A COMPARISON BETWEEN THE BOUNDARY ELEMENT METHOD AND THE HIERARCHICAL FINITE ELEMENT METHOD FOR ONE-DIMENSIONAL ELASTODYNAMICS

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Abstract. *This paper presents a comparison between the Boundary Element Method (BEM) and the Hierarchical Finite Element Method (HFEM) for one-dimensional elastodynamics. The formulation of both the BEM and the HFEM for the problem being addressed is presented, and the main differences between them are discussed. The method of Houbolt is used for the time integration procedure for both formulations. Finally, one example is solved numerically and the results are compared with the analytical solution.*

Keywords*: finite element method, boundary element method, elastodynamics, wave propagation.*

1. INTRODUCTION

Many problems from physics and engineering can be modeled as boundary value problems and by initial value problems governed by partial differential equations. However, analytical solution of such problems is possible only in a few cases, when simple geometries of the domain and simple boundary and initial conditions are assumed. For most practical cases, however, some kind of approximate solution is necessary. In this context, numerical methods such as the Boundary Element Method (BEM)(Brebbia and Dominguez, 1992), the Finite Element Method (FEM)(Bathe, 1996), the Finite Differences Method (FDM) (Ames, 1977) and the Mesh-Free Methods (MFM) (Liu, 2003) are the most widespread techniques used for solving such problems.

A class of problems that frequently must be solved by numerical methods are those arising from elastodynamics (Timoshenko and Goodier, 1951). Elastodynamics concerns the propagation of displacement waves inside elastic media and structures, and consequently is of extreme importance for structural mechanics. Most structural designs nowadays need to take into account dynamic aspects, at least to some point. This fact is put in evidence by the number of structural failures that can be traced back to the lack of some kind of dynamic analysis.

The FEM is probably the most widespread technique used for solving general structural analysis problems. Consequently, its application for structural dynamics is straightforward. Traditionally, the FEM is used together with time integration methods such as the Central Difference Method, the Houbolt Method, the Newmark Method or the Modal Superposition Method (Bathe, 1996; Hughes, 1987). However, it is well known that the standard linear FEM may give poor results for higher structural modes (Arndt et al., 2010; Carey and Oden, 1983). This can lead to poor results when solving the wave equation (Torii and Machado, 2010; Bathe, 1996) unless higher order approximations are used. Hierarchical Finite Element Methods (HFEM) (Solin et al., 2004) are probably the most efficient way of constructing higher order approximation for the FEM, and its accuracy for the problem being addressed has been demonstrated by Torii and Machado (2010).

The BEM, on the other hand, is capable of giving very accurate results for the wave equation even when low order approximations are used, as demonstrated by Carrer and Mansur (2009). Consequently it seems that the BEM is more appropriate for the solution of the wave equation problems, in its natural form, than the FEM. However, some structures such as trusses and frames are difficult to model using the BEM, and consequently it cannot be readily applied to some problems. This enforced the use of the FEM in some cases, or in cases that modeling using FEM is easier.

Here, the formulation of both the FEM and the BEM for the problem of one-dimensional elastodynamics is presented and the main differences between them are discussed. The main goal of this paper is to highlight the advantages and disadvantages of each method for the problem being addressed. A complete essay on the FEM is presented by Bathe (1996) and by Hughes (1987), while the BEM is discussed in details by Brebbia and Dominguez (1992) and Brebbia et al. (1984).

2. ONE-DIMENSIONAL WAVE PROPAGATION

The one-dimensional wave propagation problem is governed by the following partial differential equation (PDE) (Timoshenko and Goodier, 1951; Kreyszig, 2006):

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + f(x,t) \;\; \forall x \in \Omega, \tag{1}$$

where u is the displacement, c is the wave propagation velocity, t is the time, $f(x,t)$ is a time varying force and Ω is the domain of the problem. Together with the PDE it is also necessary to prescribe initial and boundary conditions, that are discussed in more details by Kreyszig (2006).

