ANALYSIS OF EXTREMA, OVERSHOOT AND UNDERSHOOT IN THE LINEAR CONTROL CONTINUOUS-TIME SYSTEM WITH REAL MULTIPLE ZEROS AND REAL POLES

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Abstract. *This paper deals with the problem for the number of extrema, that which may occur in the step- response of a stable linear system with k real multiple zeros and n real distinct poles. Some simple sufficients conditions and necessary conditions are presented for analyses when zeros located between the dominant and fastest pole does not cause extrema in the step-response. Sufficient conditions for existence of the overshoot and extended type r_u undershoot in the step-response of the continuous time transfer functions, based on their poles and zeros, are presented. The authors also present a class of linear control stable continuous time-system of minimum phase that exhibits undershoot in the step response. Simple examples illustrate and complement the main results of this paper. These conditions require knowledge of the pole-zero configuration of the corresponding transfer-function.*

Keywords*: Extrema, overshoot, undershoot, pole, zero.*

1. INTRODUCTION

Automatic control has played a vital role in the advance of engineering and science. In addition to its extreme importance in space-vehicle systems, missile-guidance systems, robotic systems, automatic control has become an important and integral part of modern manufacturing and industrial processes (Franklin, 1991; Dorf, 2001; Ogata, 2005).There exist some control problems, such as machine tool axis control and trajectory-following in robotics, where the step-response cannot exhibit local extrema. Several works have been done to clarify the influence of the zeros on the transient part of a step-response (Stewart, J., 2006; Darbha, S., 2003; El-Khoury *et alii*, 1993; Howell, 1997; Rachid, 1995; Leon de la Barra, 1994; Reis *et alii*, 2010: a-e, 2009, 2008:a-b, 2007, 2005:a-b).

El-Khoury et al (1993) obtained an upper bound on the number of extremes of the step-response of a linear system with real distinct poles, complementing the existing results for lower bounds (Widder, 1934). These results contribute to the fact that zeros located between the dominant pole and the pole faster can cause extreme. Rachid (1995) presents a sufficient condition for extrema of the step-response. Proved that every real zero related to a real pole and that this relation, the zero is located to the left of this pole does not contribute to the extreme step-response. Stewart, J. (2006) examines overshoot and reverse reaction associated with non-minimum phase zeros.

In Reis (2001, 2002, 2003, 2004, 2004-a) are presented necessary and sufficient condition for the existence of extremes, overshoot and undershoot in the step-response of second order continuous time transfer functions and same class the control systems of the third order, based on its real poles and zeros. This works results the news necessary condition and sufficient condition for the existence of extremes in the continuous-time system of *n* order with distinct real poles and distinct or multiply real zeros (Reis, 2010:a-e, 2009, 2008:a-b, 2007, 2005:a-b; Silva, 2008). These results are extensions of the works the El-Khoury *et alii* (1993) and Rachid (1995). These conditions permit to avoid when the zeros located between the dominant and fastest pole not cause extreme in the step-response. Are proved that negative real zeros also cause undershoot in the step-response. It is important because in literature undershooting phenomenon is association a positive zeros.

These results are important but they cannot offer a complete relation between the relative locations of the poles and zeros of the plant and controllers and the existence of extremes (overshoot and undershoot). For example, the determination of the exact number of extremes remains an open problem (El-Khoury, 1993). In the opinion of the authors, this note provides new insight about the correlation between poles and zeros of a scalar continuous-time transfer function and the nature of the extremes overshoot and undershoot in its step-response. These results do not constitute the final understanding of this connection, but they certainly complement, clarify and expand the various points, which have been subject of recent discussion in the literature**.** Furthermore the results presented can have many control engineering applications, especially in controller synthesis. In fact, they can be used to design a controller ensuring no overshoot and undershoot for the closed-loop step-response for a linear minimum-phase system (Rachid, 1995).

The paper is organized as follows. Section 2 contains definitions and background material. In Section 3 main results, which qualitatively correlate real zeros and extremes are presented. Applications are presented in the Section 4 and concluding remarks are given in Section 5.

2. PRELIMINARES

In this paper consider a SISO linear control stable continuous-time system with n real distinct poles and k_1 multiply real zeros characterized by their continuous-time strictly proper transfer function *G(s):*

$$G(s) = (-1)^{n+m} K \frac{\prod_{i=1}^{k_1} (s - z_i)^{r_i}}{\prod_{j=1}^{n} (s - \lambda_j)} \tag{1}$$

with:

