# Linear Stability Analysis of Two-Phase Flows in Pipeline Riser Systems

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***Abstract.*** *Offshore oil production facilites with the pipeline preceeding the riser with a downward inclination angle may lose their steady operation regime along their life span due to a decrease in the ratio between gas and liquid flow rates. Stability analysis of an appropriate model for the multifase flow in pipeline-riser system reveals the regions in the parameter space which have a steady state, or a stable steady solution in the system dynamics language. We consider the no pressure wave model for the two-fase flow in a pipeline-riser system with vertical riser. We perform numerically the linear stability analysis of this flow model. We derive the steady solution perturbation governing equations and a spectrum approximation of this linear boundary value problem is evaluated numerically. We obtain the stable and unstable regions for the steady solution in the system parameter space for a pipeline-riser system presented in the literature. We compare this result with the stability boundary in the system parameter space obtained throught numerical time simulations.*

***Keywords:****hydrodynamic instability, numerical linear stability analysis, two-phase flows, pipeline-riser system, stability boundary*

## 1. INTRODUCTION

Intermitent flow regimes may appear in offshore production facilities during their life span due to low rate between the gas and liquid mass flow rate and when the pipeline preceeding the riser has an downward inclination angle. The probability of intermitent flow regimes instead of a steady state grows with time, since the gas stock in the reservour reduces along the reservour life span. These intermitent flow regimes have a cyclic nature, like for example, the severe slugging phenomenon, and they may have a tremendous impact in oil production. They may cause reservoir flow oscillations, high average back pressure at the well head, high instantaneous flow rates, which are difficult to control and eventually may cause the offshore oil production facility shutdown. Therefore, it would be of interest to know the regions in the pipeline-riser system parameter space where the two-phase flow has a steady state operation regime.

Linear stability analysis of an appropriate model of the two-phase flow in the pipeline riser system should provide tools to identify the regions in the pipeline-riser system flow parameter space where the two-phase flow has a steady state or not. To perform the linear stability analysis of a dynamic system, we need, at first, to identify the steady states. Then, we linearize the dynamic system governing equations with respect to a steady solution we would like to know to be stabe or not. The linearized governing equations govern how perturbations of the chosen steady solution evolves with time. If perturbations grows with time, the chosen steady solution is unstable, and otherwise it is stable. The growth rate of the perturbations of the chosen steady solution is given by the real part of the eigenvalues of the spectrum associated with the linearized governing equations. If all eigenvalues have negative real part, the chosen steady solution is stable, but if at least one eigenvalue has positive real part, the chosen steady solution is unstable. Once we are able to decide if a steady state is stable or not for a configuration of the pipeline-riser system flow parameters, we can search in the system parameter space for the stability boundary, which is the boundary between the regions where the considered steady solution is stable and the regions where it is unstable.

The objective of this work is to obtain a numerical approximation for the boundary in the pipeline-riser system flow parameter space between the regions where the two-phase flow has a steady state and the region where a steady state is not possible.

In the next section, we discuss the model for two-phase flow in pipeline-riser systems adopted in this work. This model is basically the model presented in (Baliño *et al.*, 2007) and is reproduced here to make the paper self contained. The third section describes the linear stability analysis of the two-phase flow model for pipeline-riser systems. We presente the

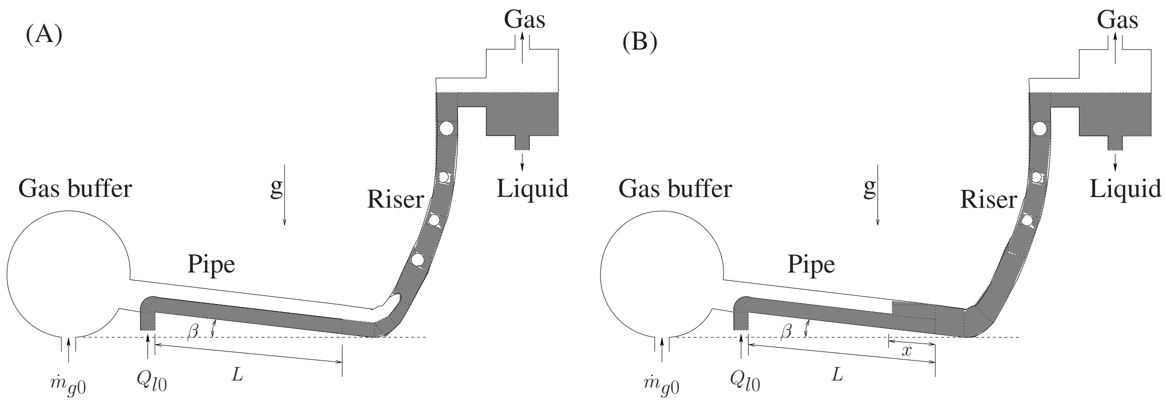


Figure 1. Part (A) - First configuration: $x = 0$. Part (B) - Second configuration: $x > 0$.

governing equations for the steady solutions. Then, we obtain the steady solution perturbation governing equations, which are linear partial-differential algebraic equations (PDAEs) with variable coeficients, and without an analytic solution. We apply the method of lines and Laplace transform to these equations and reduce them to a system of algebraic equations, from which we obtain numerically an approximation for the spectrum of the steady solution perturbation governing equations. With the spectrum we are able to decide about the stability of the steady solution. In the fourth section, we presente results from the numerical linear stability analysis applied to a vertical riser described in (Taitel *et al.*, 1996). The fifth section presents a discussion and conclusions.

## 2. TWO-PHASE FLOW MODEL

The pipeline-riser system is composed basically of two parts. The pipeline plus a gas buffer and the riser (see Fig. 1). The pipeline and the riser are connected at the bottom of the riser. The pressure at the top of the riser is assumed given and we have liquid and gas mass flowing into the pipeline.