When considering one-dimensional elastodynamics (only longitudinall displacements are allowed inside a linear structural element) the wave velocity is given by (Timoshenko and Goodier, 1951)

$$c = \sqrt{\frac{E}{\rho}}, \tag{2}$$

where E is the Young Modulus of the material and ρ is the density of the material. That is, one-dimensional elastodynamics for longitudinal displacements is governed by the wave equation (Kreyszig, 2006).

In order to solve the problem being addressed one needs to know, apart from Eq. (1), the initial and the boundary conditions to which the problem is subject. The boundary conditions will state whether the boundaries of the domain are allowed to displace or not. The initial conditions, on the other hand, will state the displacements and velocities and accelerations at initial time. In the context of FEM and BEM, however, it is customary to uncouple the x-t relation in order to apply very general solutions techniques. The main reason behind this choice is that both FEM and BEM are generally formulated for elliptic problems, but the wave equation is actually a hyperbolic problem.

2.1. Time integration scheme

The common basis of all time integration methods is that the continuous time variations are approximated by some discrete update rule. That is, we first assume that the solution will be computed at discrete time instants t_i and then apply some approximate update rule. The time integration scheme used here is the Houbolt method (Bathe, 1996), that is defined by

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\Delta t^2}\left[2\mathbf{u}_{n+1} - 5\mathbf{u}_n + 4\mathbf{u}_{n-1} - \mathbf{u}_{n-2}\right], \tag{3}$$

where $\ddot{u}$ represent the second time derivative of the displacements (accelerations), Δt is the time step selected and the indices of $n+1$, n, $n-1$, $n-2$ indicate time steps.

Substituting Eq. (3) into Eq. (1) one gets

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\ddot{u}(x,t_{i+1}) + f(x,t_{i+1}), \tag{4}$$

that does not depend upon t anymore, but only on the approximation to $\ddot{u}$.

Equation (4) can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} = F(x) \tag{5}$$

where

$$F(x) = \frac{1}{c^2}\ddot{u}(x,t_{i+1}) + f(x,t_{i+1}). \tag{6}$$

Note that Eq. (5) is elliptic (time variations have been removed by using the time integration approximation) and consequently its numerical solution is simpler. Besides, Eq. (5) can now be solved by techniques used to solve static structural analysis just by considering f as defined in Eq. (6) as a body force. That is why time integration schemes are so popular. Once the time discretization is applied the dynamic problem is reduced to a series of static problems that can be solved by standard procedures designed for elliptic problems.

Before continuing, it is important to mention that boundary value problems of the type of Eq. (5) have boundary conditions of the type

$$\begin{cases} u(x) = g(x) & \forall x \in \Gamma_1 \\ \dfrac{\partial u(x)}{\partial x} = h(x) & \forall x \in \Gamma_2 \end{cases}, \tag{7}$$

where the boundary of the problem is $\Gamma = \Gamma_1 \bigcup \Gamma_2$. In this case, the boundary condition defined for *u(x)* are called essential boundary conditions, while the ones defined for its derivative are called natural boundary conditions (Bathe, 1996; Reddy, 1998).

2.2. Weak form – Finite Element Method

The standard procedure for solving elliptic problems using the FEM is as follows (Carey and Oden, 1983; Reddy, 1998). First, multiply the differential equation from Eq. (5) by some test function v and integrate in the domain to get

$$\int_\Omega \frac{\partial^2 u}{\partial x^2} v d\Omega = \int_\Omega vFd\Omega \tag{8}$$

and then apply Green's Theorem and rearrange to get

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} d\Omega = \left[v\frac{\partial u}{\partial x}\right]_\Gamma - \int_\Omega vFd\Omega\,. \tag{9}$$

Equation (9) is known as the weak (or variational) form of the problem and is the starting point of the FEM. Note that the problem from Eq. (9) reduces the order of continuity required for the variables on the solution, since now it must respect the original problem in an integral sense. That is why this form is called the weak form of the problem. Besides, note that the natural boundary conditions appear on Eq. (9), but the essential boundary conditions do not.

