$$B(\chi) = \eta\left[\beta^2 \sinh(\alpha\chi)/\alpha + \alpha^2 \sin(\beta\chi)/\beta\right] \tag{15}$$

$$D(\chi) = \eta\left[\sinh(\alpha\chi)/\alpha - \sin(\beta\chi)/\beta\right] \tag{16}$$

$$G(\chi) = \eta\left[\alpha \sinh(\alpha\chi) + \beta \sin(\beta\chi)\right] \tag{17}$$

where $\eta = 1/\left(\alpha^2 + \beta^2\right)$

Eq.(13) and Eq.(14) can be represented in matrix formulation as indicated by Eq. (18).

$$\begin{bmatrix} B(1) + cG(1-\chi)B''(\chi) & D(1) + cG(1-\chi)D''(\chi) \\ B''(1) + cG''(1-\chi)B''(\chi) & D''(1) + cG''(1-\chi)D''(\chi) \end{bmatrix} = \begin{bmatrix} Y_1'(0) \\ Y_1'''(0) \end{bmatrix} \tag{18}$$

and then, setting the determinant of the 2 by 2 matrix to zero and solving for ω, the natural frequencies of the beam can be found. Again, the authors found the same equation misspelled in Binici (2001) paper, the 2nd row 2nd column term has the G term without the second derivative.

In the following example, it is analyzed two types of simple supported beam using the simplified model and the complete model. One of them beam has spam to length ratio of *h/L*=0.1 representing a long spam beam. The second beam has a spam to length ratio of *h/L*=0.7 representing a short beam. The main idea is to compare the behavior of the models on these two situations. The damage magnitude (*a/h*) was varied on values of 0.1, 0.3 and 0.5. The crack position ($\chi = x/L$) may vary continuously along the beam axis limited to a value of 0.5 since in this case there is symmetry.

Figure 2 shows comparisons for the damaged to undamaged 1st frequency ratio along damage position, without axial forces. Figure 3 shows the same comparisons of the Figure 2 including axial forces. Figure 4 and 5 shows the same comparisons of Figures 2 and 3 for the 2nd frequency. The results obtained by Binici (2005) are the same values obtained by this paper using the long beam and simple model.

