AN APPROACH TO RELIABILITY-BASED SHAPE AND TOPOLOGY OPTIMIZATION OF TRUSS STRUCTURES

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Abstract. *This paper deals with the reliability based geometry and topology optimization of truss structures. It presents an approach to optimize simultaneously the geometry and topology of statically undetermined trusses considering the acting forces and the yielding stress of the bars as random variables. Based on the assumptions of linear structural behavior and independent and normally distributed random variables, the optimization problem is posed in such a way that its computational cost is similar to a standard deterministic optimization problem, which is the main contribution of this paper.*

Keywords*: reliability based optimization; truss structures; geometry optimization; topology optimization.*

1. INTRODUCTION

Methods for the optimization of truss topology, where the members areas are taken as design variable, are well established and there is a rich literature on this subject (Hemp, 1973; Pedersen, 1970; Pedersen, 1993). The problem of truss shape optimization, where the nodes positions are taken as design variables, is also addressed in literature (Achtziger, 2006; Achtziger, 2007; Kocvara and Zowe, 1996), but not as extensively. One of the main reasons why topology optimization was studied more frequently than shape optimization, also called here geometry optimization, is that the topology optimization problem can, in most cases, be stated as a linear programming problem. For such cases, very efficient methods are available and it is possible to guarantee certain important mathematical properties of the solution, such as existence of global optimum (Nocedal and Wright, 1999; Rao, 1996; Arora, 2004). The truss geometry optimization problem is, instead, non linear by its nature, and therefore it needs to be solved by nonlinear optimization methods, which are in general more complex and computationally demanding than the linear programming techniques. Besides, the geometry optimization of a truss may lead to a non convex optimization problem, and several local minima may exist (Achtziger 2006; Achtziger 2007). Some strategies have been developed to deal with this optimization problem, and according to Achtziger (2007) these strategies can be grouped as: *simultaneous optimization of truss topology and geometry*, *alternating optimization* and *implicit programming optimization.* An example of alternating optimization is presented by Torri and Biondini (2009), while an example of simultaneous optimization of truss topology and geometry is presented in the forthcoming paper by Torii et al. (2011).

In deterministic optimization, however, the uncertainties involved in the design problem, such as those affecting material properties and loads, among others, are not considered. Robust optimization or reliability based optimization (RBDO) methods are usually employed to take such uncertainties into account (see, for instance, Beyer and Sendhoff (2006), Schuëller and Jensen (2009)). The former has as main goal the minimization of the variability of some parameters related to system response due to its uncertainties. For example, Calafiore and Dabbene (2008) applied this concept in the field of design of truss structures.

The main goal of the RBDO is to optimize structures guaranteeing that its probability of failure is lower than a certain level chosen *a priori* by the designer. Nakib (1991), Thampan and Krishnamoorty (2001), Togan and Daloglu (2006) dealt with the RBDO of truss structures, regarding only the topology or size optimization of such structures. Lagaros et al. (2007) pursued the reliability based robust optimization of trusses grouping at the same time the goals of the robust and RBDO.

However, only a few papers have dealt with the reliability based shape and topology optimization of truss structures (Torii et al, 2011; Morutsu and Shao, 1990; Stocki et al., 2001). This paper addresses the reliability based optimization of geometry and topology for statically indeterminate trusses, taking into account the uncertainties on the applied forces as well as the yielding stresses. In this approach the applied forces and the yielding stresses are modeled as random variables, and the failure constraints are expressed in probabilistic terms. This approach is described in details by Torii et al. (2011), and here we present only the main ideas of the approach.

The main contribution of this paper is that, based on the assumptions of linear structural behavior and independent and normally distributed random variables, the RBDO problem is posed in such a way that the reliability of the structure is accessed directly, without using iterative methods such as a First Order Reliability Method (FORM) or Monte Carlo Simulation (MCS) (Haldar and Mahadevan, 2000). This represents a significant reduction on the computational effort involved. A complete description on how sensitivity analysis can be pursued is presented by Torii et al. (2011).