We now assume that an approximate solution u_h will be sought that is of the form

$$u_h = \sum_{i=1}^{N} u_i \phi_i \tag{10}$$

and assume test functions of the form

$$v_h = \sum_{j=1}^{N} v_j \phi_j\,. \tag{11}$$

Since the essential boundary conditions do not appear in Eq. (9), one must enforce that both u_h and v_h respect these conditions. The most common approach for ensuring this is by choosing u_h and v_h that automatically respect the essential boundary conditions.

Substituting Eq. (10) and Eq. (11) into Eq. (9) one obtains the following system of linear equations:

$$\mathbf{Ku} = \mathbf{F}\,. \tag{12}$$

The whole procedure is described in details by texts on the FEM (Bathe, 1996; Hughes, 1987). Assuming the force vector **F** is as given in Eq. (6) the system of linear equations from Eq. (12) can be rewritten as

$$\mathbf{Ku} + \mathbf{M\ddot{u}} = \mathbf{f}(t)\,. \tag{13}$$

The problem can then be solved using the Houbolt method, or any other time integration method, by solving Eq. (13) for each time step, where the force vector **f** is a vector of applied forces. More details on the application of time integration schemes for dynamic analysis are discussed by Bathe (1996) and Hughes (1987).

By using standard FEM procedures one is able to build the functions used to approximate the solution given by Eq. (10). In a standard FEM, polynomials of degree up to two are generally used, since generating polynomials of higher order can be cumbersome. However, the Hierarchical approach (HFEM) allows one to easily build approximations of arbitrary order, by making use of Lobatto polynomials. A detailed presentation on the HFEM is given by Solin et al. (2004) and the application of hierarchical basis to the problem being addressed is discussed in details by Torii and Machado (2010).

2.3. Integral form – Boundary Element Method

If Green's Theorem is applied to Eq. (9) then the problem is weakened (posed under less strict conditions) once more (Brebbia and Dominguez, 1992). This gives

$$\int_{\Omega} u \frac{\partial^2 v}{\partial x^2} d\Omega = \left[u \frac{\partial v}{\partial x} \right]_{\Gamma} - \left[v \frac{\partial u}{\partial x} \right]_{\Gamma} + \int_{\Omega} vF d\Omega , \tag{14}$$

that is known as integral form of the problem. Integral formulations have been subject of research for many years and there is a rich literature in the subject (Roach, 1970; Tricomi, 1957). Besides, Eq. (14) is the starting point of the BEM.

Note that one is able to get rid of the first domain integral from Eq. (14) in the case that the test function v^* is chosen to satisfy

$$\frac{\partial^2 v^*(x,\xi)}{\partial x^2} = \delta(x-\xi) , \tag{15}$$

where δ is the Dirac's delta (Brebbia and Dominguez, 1992).

Even if it seems difficult to conceive such a test function v^* that satisfy Eq. (15), it turns out that such functions are known as Generalized solutions (or Green's functions) and there is a vast literature covering them (Roach, 1970; Tricomi, 1957). the function v^* that satisfy Eq. (15) for the problem being addressed is known to be (Brebbia and Dominguez, 1992)

$$v^*(x,\xi) = \frac{|x-\xi|}{2} . \tag{16}$$

Assuming that the test function is as defined by Eq. (15) and Eq. (16) the integral form from Eq. (14) becomes

$$\int_{\Omega} u\delta(x-\xi) d\Omega = \left[u \frac{\partial v}{\partial x} \right]_{\Gamma} - \left[v \frac{\partial u}{\partial x} \right]_{\Gamma} + \int_{\Omega} v^* F d\Omega \tag{17}$$

that simplifies to

$$u(\xi) = \left[u \frac{\partial v^*}{\partial x} \right]_{\Gamma} - \left[v^* \frac{\partial u}{\partial x} \right]_{\Gamma} + \int_{\Omega} v^* F d\Omega . \tag{18}$$

Note that in the case where no body forces are present and no time variations are considered (i.e, F= 0 everywhere) then the problem from Eq. (18) could be solved without performing any domain integral.