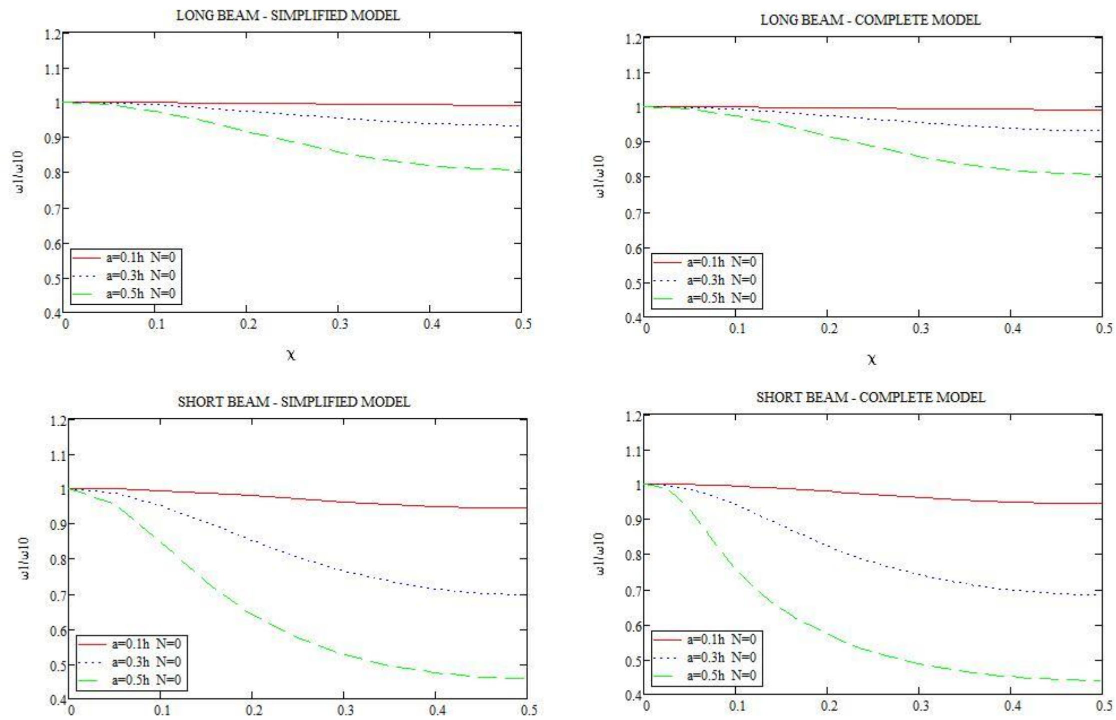

Figure 2. Damaged to undamaged 1st frequency ratio along damage position, without axial forces. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Simple Supported).

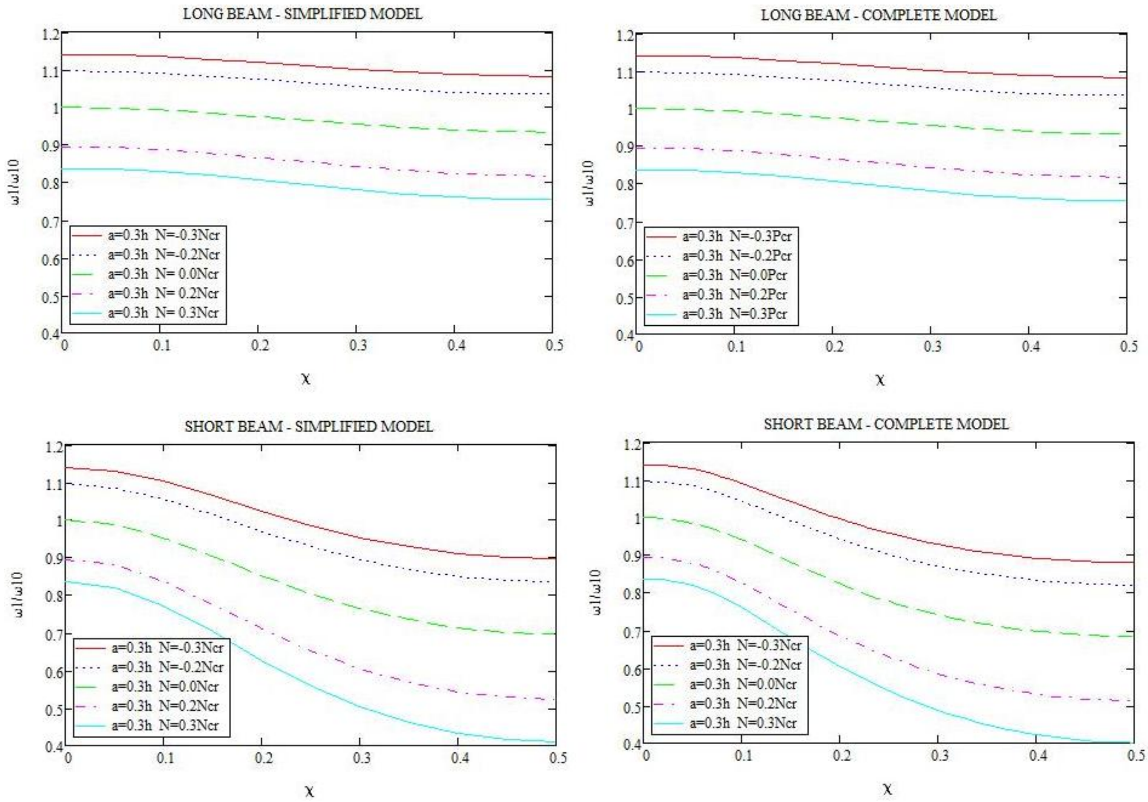

Figure 3. Damaged to undamaged 1st frequency ratio along damage position. Axial force included. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Simple Supported).

