GENERALIZED BOUNDARY ELEMENT METHODS WITH FUNDAMENTAL SOLUTION VIA THE FOURIER TRANSFORM

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Abstract. *A recent method developed by means of the spatial Fourier transform generalizes the Boundary Element Methods (BEM) to the so-called Fourier-BEM. This approach is available for all cases as long as the differential operator is linear and has constant coefficients and is possible for all variants of the BEM. Via convolution and Parceval's theorem, which states the equivalence of energy terms in the original space and in the Fourier space, the idea is to avoid the inverse Fourier transform of the fundamental solution and to work directly with the Fourier transformed fundamental solution. No inverse transform and no fundamental solution in the original space are required. Alternative Boundary Integral Equations (BIE) can be established in the Fourier transformed domain. A Galerkin approach lead to matrices identical to those obtained via the standard BIE and does not require a second integration. The elements and shape functions also can be transformed to the Fourier domain. In this work, the method is presented and then applied to heat problem to motivate the discussion and demonstrate the equivalence between traditional BEM and Fourier BEM.*

Keywords: *Fourier Transform, Fundamental Solution, Boundary Element Method, Laplace-operator.*

1. INTRODUCTION

The standard Boundary Element Method (BEM) is a powerful tool in computational mechanics. The range of applications of BEM is restricted to cases where the fundamental solution is known. Then the knowledge of a fundamental solution is crucial to solve engineering problems with BEM. It is available for a large number of cases relevant in engineering but in several problems is still not known analytically. As with the Finite Element Method (FEM), the integral equations of the BEM can be derived from weighted residuals. They consist of convolution integrals weighted with either Dirac distributions called collocation method or polynomials trial functions called Galerkin method. In traditional approach, as in Brebbia et al. (1984), Hall (1994), Bonnet (1999), and McLean (2000) the former demands for a single integration over boundary problem while the latter requires double integrations. Nevertheless, the Galerkin method is becoming more popular due to symmetry of the matrices and better convergence properties, as presented by Frangi and Bonnet (1998). Although in all cases the fundamental solution is required.

Recent approaches have been developed where the fundamental was derived in the Fourier space with transform with respect to spatial and temporal coordinates. If the differential operator is linear and has constant coefficients, its transformed form can be easily inverted, which leads directly to the Fourier transform of the fundamental solution. A work developed by Duddeck (2002) via the Fourier transform generalizes the BEM to the so-called Fourier BEM. There, new Boundary Integral Equations (BIE) are formulated, which consist only of the Fourier transformed terms and lead to equivalent matrices as in the standard BIE. It is based on Parseval's equality and the convolution theorem. The former states the invariance of energy or work with respect to the Fourier transform, and the latter relates convolution products to simple products. As show in the works presented by Duddeck (2002) and Sanches (2009), the principal idea is to avoid the inverse Fourier transform of the fundamental solution and to work directly with the Fourier transformed fundamental solution. Every term should be established in the Fourier domain. The applied Galerkin approach leads to symmetric matrices and does not require a second integration in the new method. The trial and the test functions can be easily transformed to the Fourier domain as long as they are defined on straight elements.

An example of the heat conduction applying the Laplace-operator in the Poisson equation is discussed to visualize the broad field of applications of the Fourier BEM and to demonstrate the equivalence between traditional method and the new approach.

2. THEORY OF STANDARD BEM AND FOURER-BEM

The theory of BEM is based on an equivalent weak form or reciprocity relation who includes all boundary terms known and unknown. For a n-dimensional bounded domain $\Omega \subset R^n$ with a polyhedral boundary $\partial\Omega$, we get

$$\int_{\Omega} u\Delta\phi dy = \int_{\Omega} \Delta u\phi dy + \int_{\partial\Omega} \left(u\frac{\partial\phi}{\partial\nu} - \frac{\partial u}{\partial\nu}\phi \right) d\Gamma \tag{1}$$

Considering property of the Dirac $\int_\Omega u(y)\delta(x-y)dy = u(x)$, where u is the unknown quantity and δ is the Dirac delta (the response of an infinite medium to a single force $f = \delta$ at $x = y$; f denotes known volume sources), the Somigliana's identity is derived by an inversion of the differential operator that is we replace ϕ by the fundamental solution $U(x-y)$

$$u(x)=\int_{\Omega} f(y)U(x-y)dy-\int_{\partial\Omega}[u(y)T(x-y)-t(y)U(x-y)]d\Gamma_y \tag{2}$$