The gas-liquid flow in the pipeline is assumed as always stratified. This flow behavior extends either to the whole pipeline (see part (A) of Fig. 1) or it extends until the liquid penetration position in the pipeline (see part (B) of Fig. 1). The configuration illustrated in part (A) of Fig. 1 corresponds to continuous gas flow from the pipeline into the riser and the configuration illustrated in part (B) of Fig. 1 corresponds to no gas flow from the pipeline into the riser and partial liquid flooding of the pipeline. Variables $Q_{l0}$, $\dot{m}_{g0}$, $\beta$, $L$, $g$ and $x$ are illustrated in Fig. 1 and represent, respectively, the volumetric flow rate of liquid into the pipeline, the gas mass flow rate into the pipeline, the pipeline inclination angle, the distance of the liquid inlet from the bottom of the riser, the gravity acceleration constant and the pipeline liquid flooding distance from the bottom of the riser (parts (B) of Fig. 1). We consider an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser.

In summary, we consider a set of two different configurations. The first one is illustrated in part (A) of Fig. 1. In this configuration we have stratified flow in the pipeline and continuous gas penetration from the pipeline into the riser.

The second configuration is illustrated in part (B) of Fig. 1, where we have stratified flow in part of the pipeline with liquid flooding until a distance $x$ from the bottom of the riser.

The set of governing equations is not the same for the two different configurations represented in Fig. 1. Below we give governing equations for only the configuration $x = 0$ ilustrated in part (A) of Fig. 1 above, since only this configuration has a steady solution. The second configuration comes into play only if the flow regime becomes intermittent.

### 2.1 Governing Equations for the Two-Phase Flow.

We give the governing equations for the two-phase flow in pipeline-riser system in non-dimensional form. We define the following non-dimensional variables according to the set of equations below.

$$x^* = \frac{x}{L_r}, \quad (1) \qquad P^* = \frac{P}{\rho_l R_g T_g}, \quad (3) \qquad t^* = t\frac{Q_{l0}}{AL_r}, \quad (5)$$

$$s^* = \frac{s}{L_r}, \quad (2) \qquad j^* = j\frac{A}{Q_{l0}}, \quad (4) \qquad \dot{m}^* = \frac{\dot{m}}{\rho_l Q_{l0}}, \quad (6)$$

where $L_r$ is the riser length, $A$ is the cross-sectional area of the pipeline and riser, $s$ is the space parameterization along the riser length, $T_g$ is the absolute temperature of the gas, $\rho_l$ is the liquid phase density, $R_g$ is the gas constant, $j$ stands for superficial velocity, $\dot{m}$ stands for mass flow rate, $P$ stands for pressure and $t$ stands for time. The variables with * as a superscript are non-dimensional variables.

### 2.1.1 Pipeline Governing Equations.

We first give the non-dimensional governing equation for the pipeline. We consider the gas in the pipeline behaving as a pressure cavity at non-dimensional pressure $P_g^*$, constant in position and evolving isothermically as a perfect gas. We consider a fixed control volume with the pipeline and gas buffer contours as the control volume surface. For this control volume, we obtain the mass conservation equation for each of the two phases. We have to consider two different situations at the pipeline. We have either continuous gas penetration from the pipeline into the riser ($x^* = 0$, see part (A) of Fig. 1) or partial liquid flooding of the pipeline ($x^* > 0$, see part (B) of Fig. 1). Since we are mainly intersted in the linear stability analalysis os steady solutions, we give below only the pipeline governing equations for the situation of continuous gas penetration in the riser (part (A) of Fig. 1). The situation of partial pipeline flooding comes into play only when the flow regime is intermitent. Steady solutions exist only for the situation of continuous gas penetraion in the riser ($x^* = 0$).

Below, we present the equations for the case $x^* = 0$. The liquid phase mass conservation equation is

$$-\delta \frac{d\alpha_p}{dt^*} + j_{lb}^* - 1 = 0. \tag{7}$$

Notice that in this case, the gas non-dimensional pressure $P_g^*$ is equal to the non-dimensional pressure at the bottom of the riser. Then, we use the riser bottom non-dimensional pressure $P_b^*$ instead of the gas non-dimensional pressure $P_g^*$ in the gas phase mass conservation equation, which is

$$(\delta\alpha_p + \delta_b)\frac{dP_b^*}{dt^*} + \delta P_b^* \frac{d\alpha_p}{dt^*} + P_b^* j_{gb}^* - \dot{m}_{g0}^* = 0, \tag{8}$$

where $j_{gb}^*$ is the gas non-dimensional superficial velocity at the bottom of the riser.

To close the model for the pipeline, we use an implicit algebraic relation for the pipeline void fraction $\alpha_p$ which relates it with the non-dimensional gas superficial velocity at the bottom of the riser $j_{gb}^*$, with the non-dimensional liquid superficial velocity at the bottom of the riser $j_{lb}^*$ and with the non-dimensional gas pressure $P_g^*$, and is derived from local momentum equilibrium for each phase of a stratified flow in a pipeline (Yemada and Dukler (1976), Kokal and Stanislav (1989) and others). For the case $x^* = 0$ we write

$$A_p(\alpha_p, j_{lb}^*, j_{gb}^*, P_b^*) = 0, \tag{9}$$

since in this case $P_b^* = P_g^*$. To derive these algebraic relations we assume stratified flow in the pipeline. We consider local momentum equilibrium for each phase and assume that the pressure gradient is the same for both phases. Then we eliminate the pressure gradient and end up with an algebraic relation for the quantities mentioned in the above paragraph. This procedure leads to an algebraic relation similar to Eq. (3) of Yemada and Dukler (1976).

### 2.1.2 Equations for the Riser.

For the riser, non-dimensional equations are derived from an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser. The mass conservation equation for the liquid phase is

$$-\frac{\partial \alpha_r}{\partial t^*} + \frac{\partial j_l^*}{\partial s^*} = 0, \tag{10}$$

where $j_l^*$ is the non-dimensional liquid superficial velocity along the riser and $\alpha_r$ is the void fraction along the riser. The mass conservation equation for the gas phase is

$$\frac{\partial}{\partial t^*}(P^*\alpha_r) + \frac{\partial}{\partial s^*}(P^* j_g^*) = 0, \tag{11}$$

where $P^*$ and $j_g^*$ are, respectively, the non-dimensional pressure and the non-dimensional gas superficial velocity along the riser.