- $m = \sum_{i=1}^{k_1} r_i < n,\ K = \dfrac{\prod_{j=1}^{n} \lambda_j}{\prod_{i=1}^{k_1} (z_i)^{r_i}},\ \lambda_1 < \lambda_2 < ... < \lambda_n < 0, z_1 < ... < z_{k_1}$ e $z_i \neq \lambda_j$;
- $z_i,\ i = 1, ..., k_1$ are real zeros of the *G(s)*, $z_1 < ... < z_{k_1}$;
- $\lambda_j, j = 1,...,n$ are real poles of the *G(s)* e $z_i \neq \lambda_j$;

It is convenient to classify the zeros of *G(s)* in four different sets:

$$M_1 = \{z : G(z) = 0,\ 0 < z < +\infty\},\ M_2 = \{z : G(z) = 0,\ \lambda_n < z < 0\},\ M_3 = \{z : G(z) = 0,\ \lambda_1 < z < \lambda_n\},$$
$$M_4 = \{z : G(z) = 0,\ -\infty < z < \lambda_1\}. \tag{2}$$

In addition, let m_i,for $i = 1, 2, 3, 4$, denotes the number of zeros belonging to a give class M_i, such that $m = m_1 + m_2 + m_3 + m_4$. A ***pole bracket*** is the open interval $(\lambda_{i-1}, \lambda_i)$ between two distinct consecutive poles $\lambda_{i-1} < \lambda_i$ of *G(s)*. Let ***p be the number of poles brackets*** containing an odd number of zeros of *G(s)* (El-Khoury *et alii,* 1993) and let integer $\eta \geq 0$ be the ***total number of local extreme of y(t)***, for $t > 0$.

The following lemma gives a unit step-response for the system (1). The proof of this lemma follows from the expansion in partial fraction of the *G(s).*

Lema 2.1: The unit step-response of the class linear control system with G(s) as in (1) is given by:

$$y(t) = 1 + \sum_{j=1}^{n} c_j e^{\lambda_j t} \tag{3}$$

where for $k = 2, ..., n$:

- $$c_1 = (-1)^{m+1} \frac{\prod_{j=2}^{n} \lambda_j}{\prod_{i=1}^{k_1} (z_i)^{r_i}} \prod_{i=1}^{k_1} (\lambda_1 - z_i)^{r_i} \frac{\prod_{2 \leq i < j \leq n} (\lambda_j - \lambda_i)}{\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)}; \tag{4}$$
- $$c_k = (-1)^{m+k} \frac{\prod_{j=1, j \neq k}^{n} \lambda_j}{\prod_{i=1}^{k_1} (z_i)^{r_i}} \prod_{i=1}^{k_1} (\lambda_k - z_i)^{r_i} \frac{\prod_{1 \leq i < j \leq n,\ i,j \neq k} (\lambda_j - \lambda_i)}{\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)}. \tag{5}$$

The problem to find a lower bound η for the number of step-response extrema was solved by Widder (1934) and an upper bound by El-Khoury *et alii* (1993). In the end, was considered a SISO linear control stable continuous-time system with *n* real distinct poles and *m* real distinct zeros without poles at the origin of the complex plane. Rachid (1995) contributed with sufficient condition for the absence of extremes. These results are presented below.

Theorem 2.2: (Lower-bounding theorem) $m_1 + m_2 \leq \eta$.

Theorem 2.3: (General bounding theorem).
(i) $m_1 + m_2 \leq \eta \leq m_1 + m_2 + m_3 - p;$
(ii) $parity\,\eta = parity(m_1 + m_2) = parity(m_1 + m_2 + m_3 - p)$, where $parity\,(x) = 0$ if x odd and $parity\,(x) = 1$ if x even, x be an integer.

Theorem 2.4: (Rachid, 1995**)** The step-response of system (1) has no extremum for $t > 0$ if there exists a relation R satisfying the following conditions:
(i) $z\,R\,\lambda \Leftrightarrow z < \lambda;$
(ii) Each pole λ is related by R to $\nu(\lambda)$ zeros, and;
(iii) Each finite or infinite zero z is related by R to $\nu(z)$ poles,
Where z is a zero (finite or infinite), λ is a pole of G(s) and $\nu(\,)$ denotes the order of multiplicity of *()*.

In this article, it wos considered the analysis of extremes in the step-response of system (1), where its zeros are located in the class M_3, ie, between the dominante and farthest pole. The goal is to provide extensions of the theorems 2.3 and 2.5 and results in Reis (2010: e).

3. MAIN RESULTS

The theorems presented below, provide a necessary condition for zeros of class M_3 cause extremes in step-response, and a sufficient condition for the absence of extrema. In this sense, they are extensions of the theorems 2.3 and 2.5, as well as a generalization of the results presented in Reis (2010: e), and have considered a SISO linear control stable continuous-time system with n real distinct poles and m_1 real distinct zeros.