2. DETERMINISTIC OPTIMIZATION PROBLEM

The optimization problem is posed as the minimization of the volume of the structure subject to stress constraints by taking the nodal coordinates and cross section areas as design variables:

Find: **x** and **A**

that gives

$$\min V(\mathbf{x},\mathbf{A}) = \mathbf{A}^T\mathbf{L}(\mathbf{x}), \tag{1}$$

subject to

$$g_j = +\sigma_j - \sigma_t \leq 0 \quad (j = 1,2,\dots,m), \tag{2}$$

$$g_{j+m} = -\sigma_j + \sigma_c \leq 0 \quad (j = 1,2,\dots,m), \tag{3}$$

where V is the volume of the structure, **x** is the vector of nodal coordinates, **A** is the vector of member areas, **L** is the vector of member lengths, g_j are stress constraints, σ_j is the stress on member j, σ_t is the yielding stress in tension, σ_c is the yielding stress in compression and m is the number of members subjected to stress constraints. In this paper, buckling constraints are not introduced.

For convenience, the design variables **A** and **x** can be grouped into a single design vector **X**, and the constraints from Eq. (2) and Eq. (3) can be grouped into a single vector of constraints **g**. In this way, the previous problem is rewritten as follows:

Find: **X**

that gives

$$\min V(\mathbf{X}) = \mathbf{A}^T\mathbf{L}, \tag{4}$$

subject to

$$\mathbf{g} \leq \mathbf{0}, \tag{5}$$

where **g** is a vector with $2m$ components since there are two constraints defined for each bar of the structure.

Bounds on design variables are defined as shown in Fig. 1, by prescribing how far each node can be moved from its original position. Sensitivity analysis can then be carried out using some finite difference scheme or the adjoint method as discussed by Torii et al. (2011).

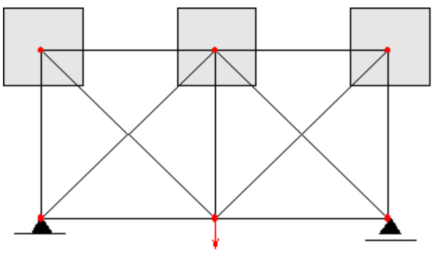


Figure 1: Bounds on nodal coordinates defined locally for each node.

2.1. Alternative loading conditions

When a set of s alternative loading conditions $\{\mathbf{F}_1,...,\mathbf{F}_k,...,\mathbf{F}_s\}$ is considered, the structural response will be defined by a set of nodal displacements $\{\mathbf{u}_1,...,\mathbf{u}_k,...,\mathbf{u}_s\}$, and the optimization problem from Eq. (4) and Eq. (5) becomes

Find: X

that gives

$$\min V(\mathbf{X}) = \mathbf{A}^T\mathbf{L}, \tag{6}$$

subject to

$$\mathbf{g}_k \leq \mathbf{0} \quad (k = 1,...,s), \tag{7}$$

where the vectors $\mathbf{g}_k$ from Eq. (7) are defined for each loading condition in Eq. (2) and Eq. (3).

3. PROBABILISTIC OPTIMIZATION PROBLEM

Consider the applied forces **F** and the yielding stresses σ_c, σ_t to be random variables with known density distribution. For convenience of notation, they are grouped into the random variable vector $\mathbf{\Xi}$. The optimization problem still searches for the minimum volume structure, but now subject to a minimum reliability level of the structure. A component level reliability constraint is considered, instead of dealing with the probabilistic failure constraint at the system level. In other words, a minimum reliability level is enforced for each bar. For a single loading condition we have the following problem:

Find: **X**

that gives

$$\min V(\mathbf{X}) = \mathbf{A}^T\mathbf{L}, \tag{8}$$

subject to

$$G_j = P_0 - P\left(g_j(\mathbf{\Xi}) \leq 0\right) \leq 0 \quad (j = 1,...,m), \tag{9}$$

where $P(\cdot)$ is the probability of the constraint to be fulfilled and P_0 is a minimum probability level of the constraint or its reliability. Note that now the constraints of the optimization problem are affected by the random variable vector $\mathbf{\Xi}$, becoming themselves random variables. The constraint from Eq. (9) states that the probability of $g_j(\mathbf{\Xi})$ being respected must be bigger than a minimum probability P_0.

