GENERALIZED SOLUTION FOR THE LAMINAR, FREE CONVECTION FLOWS ALONG A VERTICAL FLAT PLATE

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Abstract. *The classical boundary layer similar solution for natural convection flows along a vertical flat plate is valid only in the regions far form the leading edge, where the flow pattern can be considered as fully developed. Thus, a simplified solution which can describe accurately the flow in the leading edge vicinity is of great scientific interest. In this paper, a new general formulation of the free convection boundary-layer flow is obtained using the generalized boundary layer formulation and the similarity analysis approach. The partial differential equations are reduced to an ordinary differential equation through an adapted change of coordinates. The resulting ordinary differential equation will be numerically solved and the velocity and temperature profiles obtained will be used to describe the near leading edge behavior of the heat and the momentum transfer.*

Keywords: *Wind Turbines, Diffusers, BEM Method, Glauert's Model*

1. INTRODUCTION

The laminar, natural convection boundary-layer development along a vertical and semi-infinite flat plate is an important problem in the study of heat transfer in external surfaces. In spite of its importance, the complete solution of this problem is still not available. The main unanswered question is related to the behavior of the velocity and the temperature fields near the flat plate leading edge. It is well know that the classical boundary layer similar solution is valid at the regions far from the leading edge where the Grashof number is high and the flow pattern can be considered as fully developed. In an effort to obtain a better approximation for moderate Grashof number, Yang and Jerger (1964) developed a second order boundary layer correction solution using the method of matched asymptotic expansion. Their correction contributes with a negative value term for the higher order correction of Nusselt number, indicating that the value of the integrated heat transfer diminishes as the leading edge approaches. Other important work was developed by Messiter and Liñán (1976) who, using the so called "triple deck" approach, calculated the leading effects into the integrated heat transfer. In this case, the contribution of the correction is positive and furnishes an increment for the integrated heat transfer. There are several work efforts to develop efficient tools to solve the problem, Pantokratoras (2001) analyzed the problem taking into account the temperature dependence of all physical properties and the results are obtained with the numerical solution of the boundary layers equations. Hossain (2000) developed a two-dimensional analysis of mixed convective flow in incompressible viscous fluids, and the temperature dependent viscosity in a fluid impermeable. The results show the effects due to the viscosity variation parameter using an implicit finite difference method. In the present work, an alternative formulation for the laminar natural convection phenomenon on a vertical flat plate is proposed. The concept of principal equation proposed by Kaplun (1967) was applied to the Cartesian system Navier-Stokes equations to obtain a generalized boundary layer formulation. This formulation requires no division between inner and outer solution, which precludes any type of matching procedure or some kind of viscous-inviscid interaction. In a next step, a quasi similar ordinary differential equation is obtained from the generalized boundary layer formulation (this formulation is described in detail by Cruz (2002)). It is shown that the quasi-similar equation can describe the flow near the leading edge. A comparison of the results of the present formulation with the theories of Yang and Jerger (1964) and Messiter and Liñán (1976) will be made.

The outline of this paper is as follow: in section 2, the Generalized Boundary Layer Equation formulation is described. With this formulation, the Navier-Stokes system can be simplified using asymptotic analysis and a system of 4 differential equation (momentum at x and y-direction, continuity and heat transfer) is obtained. In section 3, the previous system is recasted in two differential equation by the use of similarity analysis. In next section, some results are presented and compared with other classical similarity analysis. At last section, some conclusions are drawn.

2. Problem Description

We consider a laminar, steady-state flow of a incompressible Newtonian fluid in the vicinity of a semi infinite flat plate shown if Figure (1). The fluid far from the plate is at rest with a temperature T_∞. The temperature at the plate has a fixed value of T_w. The coordinate x is taken parallel at plate and y position is measured in normal direction. Considering the plate as infinite in direction normal to $x - y$ plane, the problem can be treated as bi-dimensional one. The temperature is $T \equiv T(x, y)$ with $T_\infty \leq T \leq T_w$. The velocity components are $u \equiv u(x, y)$ in x direction and $v \equiv v(x, y)$ in y direction.