Theorem 3.1: Let a SISO linear control stable continuous-time system with n real distinct poles and k_1 multiply real zeros characterized by their continuous-time strictly proper transfer function *G(s) (5)* and step-response (6) – (8). If $m = m_3$ and zeros of the class M_3 cause extremes in step-response then $\left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1} - z_i}{\lambda_n - z_i}\right)^{r_i}\right| > 1.$

Theorem 3.2: Under assumptions of the theorem 3.1, if $m = m_3$ and $\left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1} - z_i}{\lambda_n - z_i}\right)^{r_i}\right| < 1$, then zeros in class does not contribute to extremes in response to unit step.

Corollary 3.1: If $m = m_3$ and $z_i \in \left(\lambda_1, \frac{\lambda_{n-1} + \lambda_n}{2}\right)$, $\forall i = 1, ..., m$, then $y(t)$ has no extreme.

Observation:
(a) The theorem 3.1 gives a necessary condition for zeros of class M_3 cause extremes in step-response. Theorem 3.2 provides a sufficient condition for the absence of extremes.
(b) The theorem 3.2 is an extension of theorem 2.4. In fact, if the m_3 zeros of the class M_3 are related to the m_3 poles as in Theorem 2.4, then $\left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1} - z_i}{\lambda_n - z_i}\right)^{r_i}\right| < 1$ since there are at least $m_3 - 1$ zeros to the left of the pole λ_{n-1}. Therefore $y(t)$ does not show the extremes by theorem 3.2;

For prove the theorems 3.1 and 3.2 it is convenient to make the following analysis. It follows from the lemma 2.1 that:

$$\frac{y'(t)}{c_1 \cdot \lambda_1 e^{\lambda_1 t}} = 1 + \sum_{j=2}^{n} \frac{c_j \lambda_j}{c_1 \lambda_1} e^{(\lambda_j - \lambda_1)t}\,. \tag{6}$$

In the equation (6), define:

$$c_i^{(1)} = c_i \lambda_i\,, \text{ for } i = 1, \ldots, n, \tag{7}$$

$$d_j^{(1)} = \frac{c_j^{(1)}}{c_1^{(1)}} \text{ e } \lambda_j^{(1)} = \lambda_j - \lambda_1 \text{, for } j = 2, \ldots, n. \tag{8}$$

After the substitution of (7) and (8), it follows that $\dfrac{y'(t)}{c_1 \cdot \lambda_1 e^{\lambda_1 t}} = 1 + \sum_{j=2}^{n} d_j^{(1)} e^{\left(\lambda_j^{(1)}\right)t}$.

Define the follow function $f_1(t)$:

$$f_1(t) = 1 + \sum_{j=2}^{n} d_j^{(1)} e^{\left(\lambda_j^{(1)}\right)t}. \tag{9}$$

The equations (6) – (9) follows that:

$$\frac{y'(t)}{c_1^{(1)} e^{\lambda_1 t}} = f_1(t) \cdot \tag{10}$$

The relation (10), it follows that the critical points of $y(t)$ are points t_o in $[0, +\infty)$ for that $f_1(t_o) = 0$. To analyze the number of real roots of the function $f_1(t$ follows the equation (9) that:

$$\frac{f_1'(t)}{c_2^{(2)} e^{\lambda_1^{(1)} t}} = f_2(t) \tag{11}$$

where:

$$f_2(t) = 1 + \sum_{j=3}^{n} d_j^{(2)} e^{\left(\lambda_j^{(2)}\right)t}, \tag{12}$$

$$c_i^{(2)} = d_i^{(1)} \cdot \lambda_i^{(1)}, \text{ for } i = 2, \ldots, n, \tag{13}$$

$$d_j^{(2)} = \frac{c_j^{(2)}}{c_2^{(2)}} \text{ e } \lambda_j^{(2)} = \lambda_j^{(1)} - \lambda_2^{(1)}, \; j = 3, \ldots, n. \tag{14}$$

Follow of the (11) that the critical points of $f_1(t)$ are roots of the function $f_2(t)$. Continuing with this process, we get:

$$f_k(t) = 1 + \sum_{j=k+1}^{n} d_j^{(k)} e^{\left(\lambda_j^{(k)}\right)t} \quad \forall 2 \le k \le n-1, \tag{15}$$

$$c_j^{(k)} = d_j^{(k-1)} \lambda_j^{(k-1)}, \text{ for } j = 2, \ldots, n, \tag{16}$$

$$d_j^{(k)} = \frac{c_j^{(k)}}{c_k^{(k)}}, \quad \lambda_j^{(k)} = \lambda_j^{(k-1)} - \lambda_k^{(k-1)}, \; j = 3, \ldots, n, \tag{17}$$

$$f_k(t) = \frac{f_{k-1}'(t)}{c_k^{(k)} e^{\lambda_k^{(k-1)} t}}, \quad k = 2, \ldots, n-1. \tag{18}$$

From equation (18), it follows that the critical points of the function $f_{k-1}(t)$ are the roots of the function $f_k(t)$ for all $k = 2, \ldots, n-1$. Note that if $k = n - 1$, equations (15) and (18) are written as:

$$f_{n-1}(t) = 1 + d_n^{(n-1)} e^{(\lambda_n - \lambda_{n-1})t} \quad \text{e} \tag{19}$$

$$f_{n-1}(t) = \frac{f_{n-2}'(t)}{c_{n-1}^{(n-1)} e^{\lambda_{n-1}^{(n-2)} t}}. \tag{20}$$

Follow then, the following results related to function $f_{n-1}(t)$ in (19).

