ASYMPTOTIC SOLUTIONS FOR A LAMINAR MINING SLURRY FLOW THROUGH A PIPE

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Abstract. *In the industry the pipeline system is the solution for transport non-newtonian fluid. This work shows a study of the fully developed laminar flow through a pipe with constant circular cross section. In this case is considered the formation of the core flow. The non-newtonian behavior was modeled by Carreau-Yasuda with yield stress. By dimensionless process of the flow equations a dimensionless parameter similar to Deborah number appears. The asymptotic solutions for velocity, flow rate and apparent viscosity are showed as solution of the problem. The flow was studied for low and high Deborah number range. The results show the relation between flow rate and apparent viscosity as function of Deborah for various magnitude of yield stress.*

Keywords:*Non-Newtonian, Carreau-Yasuda, Yield Stress, Pipe Flow, Asymptotic Solutions*

1. INTRODUCTION

Most of the fluids on the world are defined as non-newtonian fluids. In this kind of fluid the relation between shear rate and shear stress is not linear as newtonian fluid. The viscosity of fluid can be modeled by an empiric model. This models are designed as generalized newtonian model.

The interest of this work is a laminar flow of a ore slurry through a pipe with circular section area. This is justified by the fact of the most of the non-newtonian fluids are transported throught pipes due to environment and economical aspects (Cunningham, 2008). Some examples are the petroleum derived and the minerium slurry.

Due to the Brazil was the first country (2007) in transporting the bauxita slurry by long-distance pipline (Netherland, 2008) is important study the behaviour of the bauxita in flow through pipelines. The Carreau-Yasuda model was used to model the viscosity as function of the shear rate. The parameters for the model were extracted from Nascimento and Sampaio (1989) experimental studied where bauxite with 50% w-w from Pará was rheologically studied. The smaller size of particles were about 37 μm.

The solutions was obtained for the low and high Deborah regime. The concentration of the particles at the core of the pipe was considered. This situation is designed by core-flow. The particles at core are transported as a rigid body with a radius designed by core-radius. In the core-flow situation the flow rate decrease because the area avaiable for flow decrease. But for higher Deborah the core-radius decrease and the flow rate tends to the flow rate in the normal situation (without core flow). The results are obtained by numerical and analytical manner. In the numerical way the shear rate was obtained by Newton-Raphson, the velocity profile by Runge-Kutta 2nd Order and the flow rate by Trapezoidal method. The asymptotic solutions was obtained by the sucessive substitutions.

2. MATHEMATICAL MODEL

In this section the fully developed laminar unidimensional flow of a slurry through a pipe is described. Firstly is show the unidirectional flow through a pipe derived by a gradient of the pressure. The governing equation for slurry flow was obtained inserting the viscosity model into the newtonian constitutive equation .This equation is then non-dimensionalized by the characteristic length scale of the pipe and the constant pressure gradient of the flow. From the process of non-dimensionalization appears a dimensionless group similar to the Deborah number and some relations which will be explained later.

2.1 Unidirectional Flow

For a circular section the continuity and momentum equations are written in cylindrical coordinates. The flow is assumed unidirection and the velocity is reduced to $u_\theta = 0, u_r = 0$ and $u_x = u_x(r)$, where θ is the angular direction, r is the radial direction and x is the axial direction. Therefore for a fully developed flow through a pipe with length dimension greather than the diameter the velocity does not vary both in radial and angular directions.For an axisymetric condition all derivatives of the constitutive equation are zero at angular direction. This results in $\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{0}$ and

$$-\nabla p + \nabla . \boldsymbol{\tau} = \mathbf{0} \, , \tag{1}$$

where p is the pressure field and $\boldsymbol{\tau}$ is the deviatoric part of the stress tensor given by $\boldsymbol{\tau} = 2\eta \boldsymbol{D}$, η is the dynamic viscosity coefficient and $\mathbf{D}$ is the symmetric part of the gradient velocity tensor which is equal to

$$\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right).$$

From the Eq. (1) is possible say that for the flow through a pipe the viscous forces are balanced by a pressure gradient and $\boldsymbol{\tau}$ model not predicts the elastic efects (Phan-Thien, 2002). For an unidirection flow the Eq. (1) in cylindrical coordinate is reduced to