3.1. Reliability evaluation

The random variable $g_j(\mathbf{\Xi})$ (i.e., the constraints are now given by random values) is constructed based on a linear structural behavior, as it occurs in most structural optimization procedures (Hemp, 1973; Pedersen, 1970; Pedersen, 1973; Achtziger, 2006; Achtziger 2007; Martínez et al., 2007; Torii and Biondini, 2009; Pereira et al., 2004), and by assuming **F** and $\boldsymbol{\sigma}$ to be independent normal random variables.

If for a given applied force vector $\mathbf{F}_0$, the resulting stress in a given member is σ_0, then for an arbitrary applied force vector **F**, obtained by the multiplication of $\mathbf{F}_0$ by a scalar, the stress in that member is (by the principle of superposition from structural mechanics)

$$\sigma = \frac{\sigma_0}{\|\mathbf{F}_0\|}\|\mathbf{F}\| = \frac{\sigma_0}{F_0}F\,, \tag{10}$$

where $\|\cdot\|$ denotes the norm of a vector, and F and F_0 are the norms of $\mathbf{F}$ and $\mathbf{F}_0$, respectively. Equation (10) can be substituted into Eq. (2), giving

$$g(\mathbf{\Xi}) = \frac{\sigma_0}{F_0}F - \sigma_t \le 0\,, \tag{11}$$

where the index j has been dropped for convenience.

Since both the applied forces and yielding stresses are considered as independent normal random variables, a linear combination of them is also a normal random variable. Consequently, by denoting μ_1 and s_1 as the mean value and the standard deviation of the applied force, respectively, and μ_2 and s_2 as the mean value and the standard deviation of the yielding stress, respectively, $g(\mathbf{\Xi})$ as given by Eq. (11) is a normal random variable with mean

$$\mu_{g(\Xi)} = \frac{\sigma_0}{F_0}\mu_1 - \mu_2 \tag{12}$$

and standard deviation

$$s^2_{g(\Xi)} = \left(\frac{\sigma_0}{F_0}\right)^2 s_1^2 + s_2^2\,. \tag{13}$$

The normalized value of $g(\mathbf{\Xi})$ is then:

$$\bar{g} = \frac{g(\mathbf{\Xi}) - \mu_{g(\Xi)}}{s_{g(\Xi)}} \tag{14}$$

and the reliability index β (Lemaire et al., 2005) of the constraint can be evaluated by Eq. (14) by taking $g(\mathbf{\Xi}) = 0$. Thus, the reliability index is related to the probability of the constraint to be feasible by:

$$\beta = \Phi^{-1}\left[P\left(g_j(\Xi) \le 0\right)\right] \text{ or } P\left(g_j(\Xi) \le 0\right) = \Phi[\beta]\,, \tag{15}$$

where $\Phi = \Phi[\bullet]$ is the standard normal cumulative probability function. We may also relate the required minimum reliability level of the structure P_0 to the so called target reliability index β_t by:

$$P_0 = \Phi[\beta_t] \text{ or } \beta_t = \Phi^{-1}[P_0]\,. \tag{16}$$

Substituting Eq. (15) and (16), the constraint given by Eq. (9) becomes

$$G_j = \beta_t - \beta_j \le 0 \quad (j = 1,\ldots,m)\,. \tag{17}$$

The index j emphasizes that a reliability index is computed for each bar. The probabilistic optimization problem can then be solved by using Eq. (17) instead of Eq. (9).

3.2. Loading conditions given by several forces with different standard deviations

In some cases it may happen that a given loading condition is defined by a set of applied forces that have difference mean values and standard deviations, as occurs for the example from Fig. 2. In this case, the structure is subjected to a single loading condition. That is, the forces $\mathbf{F}^1$ and $\mathbf{F}^2$ are applied at the same time, but may have different mean values and standard deviations.