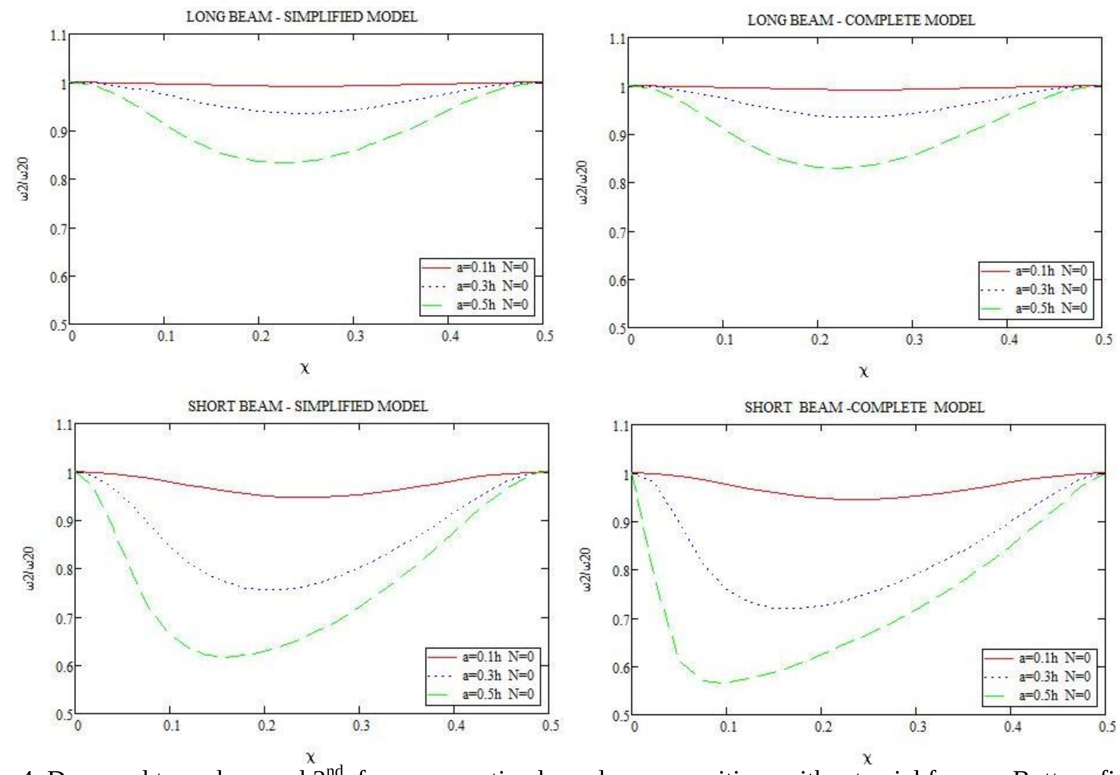

Figure 4. Damaged to undamaged 2nd. frequency ratio along damage position, without axial forces. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Simple Supported).