The transition to boundary integral equations demands a limit process of $x\rightarrow \partial\Omega$. The left-hand side of (2) is modified by a factor κ which is equal to ½ for smooth boundaries, then

$$\kappa(x)u(x)=\int_{\Omega} f(y)U(x-y)dy-\int_{\partial\Omega}[u(y)T(x-y)-t(y)U(x-y)]d\Gamma_y \tag{3}$$

In (3) u and t are approximated by a sum of piecewise polynomial trial functions η_m^u, η_m^t with coefficients u_m e t_m

$$u(y)\approx\sum_{m}^{N^u}u_m\eta_m^u(y), \qquad t(y)\approx\sum_{m}^{N^t}t_m\eta_m^t(y), \tag{4}$$

These quantities represent known and unknown boundary values. An additional weighting with these trial functions leads to the Galerkin version of the integral equation (3)

$$\begin{aligned}\int_{\partial\Omega_x}\eta_j(x)\kappa(x)u(x)d\Gamma_x &= \int_{\partial\Omega_x}\eta_j(x)\int_{\Omega} f(y)U(x-y)dyd\Gamma_x + \\ &-\int_{\partial\Omega_x}\eta_j(x)\int_{\partial\Omega_y}[u(y)T(x-y)-t(y)U(x-y)]d\Gamma_y d\Gamma_x\end{aligned} \tag{5}$$

The use of boundary quantities (4) in (5) results in

$$\begin{aligned}\int_{\partial\Omega_x}\kappa(x)\eta_j u(x)d\Gamma_x &= \int_{\partial\Omega_x}\eta_j(x)\int_{\Omega} f(y)U(x-y)dyd\Gamma_x + \\ &-\sum_{m}^{N^u}\int_{\partial\Omega_x}\eta_j(x)\int_{\partial\Omega_y}u_m\eta_m^u(y)T(x-y)d\Gamma_y d\Gamma_x + \\ &+\sum_{m}^{N^t}\int_{\partial\Omega_x}\eta_j(x)\int_{\partial\Omega_y}t_m\eta_m^t(y)U(x-y)d\Gamma_y d\Gamma_x\end{aligned} \tag{6}$$

For the case of standard BEM constructed from these BIEs (equations 3 and 6) is necessary the explicit knowledge of the fundamental solution U and its normal derivative T. The knowledge of fundamental solution is crucial to solve engineering problems with BEM. Numerical approaches have been developed but they introduce additional numerical errors to the overall approximation. The Fourier-BEM to work directly with the Fourier transformed fundamental solution. Alternative boundary integral equations can be established in the Fourier transformed domain. They lead to matrices identical to those obtained via the standard method. These approaches are presented in the following sections.

2.1. Distributional Theory Fourier-BEM

To obtain the Fourier transform of the BIE, all quantities have to be extended from a n-dimensional bounded domain Ω, with a polyhedral boundary $\partial\Omega$, to the space R^n. Formally, this can be done by defining a cutoff distribution χ which is simply one in the interior of Ω and zero outside. Then all quantities are multiplied by cutoff-distribution and finally transformed into Fourier space. Mathematically this extension and transformation is justified only in the frame of the theory of distributions (Schwartz, 1950). It simplifies the derivation of the reciprocity relations, the correct treatment of discontinuities and singularities, and the evaluation of the free terms. The Fourier transform without its distributional extension is not complete enough to treat the simplest problems (Duddeck, 2002). The main advantage of the theory is that it reestablishes differentiation as a simple and consistent procedure; all quantities are differentiable even if they exhibit severe singularities or jumps.