The gravitational acceleration g is parallel to the plate. As long as there is a difference of temperature between the wall and fluid, a buoyancy force will act in the fluid particles. Using the Boussinesq approximation, this bulk force could be treated introducing a buoyancy term into Navier-Stokes equations which, along with the mass and energy conservation equations, will result in the following set of differential equations:

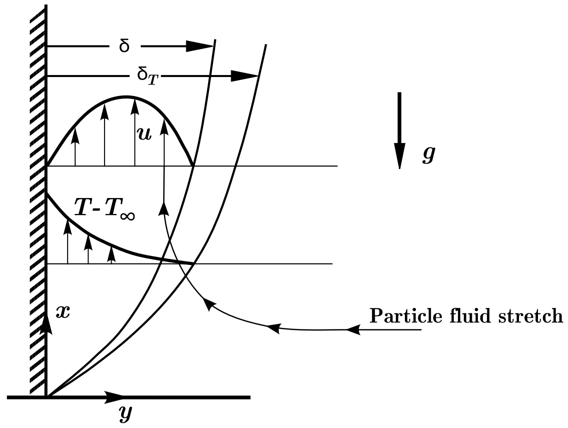

Figure 1. Natural convection phenomena

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g\beta(T - T_\infty) \quad (2)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (3)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_\infty \frac{\partial T^2}{\partial y^2} \quad (4)$$

Restrained to the following boundary conditions

$$\text{at} \quad y = 0 \begin{cases} u = 0 \\ v = 0 \\ T = T_w \end{cases} \qquad \text{at} \quad y \to \infty \begin{cases} u \to 0 \\ T \to T_\infty \end{cases} \quad (5)$$

In the above equations the variables are made non-dimensional using a characteristic length L and a characteristic velocity of the flow U (**?**). P is the non-dimensional pressure (the ratio of dimensional pressure and fluid density) and the parameters u and v represents the non-dimensional velocities. ν is the fluid kinematic viscosity. The parameter $Re = \frac{UL}{\nu}$ represents the Reynolds number.

3. The Generalized Boundary Layer Equation

The concept of distinguished limits (Kaplun, 1967) can be used to determinate the asymptotic behavior of the Navier-Stokes equation as $Re \to \infty$. The mathematical framework necessary to the obtain the high Reynolds number asymptotic behavior of the Navier-Stokes is exhaustively discussed in Cruz (2002), Silva (2003) and Silva Freire (1999). Here, just some of the relevant steps are be presented. The intermediate variables are defined as:

$$\hat{y} = \frac{y}{\eta(\varepsilon)} \quad (6)$$

$$\hat{v} = \frac{v}{\eta(\varepsilon)} \quad (7)$$

where $\varepsilon = 1/Re$. Equations (6) and (7) can be inserted into eqs. (1), (2) and (3) to obtain:

$$\frac{\partial u}{\partial x} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \quad (8)$$

$$u\frac{\partial u}{\partial x}+\hat{v}\frac{\partial u}{\partial \hat{y}}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{\eta(\varepsilon)^2}\frac{\partial^2 u}{\partial \hat{y}^2}\right)+g\beta(T-T_\infty) \tag{9}$$

$$\eta(\varepsilon)u\frac{\partial \hat{v}}{\partial x}+\eta(\varepsilon)\hat{v}\frac{\partial \hat{v}}{\partial y}=-\frac{1}{\eta(\varepsilon)}\frac{\partial P}{\partial \hat{y}}+\frac{1}{Re}\left(\eta(\varepsilon)\frac{\partial^2 \hat{v}}{\partial x^2}+\frac{1}{\eta(\varepsilon)}\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right) \tag{10}$$

Applying the η-limit onto Eqs. (9) and (10), we could estimate the order of magnitude of each term in these equations (please refer to Cruz (2002) for a detailed description of application of η-limit to this equation). From this analysis, in the limit as $Re \to \infty$, the Navier-Stokes equations behaviour is adequately described by the following set of equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{11}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty) \tag{12}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y} \tag{13}$$

and the energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_\infty\frac{\partial T^2}{\partial y^2} \tag{14}$$

Equations (11) to (14) represent a generalized form of the boundary layer theory which combines the Euler inviscid flow equations and the Prandtl classical boundary layer formulation into a single and more widespread approach. Using stream function definition

$$u=\frac{\partial \Psi}{\partial y} \qquad v=-\frac{\partial \Psi}{\partial x} \tag{15}$$

The continuity equation (Equation (11)) is automatically satisfied and for Eqs.(12) and (13) result