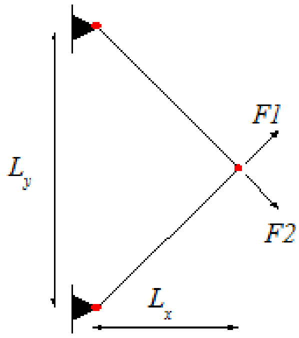


Figure 2: Example of a structure subject to one loading condition given by two forces with different standard deviations.

In order to access the reliability of the structure, it is necessary to study the effect of each force, $\mathbf{F}^1$ and $\mathbf{F}^2$, separately. Thus, considering each force at once we have (again by the superposition principle) for the applied force $\mathbf{F}^1$

$$\sigma^1 = \frac{\sigma_0^1}{\left\|\mathbf{F}_0^1\right\|}.\left\|\mathbf{F}^1\right\| = \frac{\sigma_0^1}{F_0^1}.F^1 \tag{18}$$

and for the applied force $\mathbf{F}^2$

$$\sigma^2 = \frac{\sigma_0^2}{\left\|\mathbf{F}_0^2\right\|}.\left\|\mathbf{F}^2\right\| = \frac{\sigma_0^2}{F_0^2}.F^2, \tag{19}$$

where the same notations as of Eq. (10) hold, but here the superscripts 1 and 2 represent the quantities for forces $\mathbf{F}^1$ and $\mathbf{F}^2$. That is, one structural analysis is made for each force $\mathbf{F}_0^1$ and $\mathbf{F}_0^2$, giving stresses σ_0^1 and σ_0^2 (these forces can be taken as unitary, for convenience). The stress caused by the application of $\mathbf{F}^1 + \mathbf{F}^2$ is then

$$\sigma = \sigma^1 + \sigma^2 = \frac{\sigma_0^1}{F_0^1}.F^1 + \frac{\sigma_0^2}{F_0^2}.F^2, \tag{20}$$

that can be rewritten as

$$\sigma = k_1.F^1 + k_2.F^2. \tag{21}$$

The important aspect here is that the effect inside the bar can be written as a linear combination of the separate effects of each force, as we are dealing with a linear structural analysis. This is precisely the superposition principle from structural mechanics.

Since the applied forces are Gaussian random variables, the mean value of the stress in the bar due to the combination of forces F^1 and F^2 is

$$\mu_\sigma = k_1.\mu_{F^1} + k_2.\mu_{F^2} \tag{22}$$

and its standard deviation is

$$(s_\sigma)^2 = (k_1.s_{F^1})^2 + (k_2.s_{F^2})^2, \tag{23}$$

where μ_{F^1} and s_{F^1} are the mean value and standard deviation of the force $\mathbf{F}^1$, respectively (the same notation is also used for the force $\mathbf{F}^2$). Moreover, μ_σ and s_σ are the mean value and the standard deviation of the stress inside the bar, respectively. Note that the stress is a Gaussian random variable, as it is the linear combination of two Gaussian random variables, namely the two forces.

Until now we have shown how to obtain the mean value and the standard deviation of the stress inside the bar, given two forces with magnitude F^1 and F^2 acting simultaneously. Consider now the stress constraint from Eq. (2). Since it is a linear combination of Gaussian random variables (the stress and the maximum allowable stress), we have

$$\mu_{g(\Xi)} = \mu_\sigma - \mu_{\sigma_t} \tag{24}$$

and

$$s_{g(\Xi)}{}^2 = (s_\sigma)^2 + (s_{\sigma_t})^2 \tag{25}$$

where $\mu_{g(\Xi)}$ and $s_{g(\Xi)}$ are the mean and the standard deviation of the constraint, and μ_{σ_t} and s_{σ_t} are the mean and the standard deviation of the maximum allowable stress in tension. The same reasoning holds for a constraint as defined in Eq. (3).