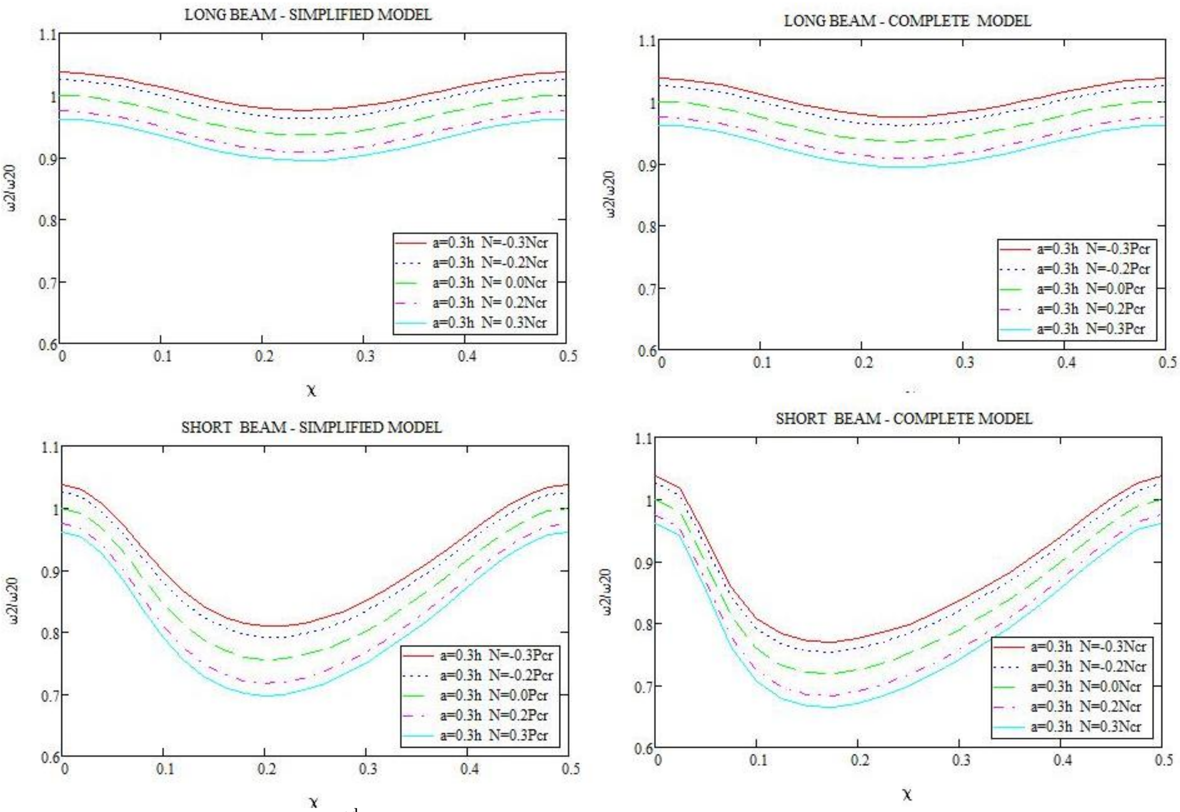

Figure 5. Damaged to undamaged 2nd frequency ratio along damage position. Axial force included. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Simple Supported).

3.2 Cantilever Beam

The analysis of a cantilever beam with only one crack is similar to the simple supported beam. However, the changes in boundary conditions at $\chi = 0$ are modified for the first beam segment. The same continuity conditions at the crack position remain true.

$$Y_1(\chi) = Y_1(0)A(\chi) + Y_1''(0)G(\chi) \tag{19}$$

$$Y_2(\chi) = Y_1(\chi) + c_1 Y_1''(\chi_1)G(\chi - \chi_1) \tag{20}$$

where $A(\chi)$ and $G(\chi)$ functions are defined as:

$$A(\chi) = \eta\left[\beta^2 \cosh(\alpha\chi) + \alpha^2 \cos(\beta\chi)\right] \tag{21}$$

$$G(\chi) = \eta\left[\alpha \sinh(\alpha\chi) + \beta \sin(\beta\chi)\right] \tag{22}$$

where $\eta = 1/\left(\alpha^2 + \beta^2\right)$

The two Eq.(12) and Eq.(13) can be put in a matrix representation as indicated:

$$\begin{bmatrix} A(1) + cG(1-\chi)A''(\chi) & G(1) + c_1 G(1-\chi)G''(\chi) \\ A'(1) + cG'(1-\chi)A''(\chi) & G'(1) + c_1 G'(1-\chi)G''(\chi) \end{bmatrix} = \begin{bmatrix} Y_1'(0) \\ Y_1'''(0) \end{bmatrix} \tag{23}$$

Setting the determinant of the matrix given on the left-hand side and solving for ω , the first two roots represents the first and second eigenfrequencies of the damaged structure. These values are compared with the closed form solution for cantilever beams with just bending effect, that is $\omega_{10} = (1.875/L)^2 \sqrt{EI/\rho A}$ and $\omega_{20} = (4.694/L)^2 \sqrt{EI/\rho A}$.

