SOME MATHEMATICAL FORMALITIES ASSOCIATED WITH THE ELASTOPLASTICITY PROBLEM

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Abstract. *The eslastoplastic models have been used for forecast of the mechanical behaviour of materials of the most several natures. A complete investigation of the non linear behaviour of structures it follows from the equilibrium path of the body. This work presents, in a mathematical rigourous framework, the definitions and theoretical results associated to the formulation (well-posed/existence and uniqueness of solution) of a general dynamic elastoplasticity problem.*

Keywords: *Elastoplasticity, existence and uniqueness.*

1. INTRODUCTION

Some matrials has a rate independent (in deformation sense) mechanical behaviour, in this case the eslastoplastic models have been widely used for forecast of these materials behaviour (see Desai (2001)). The main objective is to present the mathematical formalism inherent to the formulation of the classical elastoplastic problem for inifinitesimal strain measurement. The emphasis is on the generalized statement of a nonlinear boundary-value problem and investigation of its solvability. Conditions have been determined that ensure the existence, uniqueness, and dependence of the solution on the loads applied.

2. THE ELASTOPLASTIC MODEL

In this work the elastoplastic model is approached with isotropic hardening and mechanical damage theory. The basic components of a constitutive elastoplastic model for infinitesimal strains are: strain decomposition in plastic part (ε^p) and elastic part (ε^e), constitutive elastic equation, the yield criteria (defined from a yield function); the hardening plastic law that defines the plastic strain, characterize the internal variables evolutions and the yield limit.

Let be a body that occupies the domain Ω in an Euclidean space with a regular boundary (in the continuous normal existence sense) $\partial\Omega$. It consider the functional set U in which the elements are vector-functions that describe the displacement u of the body points. Let X denoting the admissible tensor-functions set for the Cauchy stresses σ and strains ε. In this sense it is assumed that U and X are Hilbert spaces equipped with the inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_X$ with induced norms $\|\cdot\|_U$ and $\|\cdot\|_X$, respectively. Denoting by U^* the dual space of U, it follows that a generalized boundary-value problem from the small elastoplastic strains theory can be formulated in the form of a nonlinear operator equation

$$\aleph(u) = \Bbbk \in U^*, u \in U, \tag{1}$$

where $\aleph : U \to U^*$ is the nonlinear operator associated with the plasticity theory and $\Bbbk \in U^*$ is the linear functional associated to the external loads (applied to the body) work. Unfortunately, it is impossible to give an explicity analytical expression to $\aleph$ operator, but it is possible to define the following mapping

$$\aleph(u) : v \in U \to \langle \aleph(u), v\rangle = (A(H(u))H(u), H(v))_X = (\sigma(u), \varepsilon(v))_X \tag{2}$$

where $\langle\cdot,\cdot\rangle$ represents the duality relationship on $U^* \times U$, $H : U \to X$ is a continuous linear differential (isometric between U and $X \bigcap range(H)$) operator and $A : X \to X$ is a nonlinear operator that defines the relation between stress and strain tensors (see Fig.1).

Let be the mapping $\aleph : U \to U^*$ a Fréchet differentiable operator at each point $v \in U$, i.e., $\exists \aleph'(v) : U \to U^*$ linear operator such that

$$\lim_{t\to 0} \frac{\|\aleph(v+s) - \aleph(v) - \aleph'(v)\, s\|_{U^*}}{\|s\|_U}, \forall s \in U, \tag{3}$$

where $\aleph'(v)\, s = d\aleph(v;s)$ is the differential of the Fréchet operator $v \to \aleph(v)$ at the point v with the increment $s \in U$. Assuming that $\aleph : U \to U^*$ is continuously differentiable, one has $\exists m, M \in \mathbb{R}$ such that

$$\langle \aleph'(v)\, s, s\rangle \geq m\|s\|_U^2, \forall v, s \in U; \tag{4}$$