It is possible to evaluate Eq. (24) and Eq. (25) with Eq. (22) and Eq. (23), and then obtain the reliability index as described previously. Besides, note that Eq. (24) and Eq. (25) are analogue to Eq. (12) and Eq. (13), respectively. More details are presented by Torii et al. (2011).

4. NUMERICAL EXAMPLES

In this section two numerical examples are solved in order to discuss the main aspects of the approach presented. Note that for all the following examples each figure has its own scale, and the reader can compare the different solutions by the volume of material V presented for each example. Besides, a lower bound for the areas is defined as 1E-10m^2(i.e. 0.1mm^2), in order to avoid singularity of the stiffness matrix. Finally, the optimization algorithm used here is the Sequential Quadratic Programming (SQP) (Nocedal and Wright, 1999; Rao, 1996).

4.1. Example 1 – two bar structure

The first example discussed here is the optimization of a two bar structure. Even if this example is simple from the practical point of view, it demonstrates what changes may arise when the yielding stresses have different standard deviations.

The ground structure from Fig. 3a is subjected to a single loading condition with mean F = 10,000N and standard deviation σ_F = 1,000N. The size of the structure is given by L_x = 2.5m and L_y = 5m and the Young Modulus of the material is E = 200GPa. The minimum reliability index is taken as β = 3.1. The yielding stress in tension has mean σ_t = +250MPa and the yielding stress in compression has mean σ_c = -250MPa. For the case of Fig. 3c the yielding stresses in tension and in compression have standard deviation σ_σ = 10MPa; while for the case of Fig. 3d the standard deviation of the yielding stress in tension is raised to $\sigma_{\sigma t}$ = 50MPa. The node of the applied forces is allowed to be moved up and down by the optimization algorithm.

From the results presented in Fig. 3 it can be seen that changing only the standard deviation of the yielding stresses leads to changes in the topology and geometry of the optimum structure. The structure from Fig. 3c is symmetric, since the yielding stresses both in tension and compression have the same mean value, in magnitude, and the same standard deviation. The solution of the deterministic problem, from Fig. 3b, is also symmetric. However, the structure from Fig. 3d is not symmetric because of the different values of the standard deviation of the yielding stresses. Finally, note that increasing the standard deviation of the yielding stress leads to an increase in the volume of the structure, as expected.

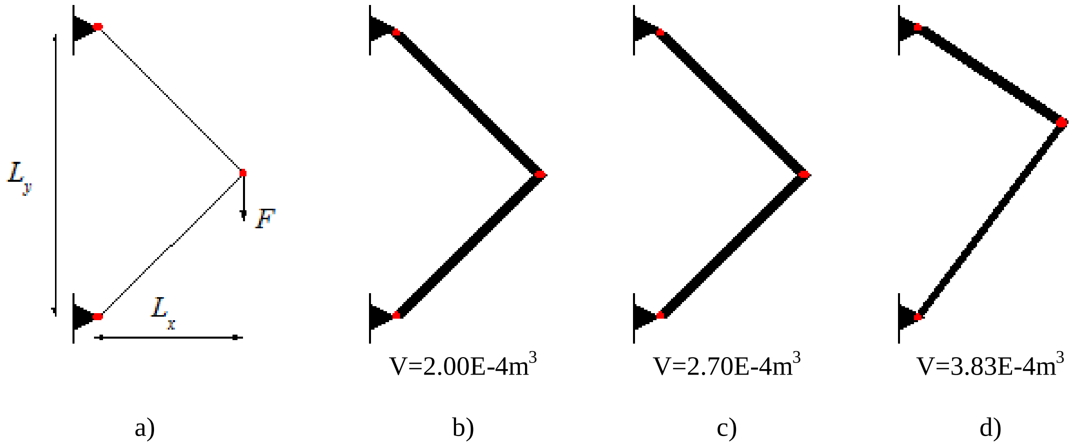


Figure 3: a) Ground structure, b) optimum solution for the deterministic problem, c) optimum solution with yielding stresses in tension and compression with the same standard deviation and d) optimum solution with yielding stress in tension with a higher standard deviation.