Lemma 3.1: The function $f_{n-1}(t)$ has a root in $(0, +\infty)$ if and only if $\left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1}-z_i}{\lambda_n - z_i}\right)^{r_i}\right| > 1$.

Idea of proof of lemma 3.1: The function $f_{n-1}(t)$ has a root in $(0, +\infty) \Leftrightarrow \left|-\frac{d_{n-1}^{(n-2)}\ \lambda_{n-1}^{(n-2)}}{d_n^{(n-2)}\ \lambda_n^{(n-2)}}\right| > 1$. Now, expressions (13), (14), (16) and (17), it follows that:

$$\left|-\frac{d_{n-1}^{(n-2)}\ \lambda_{n-1}^{(n-2)}}{d_n^{(n-2)}\ \lambda_n^{(n-2)}}\right| = \left|-\frac{d_{n-1}^{(n-3)}}{d_n^{(n-3)}}\frac{\lambda_{n-1}-\lambda_{n-3}}{\lambda_n-\lambda_{n-3}}\frac{\lambda_{n-1}^{(n-2)}}{\lambda_n^{(n-2)}}\right| = \left|\frac{d_{n-1}^{(1)}}{d_n^{(1)}}\frac{\prod_{\mathrm{i}=1}^{n-3}(\lambda_{n-1}-\lambda_{\mathrm{i}})}{\prod_{\mathrm{i}=1}^{n-3}(\lambda_n-\lambda_{\mathrm{i}})}\right| = \left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1}-z_i}{\lambda_n - z_i}\right)^{r_i}\right|$$

and then follows the proof of lemma 3.1.

Corollary 3.2: The function $f_{n-1}(t)$ has a root *in* $(0, +\infty)$ if:

(a) $m = m_1 + m_2 + m_3$ and $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, +\infty\right)$ $\forall i$ or;

(b) $m = m_3$ and $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, \lambda_n\right)$ $\forall i$ or;

(c) $m = m_3$, $z_i \in (\lambda_1, \lambda_n)$ $\forall i$ and $\left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1}-z_i}{\lambda_n - z_i}\right)^{r_i}\right| > 1$ or;

(d) $m = m_3 + m_4$, $z_i \in (-\infty, \lambda_n)$ and $\left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1}-z_i}{\lambda_n - z_i}\right)^{r_i}\right| > 1$.

As consequences of lemma 3.1 e corollary 3.2, follow the following theorems:

Theorem 3.3: Let a SISO linear control stable continuous-time system with n real distinct poles and k_1 multiply real zeros characterized by their continuous-time strictly proper transfer function *G(s) (5)* and step-response (6) – (8). Then:

(1) $y(t)$ has no extremum if:

a) $m = m_3$ and $z_i \in \left(\lambda_1, \frac{\lambda_{n-1}+\lambda_n}{2}\right)$, $\forall i = 1, ..., m$;

b) $m = m_3 + m_4$ and $z_i \in \left(-\infty, \frac{\lambda_{n-1}+\lambda_n}{2}\right)$, $\forall i = 1, ..., m$.

(2) If $m = m_3$ and $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, \lambda_n\right)$ $\forall i = 1, ..., m$, the number of the extremes the $y(t)$ will be less than m_3 if $m = m_3$ is even or will be less than $m_3 - 1$ if $m = m_3$ is odd.

Proof: Appendix

Theorem 3.4: If $m = m_1 + m_2 + m_3 + m_4$, and $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, +\infty\right)$ $\forall z_i \in M_1 \cup M_2 \cup M_3$, then the number of the extremes the $y(t)$ will be less than $m_1 + m_2 + m_3$ *if* m_3 is even or will be less than $m_1 + m_2 + m_3 - 1$ if m_3 is odd.