We assume the inertia forces small and neglect them. We consider pressure variation due to the hydrostatic force and friction. The shear stress at the riser wall was modeled using a homogeneous two-phase flow model (Kokal and Stanislav (1989)) for the fluid and a Fanning friction coefficient $f_m$. Then, the linear momentum equation is

$$\frac{\partial P^*}{\partial s^*} = -\Pi_L[1 - \alpha_r + P^*\alpha_r]\left(\sin(\theta(s^*)) + \frac{4}{\Pi_D} f_m j^* |j^*|\right), \tag{12}$$

where the non-dimensional number $\Pi_L$ is given by the equation

$$\Pi_L = \frac{gL_r}{R_g T_g}. \tag{13}$$

This non-dimensional number is the ratio between the hydrostatic pressure at the bottom of the riser when it is filled completely with liquid and the gas pressure times the ratio between the gas and liquid densities. The non-dimensional number $\Pi_D$ is defined as

$$\Pi_D = \frac{2gDA^2}{Q_{l0}^2}. \tag{14}$$

and $\theta(s)^*$ is the local riser inclination angle at position along the riser arc length, $j^*$ is the sum of the liquid and gas superficial velocities and $f_m = f_m(R_{e,m}, \epsilon_r/D)$. The quantity $\varepsilon_r$ represent the riser wall roughness, $D$ represents the riser diameter and $R_{e,m}$ is the liquid-gas mixture Reynolds number given by

$$R_{e,m} = \frac{Q_{l0}D}{A\nu_l}\frac{(1-\alpha_r+P\alpha_r)|j^*|}{1-\alpha_r+\delta_\mu\alpha_r}, \tag{15}$$

where $\delta_\mu$ is the ratio between the gas and liquid dynamic viscosities. We consider the constitutive law corresponding to the drift flux model (Zuber and Findley (1965)) to relate the void fraction along the riser with the local values of the gas and liquid non-dimensional superficial velocities. Along the riser we have the relation

$$j_g^* = \alpha_r[C_d(j_l^* + j_g^*) + U_d^*]. \tag{16}$$

For the drift flux coefficients $C_d$ and $U_d^*$ we use the following correlation based on experimental data (Bendiksen (1984))

$$C_d = \begin{cases} 1,05 + 0,15\sin(\theta(s^*)) & \text{for } |j^*| < 3,5\frac{\sqrt{gD}A}{Q_{l0}} \\ 1,2 & \text{for } |j^*| \geq 3,5\frac{\sqrt{gD}A}{Q_{l0}} \end{cases} \tag{17}$$

$$U_d^* = \begin{cases} \frac{\sqrt{gD}A}{Q_{l0}}(0,35\sin(\theta(s^*)) + 0,54\cos(\theta(s^*))) \\ \qquad \text{for } |j^*| < 3,5\frac{\sqrt{gD}A}{Q_{l0}} \\ 0,35\frac{\sqrt{gD}A}{Q_{l0}}\sin(\theta(s^*)) \text{ for } |j^*| \geq 3,5\frac{\sqrt{gD}A}{Q_{l0}} \end{cases} \tag{18}$$

The boundary conditions are the pressure $P_t$ at the top of the riser which is given, the gas mass flow rate $\dot{m}_{g0}$ and the liquid volumetric flow rate $Q_{l0}$ (see Fig. 1 for details). The boundary condition at the top of the riser in non-dimensional form is $P_t^* = P_t/(\rho_l R_g T_g)$ .

Since we are working only with non-dimensional variables, and for simplicity, from now on we omit the superscript * from the equations.

## 3. LINEAR STABILITY ANALYSIS

In this section we describe the linear stability analysis applied to the governing equations for the two-phase flow in pipeline-riser systems given in the preoviuos section. The objective of the linear stability analysis, besides being able to decide if a steady solution is stable or not, is to obtain numerically in the parameter space of the two-phase flow in pipeline-riser the regions where the considered steady solution is stable and the regions where the considered steady solution is unstable, and consequently a numerical approximation for boundary between these two regions.

An outline follows. First, we obtain the steady solutions. We will see below that we have a single steady solution. Second, we obtain the governing equations for the perturbations of the steady solution. We write the dependent variables as their steady solution value plus a perturbation and subsitute into the two-phase flow governing equations. The resulting equations are linearized with respect to the perturbation variables. The perturbation variables governing equation is a linear system of partial-differential algebraic equations (PDAEs) with variable coefficients in the space variable $s$, and therefore, without analytic solution. We transform this linear system of PDAEs in a system of algebraic equations using, first, Laplace transform to deal with the time dependence and then the method of lines to deal with space derivatives. We solve this system of algebraic equations and perform inverse Laplace transform. We obtain a solution where the time growth rate is give by the real part of the system of linear PDAEs spectrum approximation elements (eigenvalues of the eigenvalue problem associated with the linear system of PDAEs). If the real part of all the spectrum elements are negative, the steady solution is stable, but if at least one element has positve real part, the steady solution is unstable. To obtain the regions in the system parameter space where the steady solution is stable or not, we consider a mesh over the system parameter space. For each point of the mesh, we perform the steps outlined above to decide if the steady solution is stable or not.