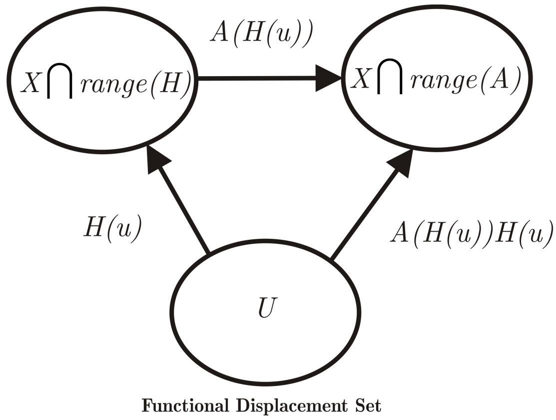

Figure 1. Hilbert Spaces

$$\|\aleph'(v)\, s\|_{U^*} \leq M\|s\|_U, \forall v, s \in U. \tag{5}$$

Note that in this case, it has insured strong monotonicity and Lipschitz continuity of the operator $\aleph$, in fact

$$\langle \aleph(v) - \aleph(w), v - w\rangle = \left\langle \int_w^v \frac{d\aleph}{d\alpha} \cdot d\alpha, v - w \right\rangle, \forall v, w \in U, \tag{6}$$

taking $\alpha = pv + (1-p)w \therefore d\alpha = (v-w)dp$ for real variable p, or $\frac{d\aleph}{dp}dp = \frac{d\aleph}{d\alpha} \cdot d\alpha = d\aleph$, thus

$$\begin{aligned}\langle \aleph(v) - \aleph(w), v - w\rangle &= \left\langle \int_0^1 \aleph'(pv + (1-p)w)(v-w)\, dp, v - w \right\rangle; \\ &= \int_0^1 \langle \aleph'(pv + (1-p)w), v - w\rangle\, dp; \\ &\geq m\|v - w\|_U^2, \forall v, w \in U,\end{aligned} \tag{7}$$

and

$$\begin{aligned}\|\aleph(v) - \aleph(w)\|_{U^*} &= \| \int_0^1 \aleph'(pv + (1-p)w)(v-w)\, dp\|_{U^*}; \\ &\leq \int_0^1 \|\aleph'(pv + (1-p)w)(v-w)\|_{U^*} dp; \\ &\leq M\|v - w\|_U, \forall v, w \in U.\end{aligned} \tag{8}$$

From this results it follows that $\aleph$ is a strongly monotonous and Lipschitz continuous operator having a strongly monotonous and Lipschitz continuous inverse operator $\aleph^{-1} : U^* \to U$ (for more details see Rockafellar (1970); Rockafellar and Wets (1997)).

Defining the nonlinear operator $\wp : X \to X$ with the use of the mapping:

$$\wp : \eta \in X \to \wp(\eta) = A(\eta)\eta \in X. \tag{9}$$

It is assumed that the operator $\eta \to A(\eta)\eta$ is continuously differentiable in a Fréchet sense, and from this definition one has

$$d\wp(\eta; \mu) = \wp'(\eta)\, \mu = dA(\eta; \mu)\eta + A(\eta)\mu, \tag{10}$$

the differential of the operator $\wp$ at the point $\eta \in X$ on the increment $\mu \in X$, and considering the continuous linear operator H, one has

$$\aleph'(v)t = d\aleph\left(v;t\right): w \rightarrow \left(d\wp\left(H\left(v\right);H\left(t\right)\right), H\left(w\right)\right)_X;$$
$$= \left(\wp'\left(H\left(v\right)\right)H\left(t\right), H\left(w\right)\right)_X = \left\langle \aleph'\left(v\right)t, w\right\rangle, \forall v,t,w \in U. \quad (11)$$

Lemma 1. *Let be $\wp : X \rightarrow X$, a continuously differentiable mapping in Fréchet sense, and the positive-definite and bounded derivative $\wp'(\eta), \forall \eta \in X$, i.e., $\exists M, m \in \mathbb{R}_+^*$ such that*

$$\left(\wp'\left(\eta\right)\mu, \mu\right)_X \geq m\|\mu\|_X^2, \forall \eta, \mu \in X, \quad (12)$$
$$\|\wp'\left(\eta\right)\mu\|_X \leq M\|\mu\|_X, \forall \eta, \mu \in X. \quad (13)$$

Then the operator $\aleph : U \rightarrow U^*$ (defined in eq.2) is strongly monotonous and Lipschitz continuous.