The one-dimensional formulation is a very particular case of the BEM since in this context one does not need to assume an approximation to the solution u_h in the boundary of the problem. This happens since it is necessary to evaluate the solution u at the boundary of the problem, that in this case is composed only by the two nodes at the extremes of the domain. The same does not hold for two and three-dimensional problems, since in these cases the boundary conditions are not defined on points anymore, but on lines and surfaces and thus one need to interpolate the solution there (Brebbia and Dominguez, 1992). Note, however, that in the problem being addressed need to approximate u inside the domain in order to carry the domain integrals and apply the time integration scheme. More details on the BEM are presented by Brebbia and Dominguez (1992) and Brebbia et al.(1984).

Writing Eq. (18) for different ξ then gives the following system of linear equations (Thoaldo and Carrer, 2010)

$$\mathbf{Hu} = \mathbf{Gq} + \frac{1}{c^2}\mathbf{M}\ddot{\mathbf{u}}\,, \tag{19}$$

where **u** is a vector of displacements and **q** is a vector of normal derivatives of **u**.

As occurs for the FEM, the problem can then be solved using the Houbolt method, or any other time integration method, by solving Eq. (19) for each time step. More details on this subject are discussed by Brebbia et al.(1984) and Carrer and Mansur (2009).

3. NUMERICAL RESULTS

The problem studied here consists a bar with length $L = 1$, cross section area $A = 1$, density $\rho = 1$ and Young Modulus $E = 1$ presented in Fig. 1. Consequently, the wave velocity is $c = 1$. The bar is constrained at both ends and is subject to the initial displacement u_{max} at the middle of the bar as depicted in Fig. 1. Besides, the initial velocities and accelerations are equal to zero at the initial time and there are no body forces. This problem can be solved analytically by applying standard procedures, see (Kreyszig, 2006).

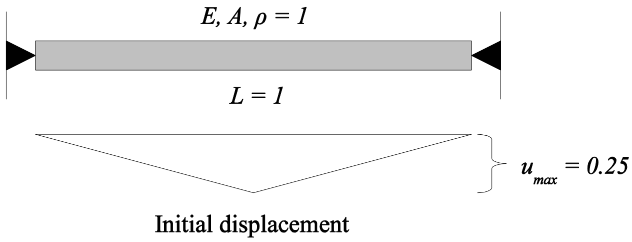
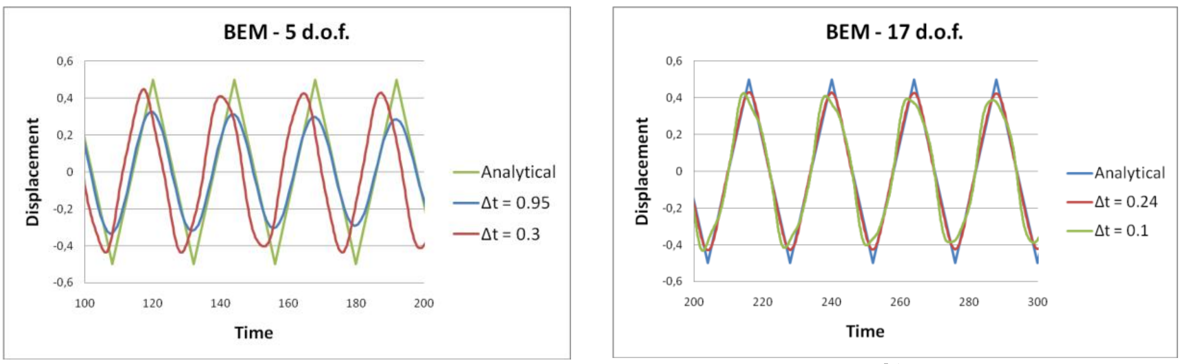


Figure 1. A bar constrained at both ends subject to an initial displacement.