### 3.1 Steady Solution

Only the situation of continuous gas penetration in the riser ($x = 0$ for the pipeline) for two-phase flows in pipeline-riser systems has a steady solution. The equations for the steady solution are given by the Eqs. (7)-(9) and by the riser governing equations with the time partial derivatives set to zero ($\partial/\partial t = 0$). Liquid mass conservation equation for the pipeline ($x = 0$) reduces to

$$j_{lb} = 1. \tag{19}$$

Gas mass conservation equation for the pipeline ($x = 0$) reduces to

$$P_b j_g = \dot{m}_{g0}, \tag{20}$$

and the pipeline void fraction $\alpha_p$ is given by Eq. (9). Liquid mass conservation for the riser reduces to

$$\frac{\partial j_l}{\partial s} = 0 \rightarrow j_l = 1, \tag{21}$$

since $j_l(s = 0) = j_{lb} = 1$ (continuity condition between pipeline and riser bottom ($s = 0$) variables) according to Eq. (10). Gas mass conservation equation for the riser reduces to

$$\frac{\partial}{\partial s}(P j_g) = 0 \rightarrow P j_g = \dot{m}_{g0}, \tag{22}$$

since $P(s = 0)j_g(s = 0) = P_b j_{gb} = \dot{m}_{g0}$ (continuity between pipeline and riser bottom ($s = 0$) variables) according to Eq. (11). The linear momentum equation for the riser used to obtain the steady solution is Eq. (12). The constitutive law corresponding to the drift flux model used to determine the steady solution is given by Eq. (16).

The main difficulty to solve the governing equations for the steady solution results from Eq. (12) which is non-linear. For general riser geometries and no further simplifying assumptions, this set of equations has to be solved numerically.

To obtain a numerical approximation for the steady solution, we reduce the system of Eqs. (21), (22), (12) and (16) to a single non-linear differential equation for the pressure. Then, we consider a grid in the interval $s \in [0, 1]$ and obtain the pressure over the nodes of this grid using, for example the Runge-Kutta method, to integrate numerically the pressure nonlinear equation. Once we have the pressure over the grid points over the interval $s \in [0, 1]$, we first obtain $j_g$ from Eq. (22) and then we obtain $\alpha_r$ from Eq. (16).

### 3.2 Perturbation Equations

Here we give the governing equations for perturnations of the steady solutions. We write the dependent varibles which appear in Eqs. (7)-(9) and in Eqs. (10)-(16) as their steady solution values plus a perturbation and substitute them into the system governing equations. We linearize the resulting equations and obtain the perturbation governing equations. Next, we perform the first step. We write the dependent variables as

$$\alpha_r(s,t) = \tilde{\alpha_r}(s) + \bar{\alpha_r}(s,t), \tag{23}$$
$$P(s,t) = \tilde{P}(s) + \bar{P}(s,t), \tag{24}$$
$$j_l(s,t) = 1 + \bar{j}_l(s,t), \tag{25}$$
$$j_g(s,t) = \tilde{j}_g(s) + \bar{j}_g(s,t), \tag{26}$$
$$P_b(t) = \tilde{P}_b + \bar{P}_b(t), \tag{27}$$
$$j_{lb}(t) = \tilde{j}_{lb} + \bar{j}_{lb}(t), \tag{28}$$
$$j_{gb}(t) = \tilde{j}_{gb} + \bar{j}_{gb}(t). \tag{29}$$

We assume $\alpha_p$ constant and equal to its steady solution value. Next, we substitute Eqs. (27)-(29) into the governing equations for the pipeline for the case $x = 0$. The liquid and gas phase conservation equations assume, respectively, the form

$$\bar{j_{lb}} = 0 \qquad (30) \qquad \text{and} \qquad \left(\frac{L}{L_r}\tilde{\alpha_p} + \frac{L_b}{L_r}\right)\frac{d\bar{P}_g}{dt} + \bar{j_{gb}}\tilde{P}_g + \tilde{j_{gb}}\bar{P}_g = 0. \qquad (31)$$

These equations are the boundary contions for the perturbation governing equation for the riser at $s = 0$. The boundary condition at the top of the riser is given by the equation

$$\bar{P}(s = 1, t) = 0. \tag{32}$$

Next, we derive the perturbations governing equation for the riser. We substitute Eqs. (23)-(26) into the riser governing Eqs. (10)-(16) and linearize the resulting equations. We use the linearized drift relation for the perturbation variables to write the perturbation for the riser void fraction $\bar{\alpha}_r(s,t)$ in terms of the perturbations for the liquid and gas superficial velocities $\bar{j}_l(s,t)$ and $\bar{j}_g(s,t)$. This allow us to eliminate one of the governing equations for the perturbations of the steady solution and the perturbation variable $\bar{\alpha}_r(s,t)$. We end up with only three partial differential equations, which are given in matrix form by

$$[M(s)]\left\{\begin{array}{c}\frac{\partial \bar{j}_l}{\partial t}\\ \frac{\partial \bar{j}_g}{\partial t}\\ \frac{\partial \bar{P}}{\partial t}\end{array}\right\} + [K(s)]\left\{\begin{array}{c}\frac{\partial \bar{j}_l}{\partial s}\\ \frac{\partial \bar{j}_g}{\partial s}\\ \frac{\partial \bar{P}}{\partial s}\end{array}\right\} + [Q(s)]\left\{\begin{array}{c}\bar{j}_l\\ \bar{j}_g\\ \bar{P}\end{array}\right\} = 0, \tag{33}$$

where the matrices $[M(s)]$ and $[K(s)]$ and the elements of matrix $[Q(s)]$ are given below by the equations

$$[M(s)] = \begin{bmatrix} \tilde{\alpha}_r^2\tilde{C}_d & -\tilde{\alpha}_r(1-\tilde{\alpha}_r\tilde{C}_d) & 0\\ -\tilde{P}\tilde{\alpha}_r^2\tilde{C}_d & -\tilde{P}\tilde{\alpha}_r^2\tilde{C}_d & \tilde{j}_g\tilde{\alpha}_r\\ 0 & 0 & 0\end{bmatrix}, \qquad (34) \qquad [K(s)] = \begin{bmatrix} \tilde{j}_g & 0 & 0\\ 0 & \tilde{j}_g\tilde{P} & \tilde{j}_g^2\\ 0 & 0 & \tilde{j}_g\end{bmatrix}, \qquad (35)$$

$$Q_{22} = \tilde{j}_g\frac{d\tilde{P}}{ds} \qquad (36) \qquad Q_{23} = \tilde{j}_g\frac{d\tilde{j}_g}{ds} \qquad (37)$$