Proof: Appendix

As consequences of theorems 3.3 and 3.4, follow the following observations:

(a) The theorem 3.3 shows a class of zeros that does not contribute to extremes in response to step and which are not covered in the classes given by theorem 2.4. In this sense, they are extensions of theorem 2.4. In fact, the items (1) - a) and b), if $m = m_3 \neq 1$, the $m_3 - 1$ zeros z_i such as $z_i \in \left(\lambda_{n-1}, \frac{\lambda_{n-1}+\lambda_n}{2}\right), \forall\, i = 1, ..., m_3 - 1$, do not contribute to the extreme step response and is not linked to any pole. Note that if $m = m_3 = 1$, apply the theorem 2.4;

(b) By theorem 2.3, $m_1 + m_2 \leq \eta \leq m_1 + m_2 + m_3 - p$. If $m_1 = m_2 = 0$, then $0 \leq \eta \leq m_3 - p$. The upper bounds found for η by theorems 3.3 and 3.4 are exactly the same, but these results improve the results provided by theorem 2.3, since they specify the locations of the zeros of the M_3 class so that they can contribute or not with the extremes of the response the step. In fact, by theorem 3.3, $\eta = 0$ if $m = m_3$ and $z_i \in \left(\lambda_1, \frac{\lambda_{n-1}+\lambda_n}{2}\right) \forall\, i$ or if $m = m_3 + m_4$ and $z_i \in \left(-\infty, \frac{\lambda_{n-1}+\lambda_n}{2}\right), \forall i = 1, ..., m$. Moreover, if $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, \lambda_n\right) \forall\, i = 1, ..., m_3$, then $0 \leq \eta \leq m_3$ if m_3 is even or $0 \leq \eta \leq m_3 - 1$ if m_3 is odd, ie if the zeros of the class M_3 are located in this subinterval, then η may take its maximum value if m_3 or $m_3 - 1$.

Proof of the theorem 3.1: Suppose that $m = m_3$ and zeros of the class M_3 cause extremes in to unit step-response. By theorem 2.3, $y(t)$ has at most $m_3 - p$ non-zero extreme in $(0, +\infty)$. Note that if $p = 0$, $\eta = m_3$ and if $p = 1$ then $\eta = m_3 - 1$, coinciding with the upper bounds found in Theorem 3.3. From equation (13), the function $f_1(t)$ will have, at most, $m_3 - p$ roosts in $(0, +\infty)$ and signal change. By Theorem 2.1, shows that $\lim_{t \to 0^+} f_{n-m_3}(t) = K$ and $f_{n-m_3-1}(0)$, $f_{n-m_3-2}(0)$, ..., $f_1(0)$ are all zero. Hence, $f_1(t)$ will have, at most $m_3 - p$ non-zero roots and at most $m_3 - p$ extremes in $(0, +\infty)$, with signal change. Continuing with the review process, $f_{n-m_3-1}(t)$ will have, at most, $m_3 - p$ non-zero extremes in $(0, +\infty)$, with signal change. Since $\lim_{t \to 0^+} f_{n-m_3}(t) = K$, $f_{n-m_3}(t)$ will have, at most, $m_3 - p$ roots in $(0, +\infty)$,which implies the existence of a maximum $m_3 - p - 1$ extremes, since $K \neq 0$. Continuing with the review process, $f_{n-1}(t)$ will have a roots in $(0, +\infty)$. By lemma 3.1, $\left|\prod_{i=1}^{k_1}\left(\frac{\lambda_{n-1} - z_i}{\lambda_n - z_i}\right)^{r_i}\right| > 1$, and this proves the theorem 3.1.

Proof of the theorem 3.2: Suppose by absurd that zeros of the class M_3 cause extremes in step-response. This contradicts the hypothesis, the theorem 3.1.

The results obtained from previous results the following theorems more general, which guarantees that, besides the zeros of the class M_4, the zeros of the class M_3, under certain conditions, does not contribute to the extreme step response.

Theorem 3.5: If $m = m_1 + m_2 + m_3 + m_4$ and $z_i \in \left(\lambda_1, \frac{\lambda_{n-1}+\lambda_n}{2}\right) \forall i = 1, ..., m_3$ then zeros of the class M_3 do not cause extremes in step-response and also, $\eta = m_1 + m_2$.

Proof: Follows directly from theorem 3.4, corollary 3.2 and the fact that $m_1 + m_2 \leq \eta$ (theorem 2.2, Widder, 1934).

4. SOME APPLICATIONS

4.1. A powerful electro-hydraulic forklift can be used to lift palletized material weighing several tons atop platform 35 feet in a construction site (Dorf, 2001). The unitary feedback systems has the open-loop transfer function:

$$G(s) = K\frac{(s+1)^2}{s\left(s^2+1\right)} \tag{20}$$

The closed-loop transfer function is:

$$H(s) = K\frac{(s+1)^2}{s^3 + Ks^2 + (2K+1)s + K}. \tag{21}$$

If $K = 7.5$, $G(s)$ has poles $\lambda_1 = -3.8508$, $\lambda_2 = -3$ and $\lambda_3 = -0.6492$ and zero $z = -1$. By theorem 3.3, $0 \leq \eta \leq 2$. Figure 1 shows the graphics of the step-response in this system.