2.2. Mathematical Equations for Extension from Ω to R^n

The Fourier transform is defined on R^n, hence we have to extend the equation (6) from Ω to R^n. Therefore, we define a cutoff-distribution (Gel'fand and Shilov, 1964)

$$\chi(x) := \begin{cases} 1 & \dots \quad x \in \Omega \\ \kappa & \dots \quad x \in \partial\Omega \\ 0 & \dots \quad x \notin \overline{\Omega} = \Omega \cup \partial\Omega \end{cases} \tag{7}$$

For smooth boundaries, χ can be expressed by a generalized multi-dimensional Heaviside-distribution $H(\psi)$

$$\chi(x) = H(\psi(x)) \qquad x \in R^n \tag{8}$$

ψ describes the boundary ∂Ω as a hypersurface. Domains with non-smooth boundaries are expressed by products and sums of these Heaviside-distributions, as presented by Duddeck (1997). The field quantities, the unknown u and the volume force f, extended over Ω, that is they vanish outside Ω. This is described by

$$u(x) \to u_\Omega := \chi(x)u(x), \qquad f(x) \to f_\Omega := \chi(x)f(x) \tag{9}$$

where u and f are C^2 (R^n). In analogy to equation (9), we need cutoff-distributions for the trial and test functions. For example, they are for a reference element

$$\chi_0(x) := H(x_1)H(1-x_1)\delta(x_2), \qquad x \in R^2 \tag{10}$$

Trial functions are obtained multiplying a C^∞ (R^n)-function $p_0(x)$, for example polynomials, with cutoff-distributions

$$\eta_0(x) := \chi_0(x)p_0(x) \tag{11}$$

The unknown a the known quantities on the boundaries are now approximated by

$$u(x) \approx \sum_m^{N^u} u_m \eta_m^u(x), \qquad t(x) \approx \sum_m^{N^t} t_m \eta_m^t(x), \tag{12}$$

The Galerkin-BIE equation (6) can now be expressed by

$$\begin{aligned} \int_{R^n} \eta_j(x) u_\Omega(x) d(x) = & \int_{R^n} \eta_j(x) \int_{R^n} f_\Omega(y) U(x-y) dy dx + \\ & - \sum_m^{N^u} u_m \int_{R^n} \eta_j(x) \int_{R^n} \eta_m^u(y) T(x-y) dy dx + \\ & + \sum_m^{N^t} t_m \int_{R^n} \eta_j(x) \int_{R^n} \eta_m^t(y) U(x-y) dy dx \end{aligned} \tag{13}$$

The factor κ is implicitly defined by the distributional representation, we have for example $\delta(x_1)H(x_1) = \frac{1}{2}\,\delta(x_1) = \kappa\delta(x_1)$. With the abbreviations

$$\langle a(x), b(x) \rangle := \int_{R^n} a(x)b(x)dx \qquad \text{and} \qquad a(x) * b(x) := \int_{R^n} a(y)b(x-y)dy \tag{14}$$

For the unknown u, we get finally as Galerkin-BIE in R^n

$$\langle \eta_j, u_\Omega \rangle = \langle \eta_j, f_\Omega * U \rangle - \sum_m^{N^u} u_m \langle \eta_j, \eta_m^u * T \rangle + \sum_m^{N^t} t_m \langle \eta_j, \eta_m^t * U \rangle \tag{15}$$

with $u_\Omega = u(x)\chi(x)$; $f_\Omega = f(x)\chi(x)$ and χ defined by equation (7).

2.3. Fourier Transform of the Galerkin-BIE

For several physical problems, we do not know the fundamental solution U required in (3) and (6). But, if the coefficients of a differential operator are constant, we can always derive the Fourier transform $\hat{U}(\hat{x})$ of $U(x)$. Then, the n-dimensional Fourier transform, $\mathcal{F}(\phi) = \hat{\phi}$ of ϕ is defined as

$$\hat{\phi}(\hat{x}) = \int_{R^n} \phi(x)\, e^{-i(x,\hat{x})} dx, \qquad \langle x, \hat{x} \rangle = \sum_j^n x_j \hat{x}_j \tag{16}$$

with $i = \sqrt{-1}$. The symbol ^ characterizes a transformed object, $\mathcal{F}\leftrightarrow$ denotes the link of an expression in the original space to the corresponding term in the transformed space. The principal idea of the Fourier-BEM is two well known theorems of the Fourier transform the theorem of Parceval which states the invariance of work or energy

$$\int_{R^n} u(x)\phi(x)dx = \frac{1}{2\pi^n} \int_{R^n} \hat{u}(\hat{x})\hat{\phi}(-\hat{x})d\hat{x} \tag{17}$$

and the convolution theorem, for example,

$$u(x) * \phi(x) \qquad \mathcal{F}\leftrightarrow \qquad \hat{u}(\hat{x})\hat{\phi}(\hat{x}) \tag{18}$$