$$\begin{aligned} Q_{31} =& \frac{4\Pi_L}{\Pi_D}\tilde{j}_g\left[1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r\right]\left(2f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\right.\\ &+ D_1 f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\tilde{j}\frac{Q_{l0}D}{A\nu_l}\frac{1}{1-\tilde{\alpha}_r+\delta_u\tilde{\alpha}_r}\left[\frac{(1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r)}{\tilde{j}}|\tilde{j}| - (\tilde{P}-1)\frac{\tilde{\alpha}_r^2\tilde{C}_d}{\tilde{j}_g}|\tilde{j}|\right.\\ &\left.\left.+\frac{(1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r)|\tilde{j}|}{1-\tilde{\alpha}_r+\delta_u\tilde{\alpha}_r}(\delta_u-1)\frac{\tilde{\alpha}_r^2\tilde{C}_d}{\tilde{j}_g}\right]\right) - \Pi_L(\tilde{P}-1)\tilde{\alpha}_r^2\tilde{C}_d\left(\sin\theta + \frac{4}{\Pi_D}f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\tilde{j}\right) \end{aligned} \tag{38}$$

$$\begin{aligned} Q_{32} =& \frac{4\Pi_L}{\Pi_D}\tilde{j}_g\left[1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r\right]\left(2f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\right.\\ &+ D_1 f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\tilde{j}\frac{Q_{l0}D}{A\nu_l}\frac{1}{1-\tilde{\alpha}_r+\delta_u\tilde{\alpha}_r}\left[\frac{(1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r)}{\tilde{j}}|\tilde{j}|\right.\\ &\left.\left.+(\tilde{P}-1)\frac{\tilde{\alpha}_r}{\tilde{j}_g}(1-\tilde{\alpha}_r\tilde{C}_d)|\tilde{j}| - \frac{(1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r)|\tilde{j}|}{1-\tilde{\alpha}_r+\delta_u\tilde{\alpha}_r}(\delta_u-1)\frac{\tilde{\alpha}_r}{\tilde{j}_g}(1-\tilde{\alpha}_r\tilde{C}_d)\right]\right)\\ &+ \Pi_L(\tilde{P}-1)\tilde{\alpha}_r(1-\tilde{\alpha}_r\tilde{C}_d)\left(\sin\theta + \frac{4}{\Pi_D}f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\tilde{j}\right) \end{aligned} \tag{39}$$

$$\begin{aligned} Q_{33} =& \frac{4\Pi_L}{\Pi_D}\tilde{j}_g\left[1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r\right] D_1 f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\tilde{j}\frac{Q_{l0}D}{A\nu_l}\frac{|\tilde{j}|\tilde{\alpha}_r}{1-\tilde{\alpha}_r+\delta_u\tilde{\alpha}_r}\\ &+ \Pi_L\tilde{j}_g\tilde{\alpha}_r\left(\sin\theta + \frac{4}{\Pi_D}f_m(\tilde{R}_{e,m},\epsilon/D)|\tilde{j}|\tilde{j}\right) \end{aligned} \tag{40}$$

with the Reynolds number $\tilde{R}_{e,m}$ given by the equation

$$\begin{aligned}\bar{R}_{e,m} =& \frac{Q_{l0}D}{A\nu_l}\frac{1}{1-\tilde{\alpha}_r+\delta_u\tilde{\alpha}_r}\left\{\frac{(1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r)}{\tilde{j}}|\tilde{j}|j+|\tilde{j}|\tilde{\alpha}_r P+(\tilde{P}-1)|\tilde{j}|\left[\frac{\tilde{\alpha}_r}{\tilde{j}_g}(1-\tilde{\alpha}_r\tilde{C}_d)j_g-\frac{\tilde{\alpha}_r^2\tilde{C}_d}{\tilde{j}_g}j_l\right]\right. \\ &\left. -\frac{(1-\tilde{\alpha}_r+\tilde{P}\tilde{\alpha}_r)|\tilde{j}|}{1-\tilde{\alpha}_r+\delta_u\tilde{\alpha}_r}(\delta_u-1)\left[\frac{\tilde{\alpha}_r}{\tilde{j}_g}(1-\tilde{\alpha}_r\tilde{C}_d)j_g-\frac{\tilde{\alpha}_r^2\tilde{C}_d}{\tilde{j}_g}j_l\right]\right\}\end{aligned} \quad (41)$$

### 3.3 Laplace Transform and The Method of Lines

Next, we apply Laplace transform to handle the time dependence and the method of lines to handle the spatial derivatives. We consider an evenly spaced grid of $N+1$ nodes over the interval $s \in [0,1]$. We approximate the diferential operator $d/ds$ by a finite difference operator. We use the five point formula for the finite difference operator. The final result is a system of algebraic equations with dimension $3N+2$. We consider the Laplace transform pair

$$\bar{\phi}(t) = \frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\hat{\phi}(\omega)\exp(\omega t)d\omega \quad (42) \qquad \hat{\phi}(\omega) = \int_0^{\infty}\bar{\phi}(t)\exp(-\omega t)dt. \quad (43)$$

Over the grid of $N+1$ nodes, we consider the following equations:

- For the node $k=1$, we consider Laplace transform of the gas and liquid conservation equations for the pipline and the discretization of the Laplace transform of the linear momentum equation. We have:

$$\hat{j_{lb}} = 0 \quad (44) \qquad \text{and} \qquad \omega\left(\frac{L}{L_r}\tilde{\alpha_p}+\frac{L_b}{L_r}\right)\hat{P}_g+\hat{j_{gb}}\tilde{P}_g+\tilde{j_{gb}}\hat{P}_g = 0. \quad (45)$$

$$\tilde{j}_{g,1}\sum_{m=a}^{b}(D_1^5)_{1,m}\hat{P}_m + A_{3,1}(s_1)\hat{j}_{l,1}+A_{3,2}(s_1)\hat{j}_{g,1}+A_{3,3}(s_1)\hat{P}_1 = 0 \quad (46)$$