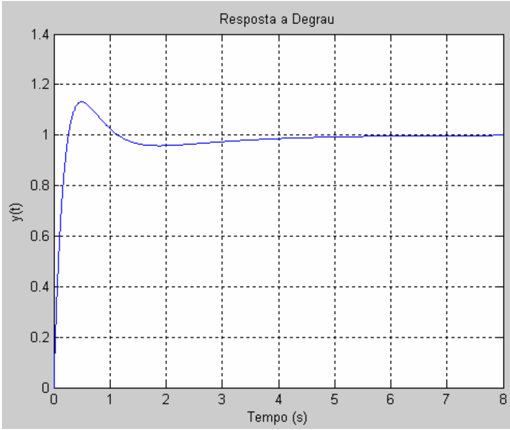


Fig. 1: Step-response if $K = 7.5$.

4.2 We consider the linear control system given by equation (1), when $n = 6$ and $m = 3$ for $G(s)$ given:

$$G(s) = \frac{\left(\frac{1}{3}s+1\right)\left(\frac{2}{5}s+1\right)^2}{\left(\frac{1}{8}s+1\right)\left(\frac{1}{7}s+1\right)\left(\frac{1}{6}s+1\right)\left(\frac{1}{5}s+1\right)\left(\frac{1}{4}s+1\right)(2s+1)}. \tag{42}$$

G(s) has zeros $z_1 = -3$, $z_2 = -2.5$ and poles $\lambda_1 = -8$, $\lambda_2 = -7$, $\lambda_3 = -6$, $\lambda_4 = -5$, $\lambda_5 = -4$ e $\lambda_6 = -0.5$. As $m = m_3 = 3$, by theorems 3.1, 3.4 and corollary 3.1 $y(t)$ has not extremes for $\prod_{i=1}^{2}\left|\frac{\lambda_5 - z_i}{\lambda_6 - z_i}\right| < 1$. Figure 2 shows the graphics of the step-response in this system. Shifting the zero z_2 and putting it in $z_2 = -0.8$ (figure 2 - (a): '+') and then moving both zeros z_1 and z_2, putting them at $z_1 = z_2 = -1$ and -0.8 (figure 2 - (b):'.'), by theorems 3.1 and 3.3, $y(t)$ has at most two extremes. Note that $G(s)$ does not satisfy the theorem 2.4. It is observed that as you approach the zeros of the pole $\lambda_6 = -0.5$, the overshoot significantly increases the value of the function $y(t)$ at the minimum becomes negative. This can be seen making the shifting of z_1 and z_2, putting them at $z_1 = -0.6$ and $z_2 = -0.5235$. Figure 2 - (b) shows this effect.

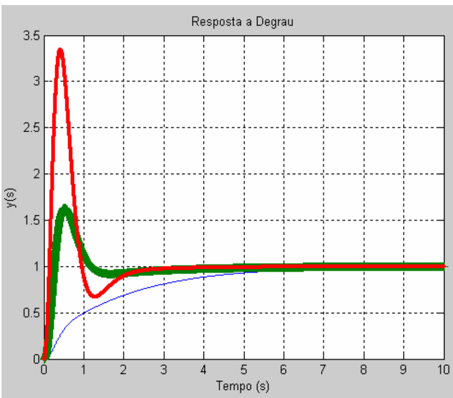


Fig. 2 - (a): (.) - (+): Step-response.

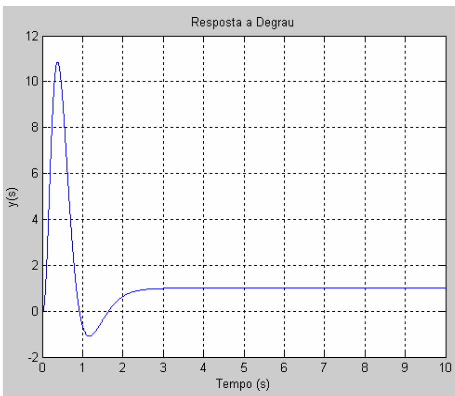


Fig. 2 - (b) Undershoot and overshoot.

This shows that the occurrence of reverse reaction can also occur, though the zeros are located in the category M_3. This fact is important since, in literature the reverse reaction is associated only with positive zeros.