$$-\frac{\partial p}{\partial x} + \frac{\partial}{\partial r}\left(\eta r \frac{\partial u}{\partial r}\right) = 0, \tag{2}$$

where the viscosity coefficient η is modeled by Carreau-Yasuda model given by (Bird et al, 1987):

$$\eta = \eta_\infty + (\eta_0 - \eta_\infty)\left[1 + (\lambda\dot{\gamma})^a\right]^{\frac{n-1}{a}}, \tag{3}$$

where η_0 and η_∞ are respectively the viscosity at zero and at infinite shear rate, λ^{-1} is the critical shear rate at which viscosity begins to decrease and $\dot{\gamma} = \partial u/\partial r$ is the shear rate. The a and n are material constants of the Carreau-Yasuda model. From Eq. (2) we see that the pressure is only a function of the axial coordinate x but by the other hand the second term of this equation is only a function of the radial coordinate r. So, the two terms of the Eq. (2) are constants because they are equal to each other and are independents for different variables. Therefore the pressure gradient is actually a constant (Bird *et al.*, 1987).

2.2 Nondimensionlization of the mathematical model

The radius of the pipe, R and η_0 amongst the pressure gradient which is a constant were selected for nondimensionalization process. From this is defined that:

$$U_c = \frac{R^2}{\eta_0}G \quad ; \quad \dot{\gamma}_c = \frac{U_c}{R} = \frac{G}{\eta_0}R, \tag{4}$$

where U_c and $\dot{\gamma}_c$ are respectively the characteristic velocity and characteristic shear rate. The quantity G is usually defined as $-\partial p/\partial x$. By integrating the Eq. (2) with respect to r and considering that $\dot{\gamma}$ is finite in the flow domain, it is possible to write that $\eta\dot{\gamma} = -Gr/2$.The Deborah number arises from the nondimensionalization process:

$$\tilde{\dot{\gamma}}\eta^* + \tilde{\dot{\gamma}}(1-\eta^*)\left[1 + \left(\tilde{\dot{\gamma}}De\right)^a\right]^{\frac{n-1}{a}} = -\frac{\tilde{r}}{2}, \tag{5}$$

where $\tilde{\dot{\gamma}} = \dot{\gamma}/\dot{\gamma}_c$, $\tilde{r} = r/R$ and

$$De = \frac{\lambda G R}{\eta_0} \quad , \quad \eta^* = \frac{\eta_\infty}{\eta_0}.$$

The parameter η^* is a ratio of the viscositity ate zero and at a infinite shear rate. If $\eta^* > 1$ then the slurry behaves as a shear thickening if not the the slurry behaves as a shear thinning. The numerical results was obtained with $\eta^* = 0.5$. The dimensionless group De may be understood as a ratio between two time scales: a time constant related to the sensivity of the fluid to shear rate, (λ), and a time scale related to the flow ($\eta_0/(GR)$. In this sense, the group $\lambda GR/\eta_0$ is similar to the Deborah number.

In the core-flow situation the core-radius is :

$$R_b = \frac{2\tau_0}{G},$$

where τ_0 is the shear rate at interface between non-newtonian fluid and the rigid body. From nondimensionalization process:

$$\tilde{R_b} = \frac{2}{De}\omega \tag{6}$$

where $\omega = (\tau_0\lambda)/\eta_0$ is a dimensionless group. It is important to note that $0 < R_b < 1$.The flow happens when $G > 1$ and therefore $De > 0$. So when De increase the R_b decreases.