3.1. Solutions given by the BEM

The problem was solved by the BEM using 5 and 17 nodal points (i.e, 5 and 17 d.o.f.). For each one of these discretizations two time steps were used. The first is obtained by the expression for an optimal time step as presented by Carrer and Mansur (2004), and the other is obtanaied with a time step that was found to be appropriate for the FEM. When using 5 nodes, times steps equal to 0.95 and 0.3 were used, and the results are presented in Fig. 2.a. When using 17 nodes times steps equal to 0.24 and 0.1 were used, and the results are presented in Fig. 2.b. Note that the time window presented in Fig. 2a is not the same as presented in Fig. 2b. That's because the solutions obtained with 17 d.o.f. are very accurate and consequently it is difficult to make comparisons for t < 200 when using this discretization.

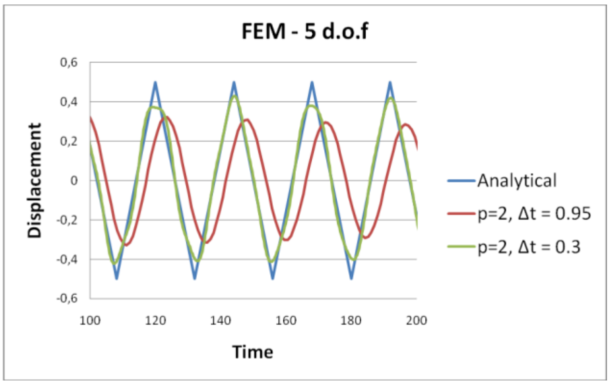


a) b)

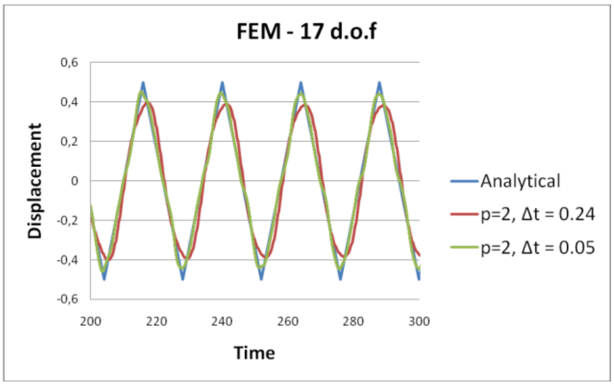
Figure 2. Displacement at the middle of the bar when using the BEM with a) 5 d.o.f. and b) 17 d.o.f. Note that the time window presented in a) is not the same as presented in b).

3.2. Solutions given by the FEM

The problem was solved by the FEM using 5 and 17 d.o.f. and considering a hierarchical finite element of order p = 2 (i.e., a quadratic finite element). For each one of these discretizations two time steps were used. The first is the same that used for the BEM, and the other is the time step found to be appropriate for the FEM. When using 5 d.o.f., time steps equal to 0.95 and 0.3 were used, and the results are presented in Fig. 3.a. When using 17 d.o.f., times steps equal to 0.24 and 0.05 were used, and the results are presented in Fig. 3.b. Note that the time window presented in Fig. 3a is not the same as presented in Fig. 3b, for the same reason discussed for the case of the BEM.



a)



b)

Figure 3. Displacement at the middle of the bar when using the FEM for p = 2 (quadratic aproximation) with a) 5 17 d.o.f. and b) 17 d.o.f. Note that the time window presented in a) is not the same as presented in b).

3.3. Comparison between the solutions given by the BEM and the FEM

From Fig. 2 and Fig. 3 one first conclusion can be drawn. Time steps that are optimum for the BEM are not necessarily suitable for the FEM. Note that when using 5 d.o.f., the BEM obtained best results for a time step equal to 0.95 while the FEM obtained best results with a time step equal to 0.3. The same occurs when using 17 d.o.f.; the BEM obtained best results with a time step equal to 0.24, while the FEM obtained best results with a time step equal to 0.05.