The application of these two theorems to equation (15) leads to the equivalences

$$\langle \eta_j, u_\Omega \rangle = \frac{1}{2\pi^n} \langle \hat{\eta}_j(-\hat{x}), \hat{u}_\Omega(\hat{x}) \rangle \tag{19}$$

$$\langle \eta_j, f_\Omega * U \rangle = \frac{1}{2\pi^n} \langle \hat{\eta}_j(-\hat{x}), \hat{f}_\Omega(\hat{x})\hat{U}(\hat{x}) \rangle \tag{20}$$

$$\langle \eta_j, \eta_m^u * T \rangle = \frac{1}{2\pi^n} \langle \hat{\eta}_j(-\hat{x}), \hat{\eta}_m^u(\hat{x})\hat{T}(\hat{x}) \rangle \tag{21}$$

$$\langle \eta_j, \eta_m^t * U \rangle = \frac{1}{2\pi^n} \langle \hat{\eta}_j(-\hat{x}), \hat{\eta}_m^t(\hat{x})\hat{U}(\hat{x}) \rangle \tag{22}$$

The equivalent to equation (15) is the BIE for the Fourier-BEM

$$\begin{aligned} \langle \hat{\eta}_j(-\hat{x}), \hat{u}_\Omega(\hat{x}) \rangle = \langle \hat{\eta}_j(-\hat{x}), \hat{f}_\Omega(\hat{x})\hat{U}(\hat{x}) \rangle - \sum_m^{N^u} u_m \langle \hat{\eta}_j(-\hat{x}), \hat{\eta}_m^u(\hat{x})\hat{T}(\hat{x}) \rangle + \\ + \sum_m^{N^t} t_m \langle \hat{\eta}_j(-\hat{x}), \hat{\eta}_m^t(\hat{x})\hat{U}(\hat{x}) \rangle \end{aligned} \tag{23}$$

The inner integrals in equation (15) are convolutions and converted by Fourier transform to simple multiplications. The outer integrals in (15) are scalar products, which are converted to equivalent scalar products by Fourier transform (Parceval). Thus the double integration for Galerkin methods is replaced by a single integration over R^n in (23).

2.4. Fourier Transform of the Fundamental Solution

The Fourier transformation of a differential equation converts the differential operator to an algebraic expression. Here, this approach is presented for the Laplacian-operator. We regard the Poisson equation in an n – dimensional bounded domain $\Omega \subset R^n$ with a polyhedral boundary $\partial\Omega$

$$\Delta u(x) = -f(x), \qquad x \in \Omega \subset R^n, \qquad \Delta = \sum_j^n \partial^2 / \partial x_j^2, \tag{24}$$

where,

$$u(x) = u_\Gamma(x), \qquad x \in \Gamma_u \subset \partial\Omega, \qquad \text{and} \qquad t(x) = t_\Gamma(x), \qquad x \in \Gamma_t \subset \partial\Omega, \tag{25}$$

As boundary conditions we assume that half of the boundary data, either u on Γ_u or its normal derivatives $t = \partial u/\partial\nu$ on Γ_t, is known, that is $\Gamma_u \cup \Gamma_t = \partial\Omega$. The Fourier transform of the differential equation (24) converts the differential operator Δ to an algebraic expression

$$-\Delta u(x) = f(x) \qquad \mathcal{F}\leftrightarrow \qquad |\hat{x}|^2\,\hat{u}(\hat{x}) = \hat{f}(\hat{x}), \tag{26}$$

with the symbol $|\hat{x}|^2 = \sum_j^n \hat{x}_j^2$. The Fourier fundamental solution $\hat{U}(\hat{x})$ as the response to a single unit force $f(x)=\delta(x)$ is obtained from

$$-\Delta U(x) = \delta(x) \qquad \mathcal{F}\leftrightarrow \qquad |\hat{x}|^2\,\hat{U}(\hat{x}) = 1, \tag{27}$$

by simple inversion

$$\hat{U}(\hat{x}) = \frac{1}{|\hat{x}|^2}. \tag{28}$$

For the complete BIE, we need the normal derivatives of U for the normal vector ν

$$T(x) = \nu \cdot \nabla U(x) \qquad \mathcal{F}\leftrightarrow \qquad \hat{T}(\hat{x}) = \nu \cdot i\hat{x}\hat{U}(\hat{x}), \tag{29}$$

This procedure can be applied to all linear and homogeneous operators; we always have the Fourier fundamental solution (Duddeck, 2002).