- For the nodes $k=2,\ldots,N$, we consider the system of equations (33). After the Laplace transform and the spatial discretization we have:

$$\omega(M_{1,1}(s_k)\hat{j}_{l,k}+M_{1,2}(s_k)\hat{j}_{g,k})+K_{1,1}(s_k)\sum_{m=a}^{b}(D_1^5)_{k,m}\hat{j}_{l,m} = M_{1,2}(s_k)\bar{j}_{l,k}(0)+M_{1,2}(s_k)\bar{j}_{g,k}(0) \quad (47)$$

$$\begin{aligned}&\omega(M_{2,1}(s_k)\hat{j}_{l,k}+M_{2,2}(s_k)\hat{j}_{g,k}+M_{2,3}(s_k)\hat{P}_k+\tilde{j}_{g,k}\left\{\sum_{m=a}^{b}(D_1^5)_{k,m}(\tilde{P}_m\hat{j}_{g,k})+\sum_{m=a}^{b}(D_1^5)_{k,m}(\tilde{j}_{g,m}\bar{P}_m)\right\} \\ &A_{2,2}(s_k)\hat{j}_{g,k}+A_{2,3}(s_k)\hat{P}_k = M_{2,1}(s_k)\bar{j}_{l,k}(0)+M_{2,2}(s_k)\bar{j}_{g,k}(0)+M_{2,3}(s_k)\bar{P}_k(0)\end{aligned} \quad (48)$$

$$\tilde{j}_{g,k}\sum_{m=a}^{b}(D_1^5)_{k,m}\hat{P}_m + A_{3,1}(s_k)\hat{j}_{l,k}+A_{3,2}(s_k)\hat{j}_{g,k}+A_{3,3}(s_k)\hat{P}_k = 0 \quad (49)$$

where $a=1$ and $b=5$ for $k=2$, $a=k-2$ and $b=k+2$ for $2<k<N$ and $a=N-3$ and $b=N+1$ for $k=N$. The symbol $(D_i^l)_{k,m}$ applied to $f_m$ stands for the coefficient which multiplies $f_m = f(s_k+(k-m)\Delta s)$ of the $l$ nodes formula for the finite difference operator of order $i$ applied to the $k$-th grid node.

- For the node $k=N+1$ we consider the two first equations of the sistem of equations (33) and the boundary condition ate the top of the riser. After the Laplace transform and the spatial discretization we have:

$$\omega(M_{1,1}(s_k)\hat{j}_{l,k}+M_{1,2}(s_k)\hat{j}_{g,k})+K_{1,1}(s_k)\sum_{m=N-3}^{N+1}(D_1^5)_{k,m}\hat{j}_{l,m} = M_{1,2}(s_k)\bar{j}_{l,k}(0)+M_{1,2}(s_k)\bar{j}_{g,k}(0) \quad (50)$$

with $k=N+1$,

$$\omega(M_{2,1}(s_k)\hat{j}_{l,k} + M_{2,2}(s_k)\hat{j}_{g,k} + M_{2,3}(s_k)\hat{P}_k + \tilde{j}_{g,k}\left\{\sum_{m=N-3}^{N+1}(D_1^5)_{k,m}(\tilde{P}_m\hat{j}_{g,k}) + \sum_{m=N-3}^{N+1}(D_1^5)_{k,m}(\tilde{j}_{g,m}\bar{P}_m)\right\}$$
$$A_{2,2}(s_k)\hat{j}_{g,k} + A_{2,3}(s_k)\hat{P}_k = M_{2,1}(s_k)\bar{j}_{l,k}(0) + M_{2,2}(s_k)\bar{j}_{g,k}(0) + M_{2,3}(s_k)\bar{P}_k(0) \text{ with } k = N+1 \tag{51}$$

$$\hat{P}_{N+1} = 0 \tag{52}$$

The equation (44)-(51) form a system of algebraic equations, which can be written in matrix form as

$$(\omega[H] + [G])\left\{\begin{array}{c}\{\hat{j}_l\}\\ \{\hat{j}_g\}\\ \{\hat{P}\}\end{array}\right\} = \left\{\begin{array}{c}\{F_1\}\\ \{F_2\}\\ \{F_3\}\end{array}\right\} \tag{53}$$

of dimension $3N+2$ for $3N+2$ variables, which are $\hat{j}_{l,1},\ldots,\hat{j}_{l,N+1}$, $j_{g,1},\ldots,\hat{j}_{g,N+1}$, $\hat{P}_1,\ldots,\hat{P}_N$. The first line of this system is equation (44). The second to the $N+1$-th line represents the discretization of the Laplace transform of the liquid phase mass conservation equation. The $N+2$-th line is equation (45) and the lines $N+3$ to $2N+2$ represent the discretization of the Laplace transform of the gas phase mass conservation equation. Lines $2N+3$ to $3N+2$ represents the discretization of the Laplace trandform of the Linear momentum equation. Vector $\{F_3\} = 0$, and the elements of the vectors $\{F_1\}$ and $\{F_2\}$ can be extracted from Eqs. (44)-(52).

### 3.4 Stability Criteria

We solve the sistem of equations (53). Since matrix $[H]$ is singular, we perform its singular value decomposition. We consider non singular matrices $[E]$ and $[F]$ such that

$$\left(\omega\left[\begin{array}{cc}[\Gamma]_{r\times r} & 0_{r\times n-r}\\ 0_{n-r\times r} & 0_{n-r\times n-r}\end{array}\right] + [E][G][F]\right)\{x\} = [E]\{F\} \tag{54}$$

where $n = 3N+2$, matrix $[\Gamma]$ has the non-zero singular values of matrix $[H]$ in its main diagonal. This matrix has dimension $r\times r$, which is the rank of matrix $[H]$. The vector $\{\{j_l\}^T\ \{j_g\}^T\ \{P\}^T\} = [F^T]\{x\}^T$ and vector $\{\tilde{F}\}^T = [E]\{\{F_1\}^T\ \{F_2\}^T\ \{F_3\}^T\}$. We define matrix $[\tilde{G}] = [E][G][F]$. If the block $[\tilde{G}_{2,2}]$ is non-singular, the equation can be written as