4.2. Example 2 – Loading conditions composed of several forces

Figure 4 presents a ground structure that is subjected to three loading conditions. However, vertical forces and horizontal forces have different mean values and standard deviations. This example illustrates the application of the method when a loading condition is composed by forces with different mean values and standard deviations.

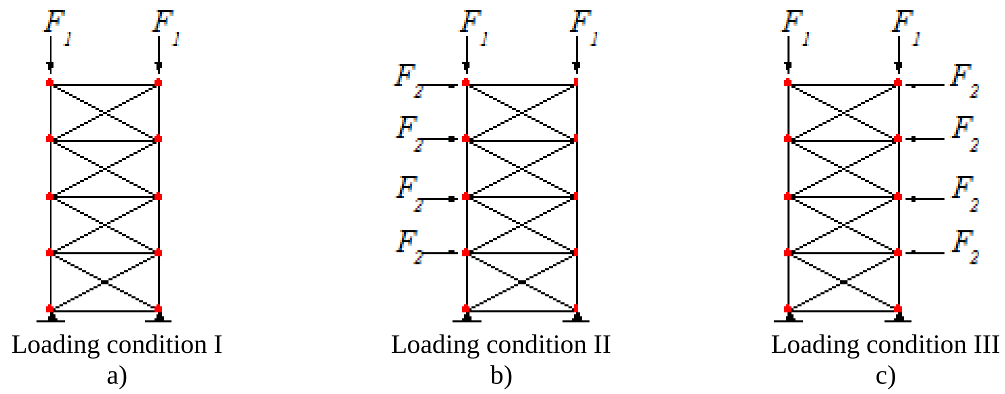
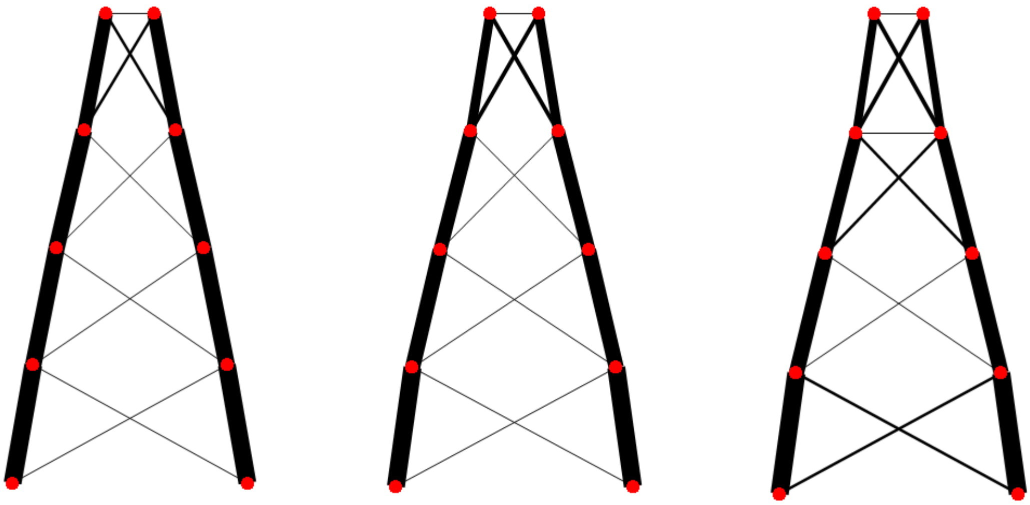


Figure 4: Ground structure subjected to three loading conditions I, II and III.

The ground structure from Fig. 4 has a total height of 4m and a total width of 2m. The material properties are the same as from example 1. The target reliability index is 3.1, and all nodes (except the nodes of the supports) are allowed to be moved left and right by the optimization algorithm, to positions as far as 0.8m from its original position. The vertical force mean value is F_1 = 1,000N and its standard deviation is σ_{F1} = 100N. For the horizontal forces F_2, three cases are studied: mean F_2 = 50N and standard deviation σ_{F2} = 5N; mean F_2 = 150N and standard deviation σ_{F2} = 15N; and mean F_2 = 250N and standard deviation σ_{F2} = 25N. Finally, symmetry of the geometry is enforced.