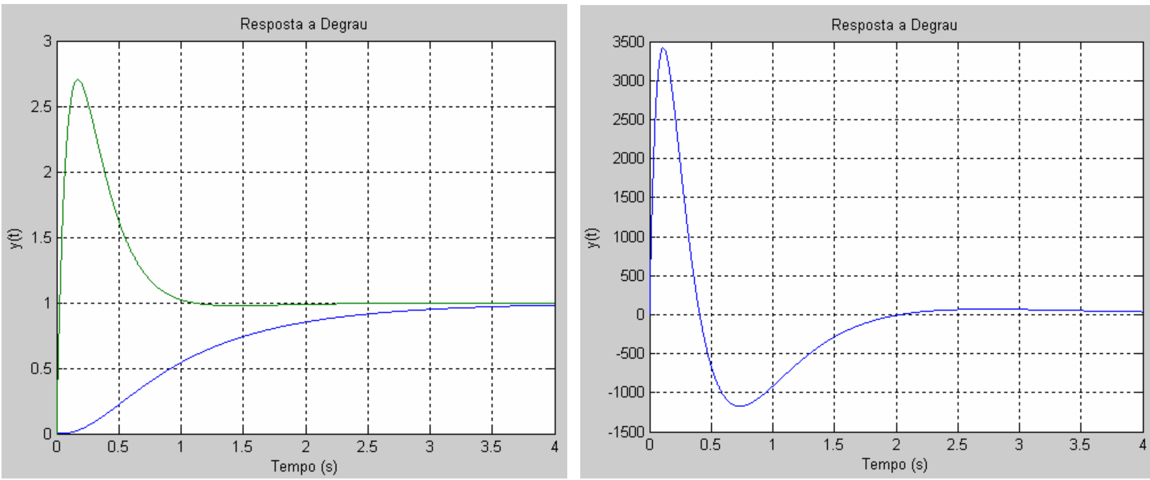
4.3 Consider a SISO linear control stable continuous-time system characterized by their continuous-time strictly proper transfer function *G(s):*

$$G(s)=\frac{\frac{K}{80T}(Ts+1)}{(s+1)\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)\left(\frac{1}{10}s+1\right)}. \tag{24}$$

The control of fuel a car uses a diesel pump that is subject to variation of parameters (Dorf, 2001). Such a system with unit negative feedback controller, has a process to control, considering (24), when $T = 2/3$. Thus, $z = -1.5$ and $z \in M_3$. By theorem 2.4 the step-response has not extremes. Figure 3 – (a) shows the graphics of the step-response. Effecting a shift in z, approaching the origin, the system changes its behavior dramatically. Moreover, changing the multiplicity of z, together with the shift to the right, the effect is the increased number of extreme in the step- response. If $T = 12/10$ e $k = 80$, $T = 10$ e $k = 80.000$ and the multiplicity of z is $m = 3$, the continuous-time strictly proper transfer functions $g(s)$ are:

$$G(s)=\frac{\left(\frac{1}{1.2}s+1\right)^3}{(s+1)\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)\left(\frac{1}{10}s+1\right)} \quad \text{and} \quad G(s)=\frac{(10s+1)^3}{(s+1)\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)\left(\frac{1}{10}s+1\right)}. \tag{25}$$

If $T = 12/10$ e $k = 80$, by theorem 3.3, $\eta = 2$. If $T = 10$ e $k = 80.000$ by theorem 3.4, $\eta = 3$ (Figure 3: (a) - (b) shows the graphics of the step-response in this systems).



(a) : *T = 12/10* e *k =80.* **(b) *T = 10* e *k = 80.000.***

Figure 3: Step-response of *H(s)* (25).

Figure 3: (b) shows the occurrence of overshoot and undershoot. This fact is important since, in literature the reverse reaction is associated only with positive zeros.

5. CONCLUSION

In this paper presented a study of the number of extremes that can occur in step-response in linear control systems stable and continuous time, with real distinct poles and real multiple zeros. It wos proved that there is a specific region on the line for the location of zeros between the pole nearest and farthest from the origin, so they do not contribute to the extreme in the step-response, ie, beyond the zeros of the class M_4, the zeros of the Class M_3 under certain conditions, does not contribute to extremes.

The results presented are necessary conditions and sufficient conditions that complement the results of theorems 2.3 and 2.4 on the relative positions of poles and zeros to zeros or not contribute to the extreme in the step-response, consisting of extensions of these theorems. in this sense, the theorems 3.3 and 3.4 presented a class of zeros which does not contribute to the extreme step response and is not covered in the classes given by theorem 2.4. furthermore, although the upper bounds found for η by theorems 3.3 and 3.4 are exactly the same, but these results improve the results provided by theorem 2.3, since they specify the locations of the zeros of the m_3 class so that they can contribute or not with the extremes of the response the step.

The authors also presented a class of linear control stable continuous time system and minimum phase that exhibits overshoot and undershoot in the step response. It was shown that the reverse reaction can also occur, though the zeros are located in the class M_3. This fact is important since, in literature the reverse reaction is associated only with positive zeros.