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y}-\frac{\partial^2 \Psi}{\partial y^2}\frac{\partial \Psi}{\partial x}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\frac{\partial^3 \Psi}{\partial y^3}+g\beta(T-T_\infty) \tag{16}$$

$$\frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial x \partial y}-\frac{\partial^2 \Psi}{\partial x^2}\frac{\partial \Psi}{\partial y}=-\frac{\partial P}{\partial y} \tag{17}$$

Summing the derivative of all terms in equation (16) in relation y with the derivative of all terms in equation (17) in relation x, we can eliminate the pressure and obtain, along with energy equation:

$$-\frac{\partial \Psi}{\partial x}\left(\frac{\partial^3 \Psi}{\partial y^3}+\frac{\partial^3 \Psi}{\partial x^2 \partial y}\right)+\frac{\partial \Psi}{\partial y}\left(\frac{\partial^3 \Psi}{\partial x^3}+\frac{\partial^3 \Psi}{\partial x \partial y^2}\right)=\frac{1}{Re}\frac{\partial^4 \Psi}{\partial y^4}+g\beta\frac{\partial T}{\partial y} \tag{18}$$

$$\frac{\partial \Psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial \Psi}{\partial x}\frac{\partial T}{\partial y}=\alpha_\infty\frac{\partial T^2}{\partial y^2} \tag{19}$$

4. The near leading edge flat plate flow and the quasi-similar equation

The classical approach to describe the laminar solution for the natural convection phenomena on a vertical flat plate was presented by Ostrach (1953). In his solution he proposed a similarity transformation for the stream function as $\Psi(x,y)=4\nu_\infty c x^{3/4}F(\eta)$. In this paper, we extend this approach and assume that the solution has the form:

$$\Psi(x,y)=4\nu_\infty c x^{3/4}\sum_{i=0}^{\infty}Gr^{-i}F_i(\eta) \tag{20}$$

And similarly for T

$$T(x,y)=T_\infty+(T_w-T_\infty)\sum_{i=0}^{\infty}Gr^{-i}\theta_i(\eta) \tag{21}$$

Where

$$\eta = y\frac{c}{x^{1/4}} \tag{22}$$

$$c^4 = \frac{g|\rho_\infty - \rho_w|}{4\nu_\infty^2\rho_\infty} \tag{23}$$

$$Gr = \frac{g\,|T_w - T_\infty|\,\beta x^3}{\nu_\infty^2} = \sigma^3 x^3 \tag{24}$$

In the above equations η is the similarity coordinate and Gr is the Grashof number. Eq. (20) represents a expansion in powers of Gr, where the functions F_i represents various orders corrections to the solution in order to account small values of Gr number. Inserting Eqs. (20) and (21) into Eqs. (18) and (19), and keeping the small order corrections terms in expansion, we obtain:

$$\begin{aligned}
&-F_0^{(4)} + F_0'F_0'' - 3F_0F_0^{(3)} - \theta_0' \\
&+\tfrac{1}{8\sqrt{Gr}}\left(-5F_0'F_0''\eta^2 - 3F_0F_0^{(3)}\eta^2 + 5\left(F_0'\right)^2\eta - 3F_0F_0''\eta + 27F_0F_0'\right) \\
&+\tfrac{1}{Gr}\left(F_1'F_0'' - 11F_0'F_1'' + 9F_1F_0^{(3)} - 3F_0F_1^{(3)} - F_1^{(4)} - \theta_1'\right) \\
&+\tfrac{1}{8Gr^{3/2}}\Big(-5F_1'F_0''\eta^2 - 17F_0'F_1''\eta^2 + 9F_1F_0^{(3)}\eta^2 - 3F_0F_1^{(3)}\eta^2 - 386F_0'F_1'\eta \\
&\qquad +9F_1F_0''\eta - 75F_0F_1''\eta - 2025F_1F_0' - 405F_0F_1'\Big) \\
&+\tfrac{1}{Gr^2}\left(F_2'F_0'' - 11F_1'F_1'' - 23F_0'F_2'' + 21F_2F_0^{(3)} + 9F_1F_1^{(3)} - 3F_0F_2^{(3)} - F_2^{(4)} - \theta_2'\right) \\
&+\tfrac{1}{8Gr^{5/2}}\Big(-5F_2'F_0''\eta^2 - 17F_1'F_1''\eta^2 - 29F_0'F_2''\eta^2 + 21F_2F_0^{(3)}\eta^2 + 9F_1F_1^{(3)}\eta^2 \\
&\qquad -3F_0F_2^{(3)}\eta^2 - 391\left(F_1'\right)^2\eta - 1358F_0'F_2'\eta + 21F_2F_0''\eta + 225F_1F_1''\eta \\
&\qquad -147F_0F_2''\eta - 15309F_2F_0' - 729F_1F_1' - 1701F_0F_2'\Big) \\
&+\tfrac{1}{Gr^3}\left(-11F_2'F_1'' - 23F_1'F_2'' - 23F_2'F_2'' + 21F_2F_1^{(3)} + 9F_1F_2^{(3)} + 21F_2F_2^{(3)}\right) \\
&+\tfrac{1}{8Gr^{7/2}}\Big(-17F_2'F_1''\eta^2 - 29F_1'F_2''\eta^2 + 21F_2F_1^{(3)}\eta^2 + 9F_1F_2^{(3)}\eta^2 \\
&\qquad -1754F_1'F_2'\eta + 525F_2F_1''\eta + 441F_1F_2''\eta - 12285F_2F_1' + 3159F_1F_2'\Big) \\
&+\tfrac{1}{8Gr^{9/2}}\left(-29F_2'F_2''\eta^2 + 21F_2F_2^{(3)}\eta^2 - 1363\left(F_2'\right)^2\eta + 1029F_2F_2''\eta - 3213F_2F_2'\right) \\
&+\tfrac{1}{Gr^4}\left(21F_2F_2^{(3)} - 23F_2'F_2''\right) \\
&= 0
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\theta_0'' + 3PrF_0\theta_0' \\
&+\tfrac{1}{Gr}(\theta_1'' + 3PrF_0\theta_1' - 9PrF_1\theta_0' + 12PrF_0'\theta_1) \\
&+\tfrac{1}{Gr^2}(\theta_2'' + 3PrF_0\theta_2' - 9PrF_1\theta_1' + 12PrF_1'\theta_1 - 21PrF_2\theta_0' + 24PrF_0'\theta_2) \\
&+\tfrac{1}{Gr^3}(-9PrF_1\theta_2' + 12PrF_2'\theta_1 - 21PrF_2\theta_1' + 24PrF_1'\theta_2) \\
&+\tfrac{1}{Gr^4}(-21PrF_2\theta_2' + 24PrF_2'\theta_2) \\
&= 0
\end{aligned} \tag{26}$$

Where $F_i^{(j)} = \frac{d^{(j)}F_i}{d\eta^{(j)}}$. In this work, we present the solution only to zeroth order term, or

$$\begin{aligned}
&F_0^{(4)} - F_0'F_0'' + 3F_0F_0^{(3)} + \theta_0' \\
&+\tfrac{1}{8\sqrt{Gr}}\left(5F_0'F_0''\eta^2 + 3F_0F_0^{(3)}\eta^2 - 5\left(F_0'\right)^2\eta + 3F_0F_0''\eta - 27F_0F_0'\right) = 0
\end{aligned} \tag{27}$$

and

$$\theta_0'' + 3PrF_0\theta_0' = 0 \tag{28}$$

The boundary conditions stated by Eq. (5) results

$$\text{at}\quad \eta = 0\left\{\begin{array}{lcl} F_0'(\eta) & = & 0 \\ F_0(\eta) & = & 0 \\ \theta_0(\eta) & = & 1 \end{array}\right. \qquad \text{as}\quad \eta \to \tilde{\eta}_\infty\left\{\begin{array}{lcl} F_0'(\eta) & \to & 0 \\ F_0''(\eta) & \to & 0 \\ \theta_0(\eta) & \to & 0 \end{array}\right. \tag{29}$$

Equation (27) represents a quasi-similar ordinary differential equation in the sense that not all of its coefficients are constants. This is an important characteristic since although this quasi-similar equation is an ordinary one or, in other words, the derivatives that are present in the equation are related to η, it still carry some information relative to the non-similar developing region near the leading edge of the flow.