The results for this example are presented in Fig. 5. As a consequence of the increase to the lateral load, for the same vertical load, there was an increase of the volume of the structure and a change of its geometry and topology. The structures designed for bigger lateral loads are clearly more fitted to resisting the increased bending moment that develops in these cases.



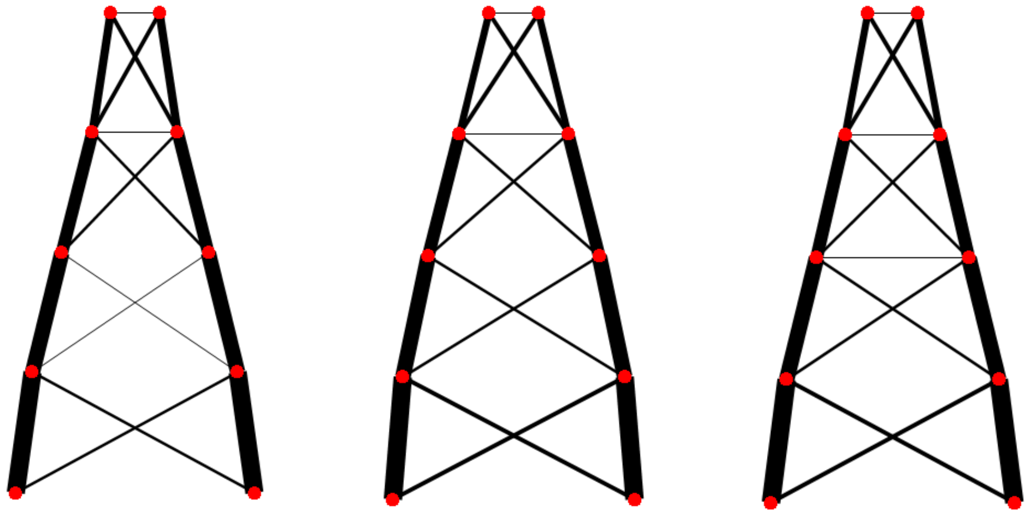
V= 5.29E-5m³
a)

V= 6.98E-5m³
b)

V= 8.75E-5m³
c)

Figure 5: Optimum solutions for increasing lateral loads from a) to c) for the ground structure from Fig. 4.

For the same ground structure from Fig. 4, another interesting example can be conceived. Taking the vertical force F_1 as defined previously and the mean of the horizontal force F_2 = 250N, we now solve the same problem for three different standard deviations: σ_{F2} = 25N, σ_{F2} = 50N and σ_{F2} = 75N. The results are presented in Fig. 6, from where it can be noted that increasing the standard deviation of the lateral load leads to an increase in the volume of the structure. Besides, the geometry and the topology change in order to resist the increased bending moments that appear in this case.



V= 8.75E-5m³
a)

V = 9.37E-5m³
b)

V=9.99E-5m³
c)

Figure 6: Optimum solutions for increasing standard deviation of the lateral loads from a) to c) for the ground structure from Fig. 4.

5. CONCLUDING REMARKS

This paper presented a formulation for the simultaneous optimization of topology and geometry of truss structures. General aspects such as constraints and several loading conditions were also discussed. The deterministic optimization scheme was then extended to the case when the yielding stresses and the applied forces are Gaussian random variables. The important aspect here is that considering these variables as Gaussian allow one to access the reliability of the structure directly, without using iterative methods such as a First Order Reliability Method (FORM) or Monte Carlo Simulation (MCS).

The numerical examples presented allows one to conclude that changes to probabilistic parameters, such as standard deviations of yielding stresses, are expected to lead to changes to both optimum topologies and geometries. For a more detailed discussion and other numerical examples the reader can consult the forthcoming paper by Torii et al. (2011).

6. REFERENCES

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