3. APPARENT VISCOSITY

The fully developed flow through a pipe may be studied by comparing the pressure gradient and the consequent flow rate, Q, for any fluid. In other words, by the classical Hagen-Poiseuille equation is possible to isolate the viscosity in such a way that:

$$\eta_{ap} = \frac{\pi\, R^4\, G}{8\, Q}$$

where Q is the flow rate and η_{ap} is an apparent viscosity. For laminar flow and Newtonian fluid the η_{ap} is a constant equal to the dynamic molecular viscosity. For a non-newtonian fluid η_{ap} is a function of the flow rate for a given pressure gradient. The characteristic flow rate Q_c is derived as:

$$Q_c = U_c \pi R^2 = \frac{\pi\, R^2\, G}{\eta_0} \ .$$

The dimensionless apparent viscosity, $\tilde{\eta}_{ap}$, is then:

$$\tilde{\eta}_{ap} = \frac{1}{8\, \tilde{Q}} \ , \tag{7}$$

where $\tilde{\eta}_{ap} = \eta_{ap}/\eta_0$ and $\tilde{Q} = Q/Q_c$.

4. FINAL DIMENSIONLESS MODEL

The constants n and a of the Carreau-Yasuda model were extracted from experimental results of Nascimento and Sampaio (1989). The values $n = 1/2$ and $a = 2$ are a good approximations for this experiment. In order to make the notation easier the tilde notation for dimensionless quantities will be suppressed. After last considerations the final dimensionless model is given by next equations:

$$\dot{\gamma} + \dot{\gamma}\, \eta^* \left[1 + (\dot{\gamma}\, De)^2\right]^{-\frac{1}{4}} = -\frac{r}{2} \ \ , \ \ r \in [0, 1] \ , \tag{8}$$

subject to the boundary condition $u(1) = 0$,

$$Q = 2 \int_0^1 u\, r\, dr \ , \tag{9}$$

and by Eq. (7).

5. NUMERICAL MODEL

The unidimensional domain was descritized in axial direction into N equality spaced control points, r_i with $N = 300$. For each value of r_i the $\dot{\gamma}$ of the Eq. (8) was derived by Newton-Raphson method. The tolerance used in the Newton-Raphson procedure was 10^{-12}. The initial guess for the method was $\dot{\gamma} = r/2$.

For obtained numerical solution is necessary solve the Eq.(8). The Newton-Raphson method was used to obtain the shear. After this procedure, the initial differential problem is rewritten in the form:

$$f(n) = \begin{cases} \frac{\partial u}{\partial r} = f(r_i) \quad , \quad i = 1, ..., N \\ \\ u(1) = 0 \end{cases} \tag{10}$$

where $\dot{\gamma}$ was replaced by $\partial u/\partial r$ and $f(r_i)$ is a numerical function (tabulation) resulting of the Newton-Raphson procedure. So the problem is described by an initial value problem, which can be solved in a straight way by a 2^{nd} Order Runge-Kutta method. The flow rate, Q, was calculated by using the Trapezoidal Rule wich is function of the velocity derived by Runge-Kutta method (Hoffman, 2001). For solve the core-flow situation the core-radius was calculated by Eq.(6). Therefore the domain can be descritized as $max(r)$ - R_b and the boundary condition for at $r \leq R_b$ is that the velocity should be equal to $u(r = R_b)$.

6. ASYMPTOTIC SOLUTIONS

The asymptotic solutions were derived by the successive substitution method (Hinch, 1995). This method consists in to generate a sequence using a recursive formula obtained from the governing equation. Using the values of the constants of the Carreau-Yassuda model the recursive formula Eq. (8) can be rewritten as:

$$\dot{\gamma}_n \, \eta^* + \frac{\dot{\gamma}_{n-1} \, (1-\eta^*)}{\left[1-(De\,\dot{\gamma}_{n-1})^2\right]^{\frac{1}{4}}} = -\frac{r}{2} \, . \tag{11}$$

For study the limit of $De << 1$, the second term of the left side was aproximated by:

$$\dot{\gamma} \, (1-\eta^*) - \frac{1}{4} \, \dot{\gamma}^3 \, \left(1-\eta^*\right) \, De^2. \tag{12}$$

Replacing the second term of the left side of the Eq.(11) by Eq.(12) was founded that:

$$\dot{\gamma}_n = \frac{1}{4} \dot{\gamma}^3_{n-1} \, \left(1-\eta^*\right) De^2 - \frac{r}{2} \tag{13}$$