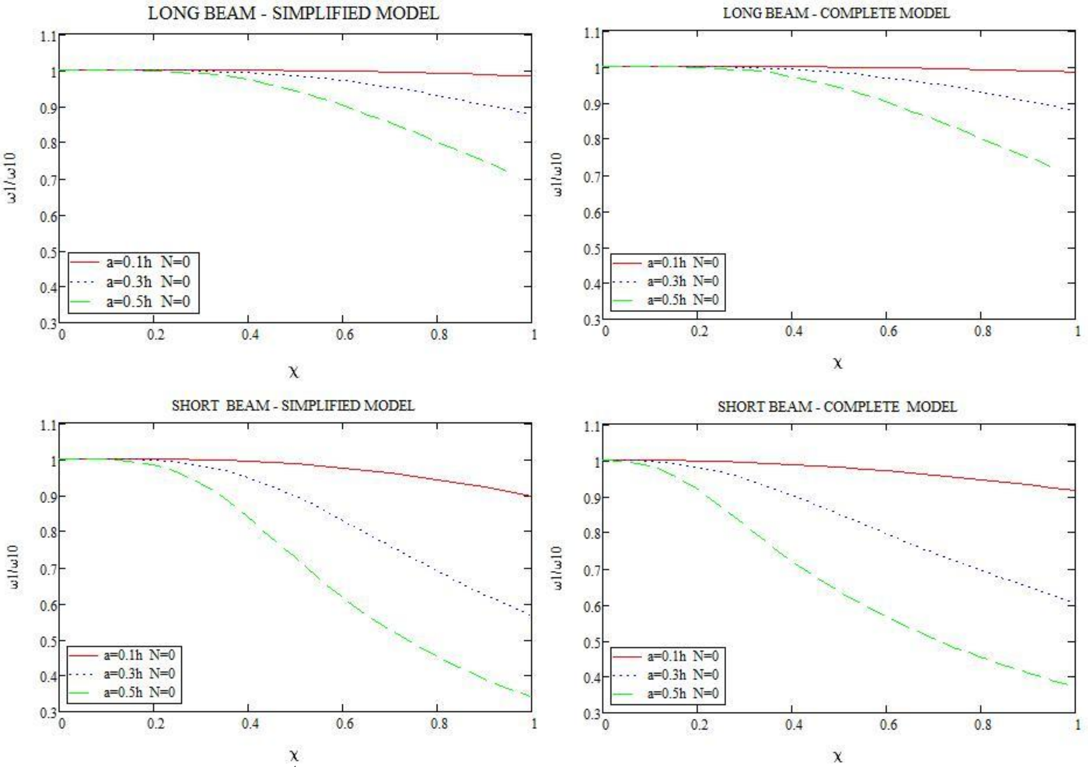

Figure 6. Damaged to undamaged 1st frequency ratio along damage position, without axial forces. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Cantilever beam).