It is important to notice that the derivatives related to ξ were disregarded on the equations below since: who is ξ?

$$\frac{\partial F_0}{\partial \xi}\frac{\partial \xi}{\partial x} \to \infty \qquad \text{as} \quad x \to 0 \tag{30}$$

Those derivatives are only important if:

$$\mathcal{O}\left(\frac{\partial F_0}{\partial \eta}/\frac{\partial F_0}{\partial \xi}\right) = \mathcal{O}(1) \qquad \text{or/and} \qquad \mathcal{O}\left(\frac{\eta}{\xi}\right) = \mathcal{O}(1) \tag{31}$$

In the present analysis η, represents a parameter analogous to the Blasius boundary non-dimensional similarity variable. Near the wall where the analysis is performed:

$$\mathcal{O}\left(\eta\right) = \mathcal{O}(1) \tag{32}$$

Then according to equation (30)

$$\mathcal{O}\left(\xi\right) = \mathcal{O}(1) \tag{33}$$

The region $\mathcal{O}\left(\xi\right)\mathcal{O}(1)$ represents the beginning of the leading edge zone, and the limit of validity of the present analysis.

5. Results and Discussion

Equations (26) and (27) were numerically solved using the FORTRAN-IMSL framework with the IVPAG subroutine. Figures 2, 3, 4 and 5 show the tangential velocity and temperatures profiles for some values of the Grashof and Prandtl numbers. The developing behavior of tangential velocity and temperature profiles is clearly seen, indicating that for moderate values of Grashof number the peak of the transformed tangential velocity profile is also small. This can be explained by the fact that, at moderate distances form the leading edge, the streamwise fluid acceleration from the **quiescent situation** is high, thus high values of the y-velocity are expected. In this situation, one the fundamental assumptions of the classical boundary layer theory is not valid. Therefore, at least the convective terms of the y-momentum equation must be retained in order to describe the flow adequately.

The v velocity convective terms are represented in eq. (27) by the terms inside the brackets, which are divided by $Gr^{1/2}$. Therefore, the influence of those terms must vanish as Gr approaches infinity and, in this case, eq. (27) reduces to the classical formulation as it is expected. What values of Gr are used here?

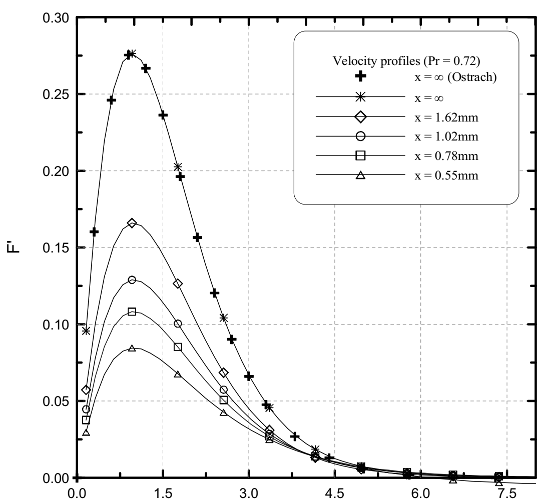

Figure 2. Velocity profiles for various distances from de leading edge locations with $\mathrm{Pr} = 0.72$.

One important parameter that can be used to compare higher order boundary layer formulations for the vertical flat plate convection phenomena is the average Nusselt numberdefinir. Figures (6) e (7) show the comparison of the present theory results and the ones of the higher order corrections contributions of Yang and Jerger (1964) and Messiter and Liñán (1976), which are described through the equations in table (1) below:

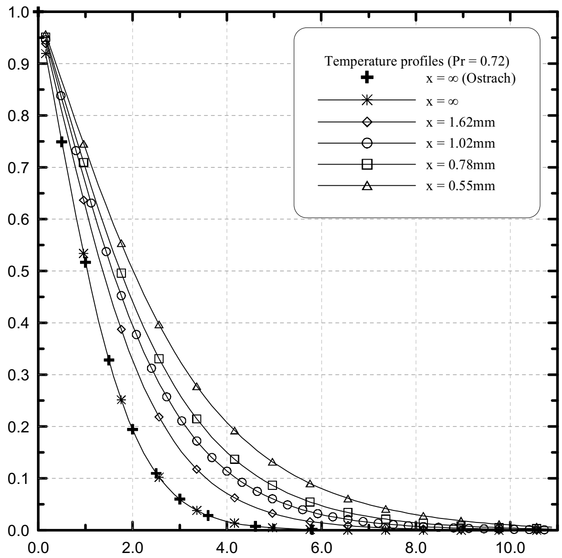

Figure 3. Temperature profiles for various distances from leading edge and $\mathrm{Pr} = 0.72$.