$$(\omega[\Gamma] + [L])\{x_1\} = \{\tilde{F}_1\} - [\tilde{G}_{2,2}^{-1}]\{\tilde{F}_2\} \tag{55}$$

where matrix $[L] = [\tilde{G}_{1,1}] - [\tilde{G}_{1,2}][\tilde{G}_{2,2}^{-1}][\tilde{G}_{2,1}]$ and we decompose the vectors $\{x\}^T = \{\{x_1\}^T\ \{x_2\}^T\}$ and $\{\tilde{F}\}^T = \{\{\tilde{F}_1\}^T\ \{\tilde{F}_2\}^T\}$. This matrix equation can be solved and its solution is

$$\{x_1\} = \frac{1}{\det[L] + \omega[\Gamma]}Adj\{([L] + \omega[\Gamma])\}\left\{\{\tilde{F}_1\} - [\tilde{G}_{2,2}^{-1}]\{\tilde{F}_2\}\right\} \tag{56}$$

where $Adj\{[G]\}$ is the matrix formed by the cofactors of matrix $[G]$. We have that $Adj\{([L]+\omega[\Gamma])\}_{k,m} = (-1)^{(k+m)}q_{k,n}\{([L]+\omega[\Gamma])\}$, where $q_{k,m}\{[G]\}$ is the determinant of the $r-1\times r-1$ matrix obtained by omitting the $k$-th line and the $m$-th column of matrix $[G]$. If we perform the inverse Laplace transform of Eq. (56), we obtain

$$\{x_1(t)\} = \frac{1}{2\pi}\left\{\sum_{k=1}^{n-r} 2\pi i\ Res\left[\frac{1}{\det\{([L]+\omega[\Gamma])\}}Adj\{([L]+\omega[\Gamma])\}\left(\{\tilde{F}_1\} - [\tilde{G}_{2,2}^{-1}]\{\tilde{F}_2\}\right)\right]_{\omega=\omega_k}\exp(\omega_k t)\right\} \tag{57}$$

where $\omega_k, k = 1,\ldots,r$ are the roots of the polynomial $\det\{[L]+\omega[\Gamma]\} = 0$. According to Eq. (57), we see that if all $\omega_k$ have negative real part, $\{x_1(t)\} \to 0$ and perturbations of the steady solution vanishes with time, which implies that the steady solution is stable. If at least one $\omega_k$ has positive real part, $\{x_1(t)\} \to \infty$, and perturbations of the steady solution grows without bound, which implies the steady solution to be unstable. This is the stability criteria.

## 4. RESULTS

We present results regarding the regions where the steady solution is stable or unstable. We consider a mesh over the $j_{l0} \times j_{g0}$ plane, where $j_{l0}$ ($j_{g0}$) is the superficial liquied (gas) velocity at the pipeline inlet. We keep all other system parameters and boundary conditions constant. Over each point of the mesh we perform numerically the stability analysis outlined above. For each point of the mesh we have a single steady solution, and we mark the mesh point as blue (red) if the steady solution is unstable (stable). We end up with the region where the steady solution is stable (unstable) defined by red (blue) dots in the $j_{l0} \times j_{g0}$ plane as illustrated in the Fig: 2 below. The stability boundary lays in the inteface between the stable steady solution region and the unstable steady solution region. We compare the linear stability analysis results with the stability boundaries in the $j_{l0} \times j_{g0}$ plane illustrated in Fig: 3. These stability boundaries are obtained from the time simulation of the two-phase flow governing equations in a pipeline-riser system. We assume a grid over the $j_{l0} \times j_{g0}$ plane. For each point we do a reasonably long enough numerical time simulation with the steady solution plus an infinitesimal perturbation as initial condition. If the numerical solution grows with time, we label this steady solution unstable, but if the numerical solution stays in a neighbourhood of the initial condition, we label it stable. By refining the grid in the region between the stable steady solution region and the unstable steady solution region, we obtained the stability curves in Fig. 3. The full line is for $L_b = 10\ m$, the dashed line is for $L_b = 5.1\ m$ and the dash dotted line is for $L_b = 1.69\ m$. The pipeline-riser system considered is described in Taitel *et al.* (1996), and the parameters value are given below. This approach to find the stability boundary is much more time consuming and computationaly intensive than the linear stability analysis.

$$
\begin{array}{lll}
P_t = 1.01325 \times 10^5\ Pa, & D = 0.0254\ m, & T_g = 293\ K, \\
L = 9.1\ m, & \theta = 90^o, & R_g = 287.336\ m^2 sec^{-2} K^{-1}, \\
L_r = 3.0\ m, & \beta = 5^o, & \rho_l = 1000\ kg.m^{-3}, \\
\epsilon = 1.5 \times 10^{-6}, & \mu_g = 1.8 \times 10^{-5}\ kg/m/s, & \mu_l = 1.0 \times 10^{-3}\ kg/m/s.
\end{array}
$$

For this numerical exercise we use $N = 50$ for the discretization of the interval $s \in [0, 1]$ and the numerical precision considered was $10^{-12}$.