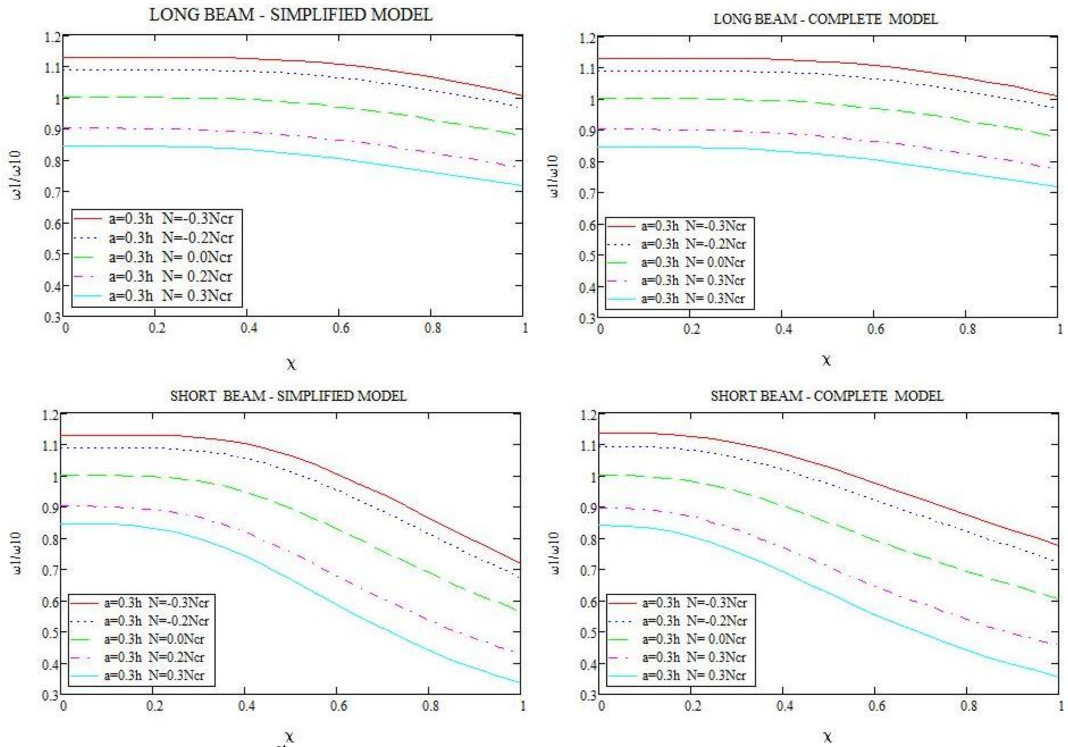

Figure 7. Damaged to undamaged 1st frequency ratio along damage position. Axial force included. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Cantilever beam).

Figure 6 shows comparisons for the damaged to undamaged 1st frequency ratio along damage position, without axial forces. Figure 7 shows the same comparisons of the Figure 6 including axial forces. Figure 8 and 9 shows the same comparisons of Figures 6 and 7 for the 2nd frequency. The results obtained by Binici (2005) are the same values obtained by this paper using the long beam and simple model.