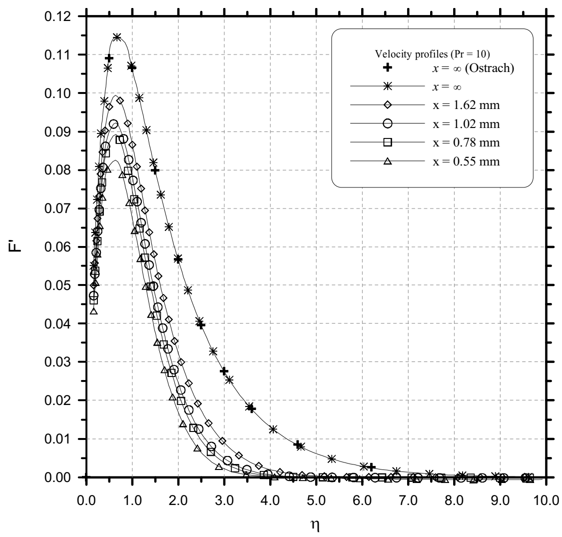

Figure 4. Velocity profiles for various distances from leading edge locations and $\mathrm{Pr} = 10$.

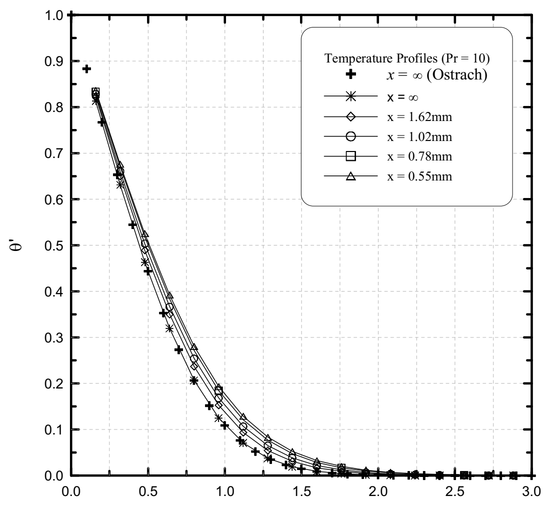

Figure 5. Temperature profiles for various distances from leading edge and $\mathrm{Pr} = 10$.

Table 1. Higher order corrections contributions of Yang and Jerger (1964) and Messiter and Liñán (1976)

$Pr = 0.72$	Yang and Jerger	$0.475Gr^{1/4} - 0.312$
	Messiter and Liñán	$0.475Gr^{1/4} + 0.623$
$Pr = 10$	Yang and Jerger	$1.102Gr^{1/4} - 0.216$
	Messiter and Liñán	$1.102Gr^{1/4} + 0.457$

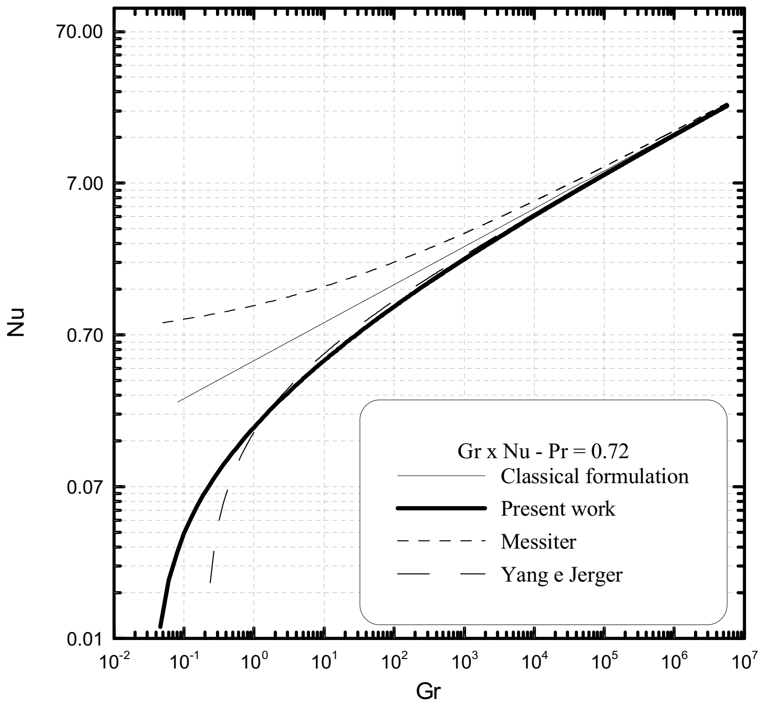

Figure 6. Comparison of average Nusselt number for $\mathrm{Pr} = 0.72$.