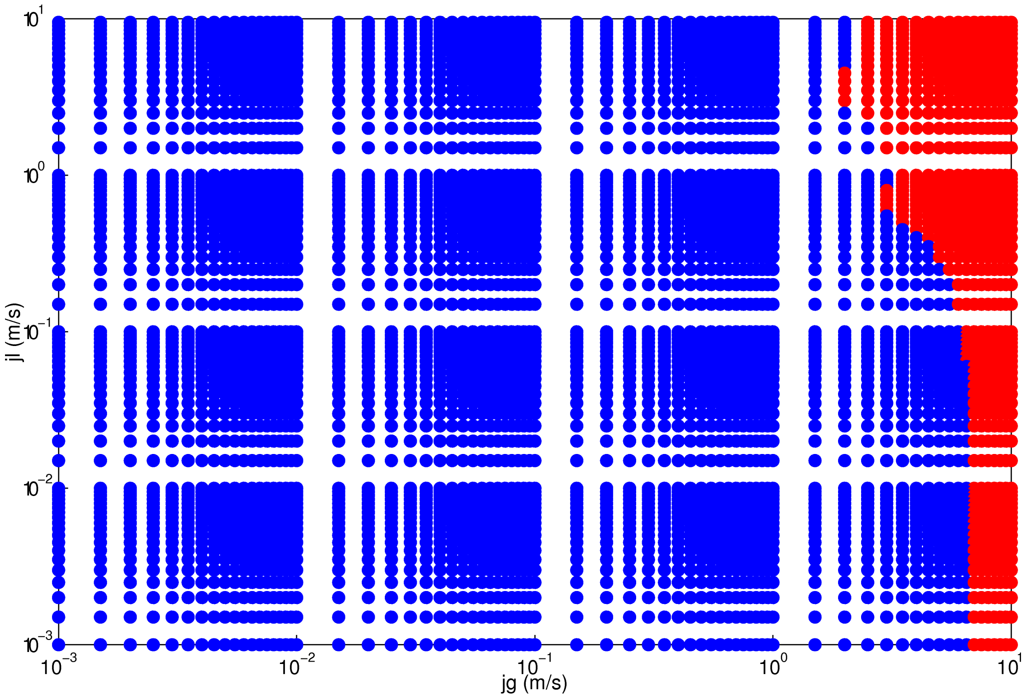


Figure 2. Stable stable steady solution region and unstable steady solution region in the $j_{l0}$ and $j_{g0}$ for $L_b = 1.69\ m$. Points in the stable steady solution region are marked red and points in the unstable steady solution region are marked blue.

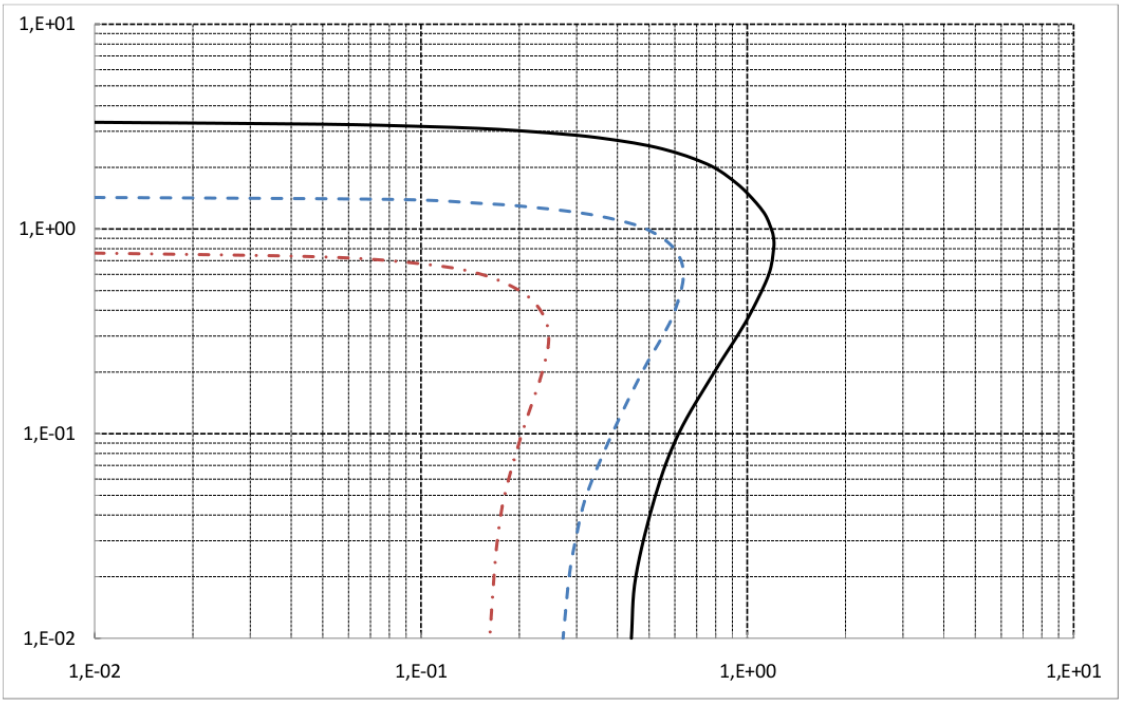


Figure 3. Stability boundaries in the $j_{l0}$ and $j_{g0}$. For $L_b = 1.69\ m$:-·-·-·-·- , for $L_b = 5.1\ m$: - - - - - - and for $L = 10\ m$: ——— .

## 5. DISCUSSION AND CONCLUSIONS

If we compare both Fig. 2 and Fig. 3 we see a very different qualitative behaviour. We expected at least to obtain the same qualitative behaviour, but the results from the numerical linear stability analysis do not capture the horizontal part of the stability boundaries illustrated in Fig. 3. Even the vertical branch of the stability boundary from the numerical stability analysis is shifted for larger values of $j_{g0}$ than what we observe in Fig. 3. These results lead us to search in the literature for application of numerical stability analysis of similar problems, like reference Hanke *et al.* (2007). They study the stability of the steady solution for an carbon-dioxide evaporator. Their model is a non-linear PDAE. Instead of linearizing the PDAE with respect to the steady solution, and then apply the method of lines to solve the linear PDAE and study its spectrum to decide about the stability of the steady solution, they apply the method of lines to the non-linear PDAE to obtain a system of non-linear DAEs. Then, they obtained the steady solution of this system of DAEs, and study its linear stability by linearizing the sistem of non-linear DAEs with respect to the steady solution. They show few linear stability analysis results that seems correct according to them.

So far we do not know why our approach to the linear stability analysis did not give the expected results. We performed a literature search about linear stability analysis for PDAEs and DAEs. We found a good number of references for DAEs in the context of non-linear circuits dynamics, but very few references about stability analysis of PDAEs. See reference Riaza (2010) and references therein for stability analysis of steady solutions of DAEs.

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