We also note that both methods are able to give accurate results. Torii and Machado (2010) have shown that the linear FEM gives very poor results in some cases, and may not be an appropriate approach for the wave propagation phenomenon. However, the HFEM using a quadratic polynomial gives satisfactory results, and thus can be viewed as a viable alternative for the problem being addressed.

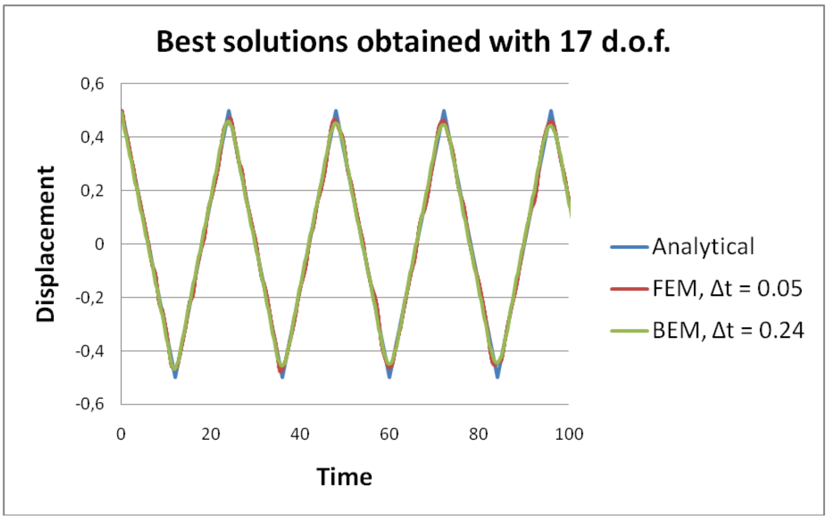
In terms of numerical accuracy one final comparison can be made. In Fig. 4 the best results obtained with each method when using 17 d.o.f. are compared. Note that the optimal time step used by the FEM is smaller than the one used by the BEM. In this case, Fig. 4b seems to show that the FEM was able to give more accurate results for its optimal time step. However, the BEM obtained very accurate results for a much bigger time step, thus resulting in smaller computational effort. Finally, from Fig. 4a it can be seen that both methods were able to achieve very accurate results for the initial time steps, and the difference between them is barely noticeable.

4. A GENERAL COMPARISON BETWEEN THE FEM AND THE BEM

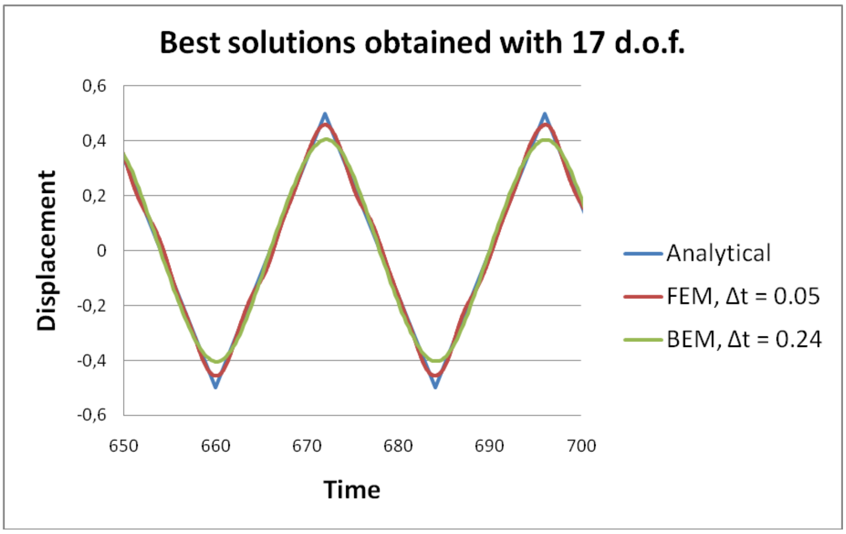
A general comparison between the FEM and the BEM is very difficult to accomplish. That's because the FEM is based on the weak form of the problem, while the BEM is based on the integral form of the problem. This fact can be viewed as the main source of differences between the two methods.