2.5. Fourier Transform of Particular Generalized Function

A number of particular generalized functions can be studied, some for their intrinsic interest and wide-spread utility, and others solely for their application to the technique of asymptotic estimation of Fourier transforms. We begin by defining integral powers, and more precisely the function $sgn\ x_k/x_k^m$. In this work, Lighthill (1966) gives as transform of $sgn\ x_k/x_k^m$ with an integer $m > 0$ the following result

$$\frac{sgn\, x_k}{x_k^m} \qquad \mathcal{F}\leftrightarrow \qquad -2\frac{(-i\hat{x}_k)^{m-1}}{(m-1)!}\left(ln|x_k| + C\right) \tag{30}$$

The constant C occurs because the functions $sgn\ x_k/x_k^m$ are determined only up to an arbitrary multiple of the Dirac-distribution $\delta^{(m-1)}$ at the singular point $x_k = 0$. This constant C has to be defined for each fundamental solution. It is well known that the constant C in the original space can be added to the fundamental solution without changing the solution. In general it is chosen to be zero. The table 1 entries for $m > 1$ was originally obtained by Lighthill (1966) and developed by Duddeck (2002)

Table 1 - Fourier pairs for $sgn\ x_k/x_k^m$

Original space	$\mathcal{F}\leftrightarrow$	Fourier space
$sgn\ x_k/x_k$	$\mathcal{F}\leftrightarrow$	$-2(ln|\hat{x}_k|)$
$sgn\ x_k/x_k^2$	$\mathcal{F}\leftrightarrow$	$-2i\hat{x}_k(ln|\hat{x}_k| - 1)$
$sgn\ x_k/x_k^3$	$\mathcal{F}\leftrightarrow$	$-2\hat{x}_k^2/2!(ln|\hat{x}_k| - 3/2)$

The Fourier transform is based on this distribution theory and therefore by far better suited for singular integrals. A number of the one and two dimensional integrations can be computed analytically by means of the equation (30).

2.6. Fourier Transform of the Trial and Test Functions

The transformed cutoff-distributions $\hat{\chi}_0(\hat{x})$ are defined for the two-dimensional reference element

$$\chi_0(x) := H(x_1)H(1-x_1)\delta(x_2) \qquad \mathcal{F}\leftrightarrow \qquad \hat{\chi}_0(\hat{x}) := \frac{i}{\hat{x}_1}\left(e^{-i\hat{x}_1}-1\right) \tag{31}$$

As described by Duddeck (2002), elements of arbitrary polynomial degree are constructed via a multiplication with $p_0(x)$ in the original domain or an analytical convolution in the transformed domain. General transformed straight elements are obtained by transformed dilation and translation operators. For straight elements and for arbitrary polynomials $p_0(x)$, the transformed trial functions are analytically know.

2.7. Construction of Fourier-BEM Matrices

The already discretized Fourier-BIE equation (23) leads to an algebraic system of j equations

$$\kappa u_j = F_j - \sum_m^{N^u} G_{jm} u_m + \sum_m^{N^t} H_{jm} t_m \tag{32}$$

where we have defined

$$F_j := \int_{R^n} \hat{\eta}_j(-\hat{x}) \hat{f}_\Omega(\hat{x}) \hat{U}(\hat{x}) d\hat{x}, \tag{33}$$

$$G_{jm} := \int_{R^n} \hat{\eta}_j(-\hat{x}) \hat{\eta}^u_m(\hat{x}) \hat{T}(\hat{x}) d\hat{x}, \tag{34}$$

$$H_{jm} := \int_{R^n} \hat{\eta}_j(-\hat{x}) \hat{\eta}^t_m(\hat{x}) \hat{U}(\hat{x}) d\hat{x}, \tag{35}$$