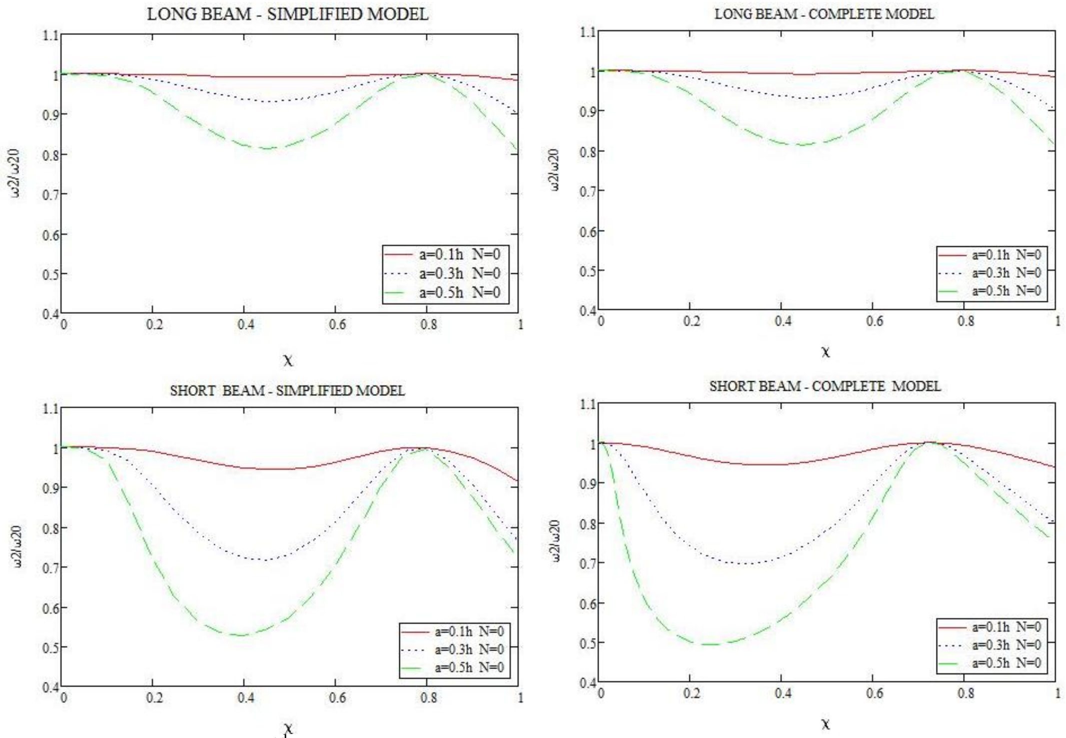

Figure 8. Damaged to undamaged 2nd. frequency ratio along damage position, without axial force. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Cantilever beam).

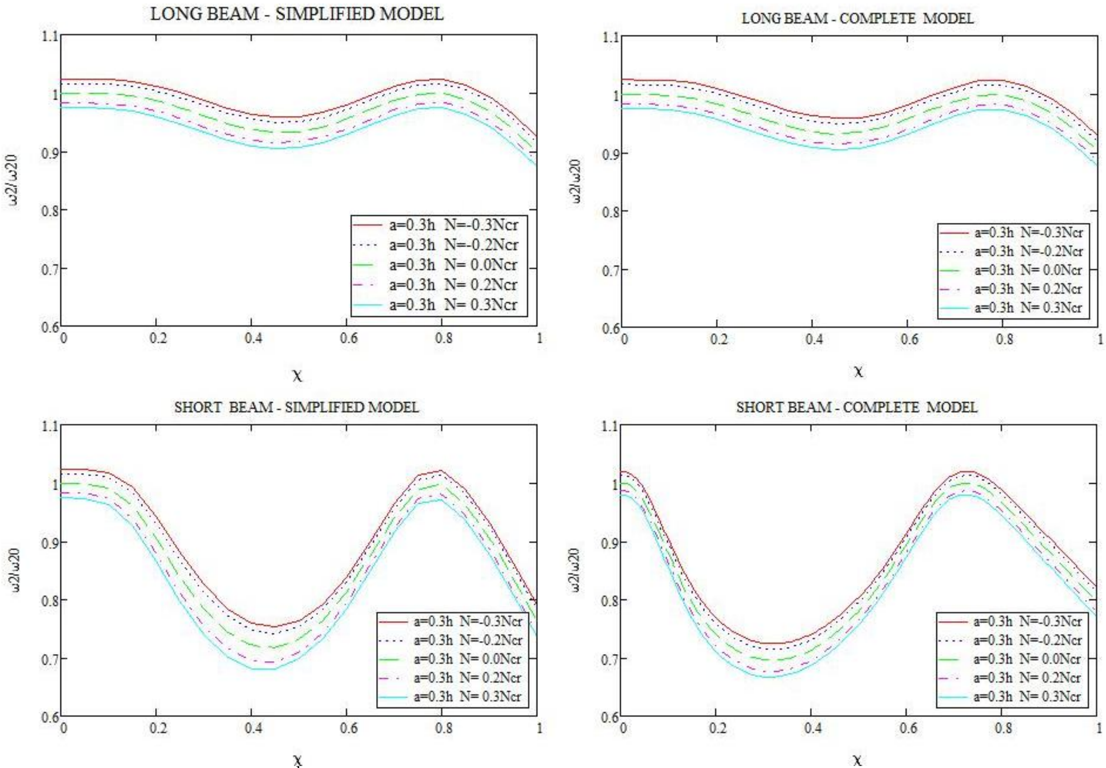

Figure 9. Damaged to undamaged 2nd frequency ratio along damage position. Axial force included. Bottom figures for short spam beam, upper figures for long spam beam. Left figures for simplified model, right figures for complete model (Cantilever beam).

4. CONCLUSIONS

Generally speaking, by the analyzed examples the simple model (including just bending and axial effects) is capable to predict the frequencies variations on both types of structures (simples supported and cantilever beam) in the case of long beams. It was observed by the examples that the complete model presented very close results in these cases (long beams).