The first difference between them is that the FEM evaluates the solution on the entire domain of the problem, while the BEM needs to evaluate it only at the boundary of the problem in the case that no time variation occurs and no body forces are present. This is a very important difference in many applications, since the BEM may not need a discretization of the domain (unless for integration purposes) and this may represent a significant save in computational effort. The FEM, on the other hand, always need domain discretizations, a procedure that may be cumbersome in some applications, mainly for complex geometries. This leads to an increase in the computational effort needed by the FEM. However, in the case being addressed this is not a vital difference, since the BEM also needs a domain discretization in order to carry the time integration procedure.

A second difference between the two methods is that the system of linear equations given by the FEM is solved, in general, for a symmetric positive definite matrix. The BEM, on the other hand, leads to the solution of a system of linear equations given by a non symmetric matrix. This is a very important feature since it is well known that solving a system of linear equations for a symmetric matrix is much easier than solving one for a non symmetric matrix (Kelley, 1995). Indeed, it can be said that for the same number of d.o.f. the system of linear equations given by the BEM needs, in general, more computational effort.



a)



b)

Figure 4. Best solutions obtained with the quadratic FEM and the BEM for two time windows a) t=[0,100] and b) t =[650,700].

From the mathematical point of view, one important difference between the two methods is the requirement placed upon the form of the approximate solution. Note that the weak form from Eq. (9) needs an approximation that have first order derivatives that can be integrated (in the Lebesgue sense)(Reddy, 1998) , i.e., $u \in H^1(\Omega)$. The integral form from Eq. (13), however, require only the approximation itself to integrable, and thus $u \in H^0(\Omega)$. This is the reason why constant elements can be used in the BEM, while the FEM needs at least linear elements.

Another aspect to be taken into account form the mathematical point of view is that the FEM is heavily based on the Galerkin Method and on Approximation Theory (at least in its more traditional formulation), while the BEM is based on the mathematical theory of integral equations and Green's functions. In this context it can be said that both methods have strong mathematical backgrounds, but that most mathematical conclusion drawn for one does not necessarily apply to the other.

From the modeling point of view, the use of the FEM can be advantageous in cases where the domain of the problem is composed of several sub domains. One classical example is the case of trusses and frames. Once a bar finite element or a beam finite element is formulated, its application to truss and frame structures is straightforward (Bathe, 1996). The BEM, however, cannot be easily applied to trusses and frames, since special procedures are needed in order to connect different sub domains.

Finally, the BEM and the FEM have two very important characteristics in common. First, from the mathematical point of view, both methods do not solve the approximate problem using its strong form. This leads to important implications in the theories behind them and computational aspects. The Finite Difference Method, for example, is solved for an approximate problem based on the strong form, and this can lead to stability issues. Second, from the computational point of view, both the FEM and the BEM use some kind of "element". This allows one to easily build approximate solutions, since the global approximations are built locally and then patched together. Note that this feature distinguishes these methods from Meshfree Methods (Liu, 2003), for example, where great effort is dedicated to the construction of appropriate approximations.

5. CONCLUDING REMARKS

The BEM and the FEM are both efficient methods for obtaining approximate solutions for problems governed by partial differential equations. The success of both methods is based on its sound mathematical foundations and its adequacy for computational implementation.

This paper discussed the main aspects of the application of both the FEM and the BEM for the problem of one-dimensional wave equation. It has been shown that the BEM and the FEM are based on different forms of the problem and this leads to most differences between them. However, both methods still have some common features, as the need for some kind of discretization. Besides, both methods are able to give very accurate results for the problem being addressed.

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