These Fourier BIE lead to the same matrix entries as those of the standard approach. Double integrations over the boundary elements are replaced by single integrations over the infinite space (the factor $(2\pi)^{-n}$ was canceled). The differentiation of the fundamental solution in the Fourier space is straight forward, i.e. replaced by a multiplication with these polynomials. The approach requires also the Fourier transforms of the test and trial functions, which can be done analytically as long as the elements are straight. Due to equivalence of the work terms in the original domain and the transformed domain which is stated by Parceval's theorem, all vectors and matrices of (32) have the same values as it would be obtained by a traditional BEM. Therefore, the further processes of the BEM algorithm can be taken without any modifications from the standard BEM.

3. FOURIER-BEM APPLYED TO THE HEAT CONDUTION PROBLEM

We consider a problem of heat conduction in a plane domain Ω which is heated by stationary interior sources f. The temperature at the boundaries is kept to zero and the isotropic conductivity K is set to one. Mathematically, this leads to the Dirichlet problem of the Poisson equation (24). As described by Duddeck (2002) this problem is best suited to demonstrating the equivalence between traditional BEM and Fourier BEM. The problem is solved in a two-dimensional domain $\Omega = [0,1]\times[0,1]$. In this case, the boundary $\partial\Omega$ was divided into four and eight elements, one and two for each side. Additionally, a stationary and uniform heat source f is assumed over Ω, then

$$f_\chi(x) := f_0\ H(x_1)H(1-x_1)H(x_2)H(1-x_2) \qquad \mathcal{F}\leftrightarrow \qquad \hat{f}_\chi(\hat{x}) : (\hat{x}) := -f_0 \frac{\left(e^{-i\hat{x}_1}-1\right)\left(e^{-i\hat{x}_2}-1\right)}{\hat{x}_1\hat{x}_2} \tag{36}$$

Taking into account that the temperature is vanishing at the boundaries the general system of Fourier BIE can be reduced to

$$0 = \left\langle \eta_j, f_\Omega * U \right\rangle + \sum_m^{N^t} t_m \left\langle \eta_j, \eta^t_m * U \right\rangle \quad \mathcal{F}\leftrightarrow \quad 0 = \left\langle \hat{\eta}_j(-\hat{x}), \hat{f}_\Omega(\hat{x})\hat{U}(\hat{x}) \right\rangle + \sum_m^{N^t} t_m \left\langle \hat{\eta}_j(-\hat{x}), \hat{\eta}^t_m(\hat{x})\hat{U}(\hat{x}) \right\rangle \tag{37}$$

To solve the first term of the equation (37) we need to consider a constant trial and test function for the flux t for the first linear boundary element

$$\eta_1^t(x) := H(x_1)H(1-2x_1)\delta(x_2) \qquad \mathcal{F}\leftrightarrow \qquad \hat{\eta}_1^t(\hat{x}) := \frac{i}{\hat{x}_1}\left(e^{-i\hat{x}_1/2}-1\right) \tag{38}$$

$$\eta_1(x) := H(x_1)H(1-2x_1)\delta(x_2) \qquad \mathcal{F}\leftrightarrow \qquad \hat{\eta}_1(-\hat{x}) := \frac{-i}{\hat{x}_1}\left(e^{+i\hat{x}_1/2}-1\right) \tag{39}$$

The fundamental solution $U(x)$ and the Fourier fundamental solution $\hat{U}(\hat{x})$ determined by equation (28) is

$$U(x) = -\frac{1}{2\pi}\ln\sqrt{x_1^2+x_2^2} \qquad \mathcal{F}\leftrightarrow \qquad \hat{U}(\hat{x}) = \frac{1}{\hat{x}_1^2+\hat{x}_2^2}. \tag{40}$$