In the case of short beams, spam to length ratio greater than 0.66, the simple model should not be used to predict the frequencies variations due to large errors when compared to the complete model.

Regarding the damage site, in all the examples analyzed, the expected point that most contributes to the frequency variations is the expected one. For example in the case of simple beams, the middle spam most contributes to the 1st frequency variations and in the case of cantilever beams, the fixed end most contributes to the 1st frequencies variations. The same behavior is noticed for the 2nd frequency variations(short beams) except for the fact that the complete model shifts the locations of the point that most contributes to the frequencies variations near to the pinned end (simple supported) and near the free end(cantilever). This shows the importance of the terms related to the shear deformation terms, rotational inertia and combinations.

Related to the axial force, the expected behavior of increase in frequency with the increase of the tension axial force was observed. In the case of long beams using the simple or the complete model, the frequency variations were the same (the lines in the graphs related to the increasing axial forces are all just shifted). However, when dealing with short beams, both models presented a frequency variation that are not proportional to the case of no axial force. This can be easily noticed by the different width of the graphs related to the maximum and minimum axial forces along the crack position. This behavior is most pronounced using the complete model.

Comparing the simple supported and the cantilever beams, it can be noticed that the complete model presents a different behavior regarding the frequencies changes. In the case of simple supported short beam, the complete model showed frequencies variations greater than the simple model when the damaged occurs near the middle-spam. In the case of a short cantilever beam, the complete model showed lower frequencies variations near the fixed end.

If one needs a value to be used for comparisons with discrete methods (Finite Elements) it is suggested to use the values obtained by the complete model regardless of the type of beam (short or long). These results were not compared with experimental ones, so some care should be taken in the used of these results. Future works will deal with the experimental validation of the obtained results.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

Aydin, K. 2008, "Vibratory Characteristics of Euler-Bernoulli Beams with an Arbitrary Number of Cracks Subjected to Axial Load", Journal of Vibration and Control, V.14, No.4, pp.485–510.

Binici, B., 2005, "Vibration of Beams with Multiple Open Cracks Subjected to Axial Force", Journal of Sound and Vibration, V. 287, pp.277-295.

Chondros, T. G., 2001, "The continuous crack flexibility model for crack identification", Journal of Fatigue Fracture Engineering Mater Structures, No. 24, pp.643-650.

Chondros, T., 1977, "Dynamic response of cracked beams", Dipl.Eng. Thesis. University of Patras, Greece.

Clough, R. W., Penzien, J., 1975, Dynamic of Structures, MacGraw-Hill, 634p.

Chondros, T., 1981, "Dynamics of cracked structures", Dr. Eng. Thesis. University of Patras, Greece.

Dimaragonas, A.D. 1996. Vibration of cracked structures: a state of the art review, Engineering Fracture Mechanics, V. 55, No.5. pp.831–857.

Owolabi, G. M., Swamidas , A. S. J., Seshadri, R., 2003. "Crack detection in beams using changes in frequencies and amplitudes of frequency response functions", Journal of Sound and Vibration, V.265, pp.1-22.

Li, Q. S., 2002, "Free Vibration Analysis of Non-Uniform Beams with an Arbitrary number of Cracks and Concentrated Masses", Journal of Sound and Vibration, V.252, No.3, pp.509-525.

Liebowitz, H. and Claus, W. D., 1968, "Failure of notched columns", Journal of Engineering Fracture Mechanics. V.1, p.379–383.

Narkis, Y.,1994, "Identification of Crack Location in Vibration Simple Supported Beams", Journal of Sound and Vibration, V. 172, No.4, pp.549-558.

Naniwadekar, M.R., Naik, S.S., Maiti, S.K.,2008, "On prediction of crack in different orientations in pipe using frequency basead approach", Mechanical Systems and signal Processing 22 693-708.

Sáez, J. F., Rubio, L., Navarro, C., 1999, "Approximate Calculation of the Fundamental Frequency for Bending Vibrations of Cracked Beams", Journal of Sound and Vibration, V. 225, No.2, pp.345-352.

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