The value of the $\dot{\gamma}_0$ was obtained making $De = 0$ in Eq.(13). The term $\dot{\gamma}^3_{n-1}$ was replaced by $\dot{\gamma}_0$ value. By this way was obtained the $\dot{\gamma}_1$ value. The dimensionless velocity profile was derived by solving the dimensionless shear rate with velocity zero at wall pipe as boundary condition. The dimensionless flow rate was calculated by integral of velocity profile for a circular cross section area with limits of the integration between zero (center of the pipe) and 1 (maximum dimensionless radius). The dimensionless expressions for shear rate, velocity, volumetric flow rate and apparent viscosity are, respectively:

$$\dot{\gamma} = -\frac{1}{32} \left(1-\eta^*\right) \, r^3 De^2 - \frac{1}{2} r \, , \tag{14}$$

$$u\,(r) = \left(\frac{1}{128} (\eta^* - 1) r^4 + \frac{1}{128} \left(1-\eta*\right) \right) De^2 - \frac{1}{4} \left(1-r^2\right) \, , \tag{15}$$

$$Q = \frac{1}{8} + \frac{1}{192} \left(1-\eta^*\right) De^2 \, , \tag{16}$$

$$\eta_{ap} = -\frac{24}{\left(\eta^* - 1\right) De^2 - 24} \, . \tag{17}$$

For the study of the limit of $De >> 1$ it was necessary to make a chance of variable. In this way we take $1/\alpha = De^2$. So, the second term of the Eq.(11) was chanced to:

$$\frac{\dot{\gamma}_{n-1} \, (1-\eta^*)}{\left[1-\left(\frac{1}{\alpha}\,\dot{\gamma}_{n-1}\right)^2\right]^{\frac{1}{4}}} \, . \tag{18}$$

Note that when $De^2 \to \infty$, $\alpha \to 0$ and:

$$\dot{\gamma}_0 = -\frac{r}{2\,(1-\eta^*)} \, . \tag{19}$$

The Eq.(18) was aproximated by the next analytical expression:

$$\dot{\gamma}^{\frac{1}{2}} \, (1-\eta^*) \, De^{\frac{1}{2}} \, .$$

The shear rate expression for high Deborah is then:

$$\dot{\gamma}_n = -\frac{\dot{\gamma}_{n-1}}{\eta^*} \left(1-\eta^*\right) De^{\frac{1}{2}} - \frac{r}{2\eta^*} \, . \tag{20}$$

So for high Deborah the dimensionless expressions for shear rate, velocity, volumetric flow rate and apparent viscosity are, respectively:

$$\dot{\gamma} = \frac{r\left(De^{-\frac{1}{2}} - 1\right)}{2\eta^*} \, , \tag{21}$$

$$u\left(r\right)=\frac{De^{-\frac{1}{2}}\left(r^{2}-1\right)-\left(r^{2}-1\right)}{4\eta^{*}}\ , \tag{22}$$

$$Q=\frac{1-De^{-\frac{1}{2}}}{8\eta^{*}}\ , \tag{23}$$

$$\eta_{ap}=-\frac{\eta^{*}}{De^{-\frac{1}{2}}-1}\ . \tag{24}$$

The results for flow rate and viscosity dimensionless are presented and commented below.

7. RESULTS

In this section results of flow rate, apparent viscosity and the comparison between the normal case and the core-flow case are presented. Figure (1) shows the flow rate for the normal case. The numeric results shows two plateaus wich are fitted by analitycal solutions. At low values of Deborah the particules of slurry are very close together. In this situation the magnitude of the viscosity is high. As Deborah increases the particules move away and the viscosity descrease until the distance of the particules do not affect anymore the viscosity. In this situation we have the plateau to high Deborah.