The integration of the first matrix (H_{11}) entry in the original space is

$$\begin{aligned} H_{11} &= \int_{R^2}\eta_1(x)\int_{R^2}\eta_1^t(y)U(x-y)d\Gamma_y d\Gamma_x \\ &= \frac{1}{2\pi}\int_0^{1/2}\int_0^{1/2}U(x_1-y_1)dy_1dx_1 = \frac{1}{2\pi}\int_0^{1/2}\int_0^{1/2}\ln\sqrt{(x_1-y_1)^2}\,dy_1dx_1 \\ &= \frac{1}{8\pi}\int_0^{1/2}\left[\ln|x_1-1/2|^2-2-4x_1\left(\ln|x_1-1/2|-\ln|x_1|\right)\right]dx_1 \\ &= -\frac{1}{16\pi}\left(2\ln 2+3\right) = -0{,}08726. \end{aligned} \tag{41}$$

The corresponding integration in Fourier space leads to the same value. Although we solve separately the two integrations

$$\begin{aligned} H_{11} &= \frac{1}{(2\pi)^2}\int_{R^2}\hat{\eta}_1(-\hat{x})\hat{\eta}_1^t(\hat{x})\hat{U}(\hat{x})d\hat{x} \\ &= -\frac{1}{(2\pi)^2}\int_{R^2}\frac{-i\left(e^{i\hat{x}_1/2}-1\right)}{\hat{x}_1}\frac{i\left(e^{-i\hat{x}_1/2}-1\right)}{\hat{x}_1}\frac{1}{\hat{x}_1^2+\hat{x}_2^2}d\hat{x}_1d\hat{x}_2 \\ &= \frac{1}{(2\pi)^2}\int_{R^2}\frac{\left(e^{i\hat{x}_1/2}-1\right)\left(e^{-i\hat{x}_1/2}-1\right)}{\hat{x}_1^2}\frac{1}{\hat{x}_1^2+\hat{x}_2^2}d\hat{x}_1d\hat{x}_2 \\ &= \frac{1}{2\pi}\int_{R^1}\frac{sgn\,\hat{x}_1\left(\cos(\hat{x}_1/2)-1\right)}{\hat{x}_1^3}d\hat{x}_1 \end{aligned} \tag{42}$$

The integration over x_2 leads to an equation (42). Reordering the equation (42), it can be written in the following form

$$\begin{aligned} H_{11} &= \frac{1}{(2\pi)^2}\int_{R^2}\hat{\eta}_1(-\hat{x})\hat{\eta}_1^t(\hat{x})\hat{U}(\hat{x})d\hat{x} = \frac{1}{2\pi}\int_{R^1}\frac{sgn\,\hat{x}_1}{\hat{x}_1^3}\left(\frac{e^{i\hat{x}_1/2}+e^{-i\hat{x}_1/2}}{2}-1\right)d\hat{x}_1 \\ &= \frac{1}{4\pi}\int_{R^1}\frac{sgn\,\hat{x}_1}{\hat{x}_1^3}e^{i\hat{x}_1/2}d\hat{x}_1+\frac{1}{4\pi}\int_{R^1}\frac{sgn\,\hat{x}_1}{\hat{x}_1^3}e^{-i\hat{x}_1/2}d\hat{x}_1-\frac{1}{2\pi}\int_{R^1}\frac{sgn\,\hat{x}_1}{\hat{x}_1^3}d\hat{x}_1 \end{aligned} \tag{43}$$

Applying the equation (30), developed by the Lighthill (1966), and considering the Fourier transform (16) in the new expression (43) we obtain the final result

$$
\begin{aligned}
H_{11} &= \frac{1}{4\pi}\int_{R^1}\frac{sgn\,\hat{x}_1}{\hat{x}_1^3}e^{i\hat{x}_1/2}d\hat{x}_1+\frac{1}{4\pi}\int_{R^1}\frac{sgn\,\hat{x}_1}{\hat{x}_1^3}e^{-i\hat{x}_1/2}d\hat{x}_1-\frac{1}{2\pi}\int_{R^1}\frac{sgn\,\hat{x}_1}{\hat{x}_1^3}d\hat{x}_1 \\
&= \frac{1}{4\pi}\hat{\Im}(1/2)+\frac{1}{4\pi}\hat{\Im}(-1/2)-\frac{1}{2\pi}\hat{\Im}(0) \\
&= \frac{1}{2\pi}\hat{\Im}(1/2) \\
&= \frac{1}{2\pi}(1/2)^2\left(ln|1/2|-3/2\right) \\
&= \frac{1}{8\pi}\left(ln\,1-ln\,2-3/2\right) \\
&= -\frac{1}{8\pi}\left(ln\,2-3/2\right)=-\frac{1}{16\pi}\left(2\,ln\,2-3\right)=-0{,}08726
\end{aligned}
\tag{44}
$$