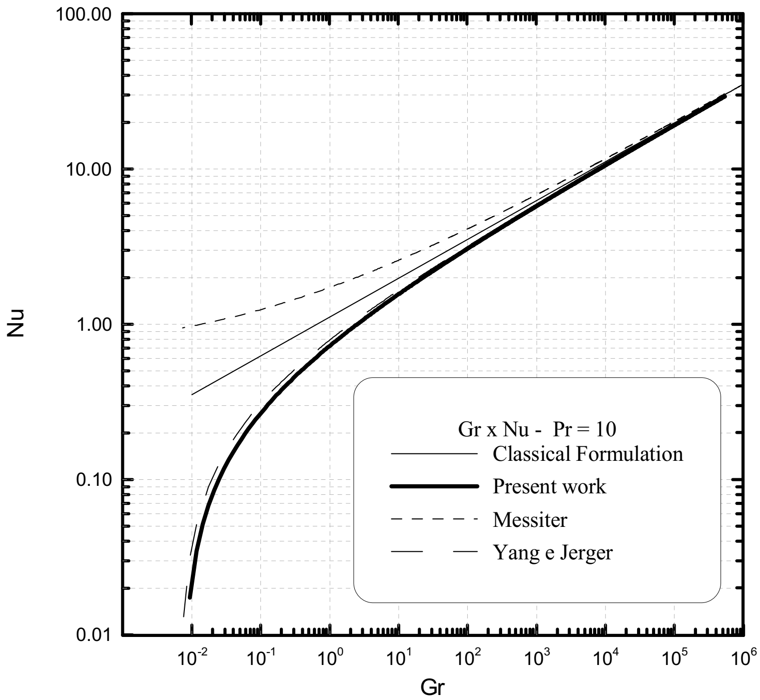

Figure 7. Comparison of average Nusselt number for $\mathrm{Pr} = 10$.

The influence of the displacement thickness into the velocity and the temperature fields were studied by Yang and Jerger (1964). Using the matched asymptotic expansions method, these authors were able to calculate the higher order displacement thickness correction influence on the average Nusselt number. A comparison of the results of these authors with the present theory for values of the Prandtl number of 0.72 and 10 shows a remarkable agreement between both approaches. This fact suggests that Eq. (27) represents a generalized version of the natural convection flat plate problem. Despite the fact that Eq. (27) is an ordinary differential equation, its quasi similar character allows the formulation to retain some information from the moderate Grashof number non-similar region. It is also important to note that this additional non-similar higher order information comes from a single differential equation which was obtained by a first order analysis.

The results of the present work are also compared with the higher order correction of Messiter and Liñán (1976) who considered an asymptotic expansion in an effort to introduce the leading edge effects into the analysis. Their approach does not take into account the influence of the boundary layer into the inviscid flow, but instead creates a near leading edge correction which must be added to the classical theory solution. It is clear from Figs. (6) and (7) that the Messiter and Liñán (1976) results are not in agreement with the present theory neither with the Yang and Jerger (1964) higher order equation.

6. Conclusions

In this work, a generalized boundary layer formulation was used to analyze the natural convection phenomena on a vertical flat plate. A quasi-similar equation was proposed which describes the flow for moderate values of the Grashof number. One major advantage of this quasi-similar equation is that, although its a ordinary differential equation, some information of the non-similar moderate Grashof number region is kept through one of the eq. (27) coefficients which explicitly depends on the Grashof number. The results of the present formulation were compared with the two natural convection boundary layer higher order theories of Yang and Jerger (1964) and Messiter and Liñán (1976). The results indicate that the perpendicular to the plate velocity influence should be considered for moderate values of the Grashof number. The results indicate that Eq. (27) contain the viscous-non viscous mutual influence phenomenon, which in the present approach is described by a single ordinary differential equation, disregarding the need of any type of laborious interactive process.

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