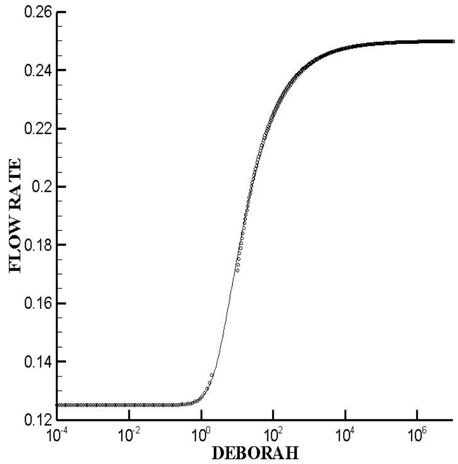

Figure 1. Numerical (line) and analytical (points) dimensionless flow rate at low and high De

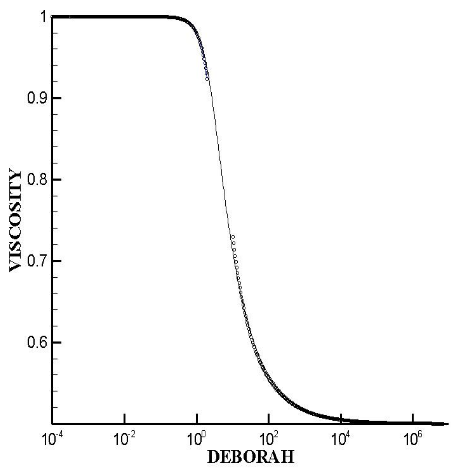

Figure 2. Numerical (line) and analytical (points) dimensionless flow rate at low and high De

Figure (3) and Fig. (4) show the flow rate behaviour in function of low and high Deborah at situation of core-flow and without core-flow. The results show that when exist concentration of particles at core region of the pipe the flow rate decrease for the same Deborah. When Deborah increases the flow rate with core-flow tends to flow rate without core-flow. It happens because when Deborah increase the radius of the core-flow decrease.

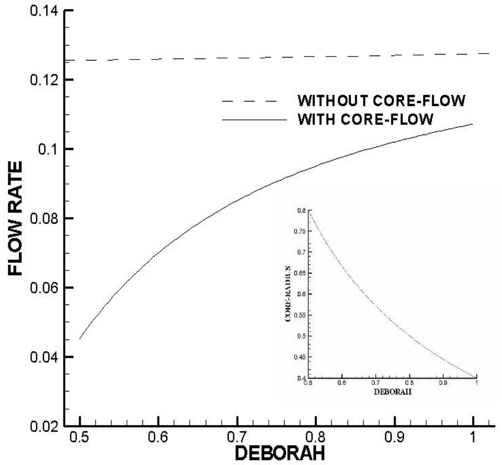

Figure 3. Flow rate at low Deborah for the normal and core-flow cases

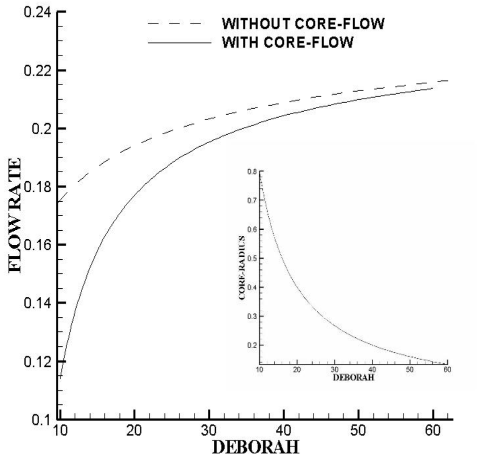

Figure 4. Flow rate at high Deborah for the normal and core-flow cases

8. CONCLUSIONS

This work studied by numerical and analytical solutions the dimensionless flow of an non-newtonian fluid through a pipe with circular cross section in the laminar regime . This study is interested in flow of the ore. The Carreau-Yassuda model is an generalized newtonian fluid model and was used for model the viscosity in function of the shear rate. The material constants of the model were obtained by bauxite slurry experimental results. The results show two plateaus: one at low Deborah values and other at high Deborah values. At low Deborah the flow rate is lower (the viscosity is higher) than high Deborah. These study considered the concentration of particles at core region of the pipe. For this situation was verified that the flow-rate decrease for the same Deborah regime. But when Deborah increase the flow rate tends to the flow rate where core-flow does not occurr. This happen because the area for the flow increases (the radius of core-flow decrease) when Deborah increases.

9. REFERENCES

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10. Responsibility notice

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