Note in expression (44), the Fourier transform of the function $sgn\ x_k/x_k^m$. The total matrix H is obtained analytically either in the original or in Fourier space as

$$
H=\begin{bmatrix}
-0{,}08726 & -0{,}03210 & -0{,}00934 & 0{,}0234 & 0{,}00079 & 0{,}00484 & -0{,}0422 & 0{,}00934 \\
 & -0{,}08726 & -0{,}04222 & -0{,}00934 & 0{,}00484 & 0{,}00079 & -0{,}00934 & 0{,}00234 \\
 & & -0{,}08726 & -0{,}03210 & 0{,}00234 & -0{,}00934 & 0{,}00079 & 0{,}00484 \\
 & & & -0{,}08726 & -0{,}00934 & -0{,}04222 & 0{,}00484 & 0{,}00079 \\
 & & & & -0{,}08726 & -0{,}03210 & -0{,}00934 & -0{,}04222 \\
 & & SIMETRIC & & & -0{,}08726 & 0{,}00234 & -000934 \\
 & & & & & & -0{,}08726 & -0{,}03210 \\
 & & & & & & & -0{,}08726
\end{bmatrix}
\tag{45}
$$

The matrix G was obtained of the same way and the standard matrix approach of the BEM was used for compute the results of the Fourier-BEM. The results of heat flux on each side of the boundary to the mesh with four and eight linear elements are presented in Table 1. These values are compared with exact solution expressed by a Fourier series and with those obtained by Duddeck (2002).

Table 1. Results Boundary Heat Flux.

Method	Boundary Heat Flux
Fourier-BEM (4 elements)	0,2489
Fourier-BEM (8 elements)	0,2611
Duddeck (2002)	0,2564
Fourier Series	**0,2628**

The coarse mesh leads to an acceptable result, as in Duddeck (2002). The global error for the mesh with eight linear elements is approximately 0.7 %. These results were compared with the exact solution expressed by Fourier series obtained by separation de variables.

4. CONCLUSIONS

The formulation and implementation of the Fourier BEM approach generalizes boundary element methods such that all cases of linear partial differential operators with constant coefficients can be treated. The traditional Galerkin boundary integral equations are reformulated by means of convolution theorem and Parceval's identity. In contrast to the standard method, all quantities, the trial and test functions as well as the fundamental solution and its derivatives are only required in Fourier space. This BIE leads to symmetric, but fully populated matrices. The matrices are evaluated directly, no inverse transform is necessary. A rigorous distributional representation of the boundary element method was showed (Schwartz, 1950). It enables the correct treatment of the jumps and singularities occurring in the BIE. The distributional concept is necessary for the differentiation of the BIE required for the symmetric Galerkin BEM. Additionally, for some physical problems, the fundamental solution is only known explicitly in Fourier space. Hence, approaches via traditional BEM encounter a lot of difficulties which are avoided in the new method. Therefore, the field of applications of the Fourier BEM is enlarged.

The scheme is valid for all linear problems where the material parameters do not depend on location or time. An example was presented and taken from simple stationary heat conduction. The matrices of the BEM were solved by traditional method and Fourier transforms (Lighthill, 1966) to show the equivalence between the BEM and the Fourier BEM. On the other hand, equivalents integral equations can be determined in this space. This leads to the identical matrices obtained in the traditional BEM. The shape functions and boundary element present can also be transformed to Fourier space. Thus, a numerical implementation similar to the standard technique can be developed.

As numerical application, we took as basis the differential Laplace operator in a problem of heat conduction (Sanches, 2009). The results showed good convergence between traditional models and new methodology. In the near future, it is expected to apply the Fourier-BEM in elasticity and dynamic problems involving anisotropic plates. At the same time, we intend to analyze mathematically the unique strengths and hyper-singularities existing in the nucleus of integral equations to convert the cases cited in this work.

5. ACKNOWLEDGEMENTS

The research leading to this article has been supported by CNPq. This is gratefully acknowledged.

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