In the authors opinion, this note provides new insight about the correlation between poles and zeros of a scalar continuous-time transfer function and the nature of the extremes in its step-response. These results do not constitute the final understanding of this connection, but they certainly complement, clarify and expand the various points, which have been subject of recent discussion in the literature. Furthermore the results presented can have many control engineering applications, especially in controller synthesis. In fact, they can be used to design a controller ensuring no overshoot and undershoot for the closed-loop step-response for a linear minimum-phase system (Rachid, 1995).

6. ACKNOWLEDGMENTS

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8. RESPONSIBILITY NOTICE

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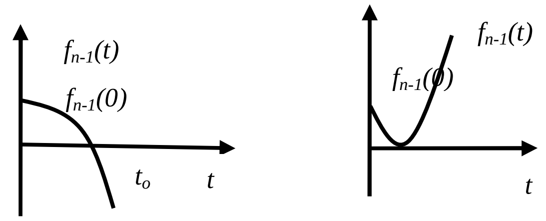
APPENDIX

Idea of proof of theorem 3.3: Suppose n is even, $m = m_3 = \sum_{i=1}^{k_1} r_i$ and $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, \lambda_n\right), \forall i$. By corollary 3.2 shows that the possible ways the graph of $f_{n-1}(t)$ are given in Figure 4, below.



(a) m_3 even (b) m_3 odd
Figure 4: The graph of $f_{n-1}(t)$.

$f_{n-1}(t)$ has a root with sign change if m_3 is even and a root no sign change if m_3 is odd. Hence, from eq. (18), $f_{n-2}(t)$ to have a critical point at t_o in $(0,+\infty)$, which is an absolute minimum point for m_3 even or an inflection point if m_3 is odd. Note that $f_{n-2}(0) > 0$ or $f_{n-2}(0) < 0$ and if m_3 is even, for $t \in (t_o, +\infty)$, $f_{n-2}(t) \to +\infty$.

Thus $f_{n-2}(t)$ has at most two roots with signal change in $(0,+\infty)$, which implies $f_{n-3}(t)$ has at most two critical points with signal change, and at most three roots with sign changes. if m_3 is odd, $f_{n-2}(t)$ is decreasing in $(0,+\infty)$ and $f_{n-2}(t) \to -\infty$. Therefore, $f_{n-2}(t)$ has at most one root with sign change in $(0, +\infty)$, then $f_{n-3}(t)$ has at most one critical point with sign change and, at most, 2 roots with sign changes. Continuing with this analysis, to proof that if m_3 is even, $f_{n-1}(t)$ has a root with sign change if, $f_{n-2}(t)$ has at most two roots with signal change, ..., $f_{n-m_3}(t)$ has at most m_3 roots with signal change. From this function, like all others vanish at the origin, they will at most m_3 roots with sign changes. Hence, $f_1(t)$ has at most roots m_3 roots in $(0,+\infty)$ with sign changes. Therefore, by (10), $y(t)$ will be at most, m_3 extreme in $(0,+\infty)$ if $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, \lambda_n\right), \forall i$. Similarly, if m_3 is odd, $f_1(t)$ will be, at most $m_3 -1$ roots in $(0,+\infty)$ with sign change, which proves item (2) of theorem 3.3. For proof of item (1), simply note that if $z_i \in \left(\lambda_1, \frac{\lambda_{n-1}+\lambda_n}{2}\right), \forall i = 1, ..., m_3$ or if $m = m_3 + m_4$ and $z_i \in \left(-\infty, \frac{\lambda_{n-1}+\lambda_n}{2}\right)$, by lemma 3.1 the function $f_{n-1}(t)$ has no root in $(0, +\infty)$ and then, consequently, the equations (20), (18), (11) and (10) we have that $y(t)$ does not possess extremes in $(0, +\infty)$. The proof for n odd is done similarly.

Idea of proof of the theorem 3.4: Suppose n is even, $m = m_1 + m_2 + m_3 + m_4$ and $z_i \in \left(\frac{\lambda_{n-1}+\lambda_n}{2}, +\infty\right), \forall z_i \in M_1 \cup M_2 \cup M_3$. By corollary 3.2, $f_{n-1}(t)$ has one roots in $(0, +\infty)$. From equations (10) - (19) is proved similarly to the proof of theorem 3.3 that $f_1(t)$ will have at most $m_1 + m_2 + m_3$ roots in $(0, +\infty)$ with sign change, if m_3 is even or have at most $m_1 + m_2 + m_3 -1$ roots in $(0, +\infty)$ with sign change *if* m_3 is odd .From equation (10), $y(t)$ will have at most $m_1 + m_2 + m_3$ extrema in $\left(\frac{\lambda_{n-1}+\lambda_n}{2}, +\infty\right)$ is m_3 is even or have at most $m_1 + m_2 + m_3 -1$ extremes is m_3 is odd, which proves the theorem 3.4.