► It takes $\eta = H\left(v\right)$ and $\mu = H\left(w\right)$, $\forall v, w \in U$. From the differential $d\aleph\left(v;s\right)$ and the positive definiteness of the operator $\wp'\left(\eta\right)$, it can write

$$\langle\aleph'(v)w, w\rangle = \left(\wp'\left(H\left(v\right)\right)H\left(w\right), H\left(w\right)\right)_X = \left(\wp'\left(\eta\right)\mu, \mu\right)_X;$$
$$\geq m\|\mu\|_X^2 = m\|w\|_U^2, \forall v, w \in U. \quad (14)$$

It follows from the norm in the U^* space

$$\|\aleph'(v)w\|_{U^*} = \sup_{s\in U}\frac{|\langle\aleph'(v)w, s\rangle|}{\|s\|_U} = \sup_{s\in U}\frac{|\left(\wp'(\eta)\mu, H\left(s\right)\right)_X|}{\|H\left(s\right)\|_U};$$
$$\leq \|\wp'(\eta)\mu\|_X, \forall v, w \in U, \quad (15)$$

thus

$$\|\aleph'(v)w\|_{U^*} \leq \|\wp'(\eta)\mu\|_X \leq M\|\mu\|_X = M\|w\|_U, \forall v, w \in U. \blacksquare \quad (16)$$

2.1 Thermodynamics of Non-reversible Process Comments

The main hypothesis that elasto-plastic formulation takes account is the multiplicative decomposition of the displacement gradient tensor $\mathbf{F}$, into a plastic part $\mathbf{F}^p$ and an elastic part $\mathbf{F}^e$, then

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p. \quad (17)$$

Considering Ω_o the reference configuration and Ω the current configuration at instant t, both bounded Lipschitz domains with regular and positive measurable boundary. Defining the smooth moving function φ that carry a point $\mathbf{x}_o \in \Omega_o$ in a point $\mathbf{x} \in \Omega$, one follows

$$\mathbf{x} = \varphi\left(\mathbf{x}_o, t\right) \text{ with } \varphi\left(\mathbf{x}_o, t\right) = \mathbf{x}_o + u \Rightarrow \mathbf{F} = \nabla\varphi\left(\mathbf{x}_o, t\right). \quad (18)$$

From this multiplicative decomposition, the velocity gradient tensor can be written

$$\mathbf{L}\left(\mathbf{x}, t\right) = \nabla\dot{u}\left(\mathbf{x}, t\right),$$

where $\dot{u}\left(\mathbf{x}, t\right)$ is the spacial description of velocity. This tensor can be decomposed into an elastic part ($\mathbf{L}^e$) and a plastic part ($\mathbf{L}^p$)

$$\mathbf{L}\left(\varphi\left(\mathbf{x}_o, t\right), t\right) = \dot{\mathbf{F}}\left(\mathbf{x}_o, t\right)\mathbf{F}^{-1}\left(\mathbf{x}_o, t\right) = \mathbf{L}^e + \mathbf{L}^p \quad (19)$$

where $\mathbf{L}^e = \dot{\mathbf{F}}^e\left(\mathbf{F}^e\right)^{-1}$ and $\mathbf{L}^p = \mathbf{F}^e\dot{\mathbf{F}}^p\left(\mathbf{F}^p\right)^{-1}\left(\mathbf{F}^e\right)^{-1}$.

The deformation rate tensor, $\mathbf{D} = sym(\mathbf{L})$, can be decomposed too

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p. \quad (20)$$

Since that each displacement gradient tensor part admits polar decomposition, then

$$\mathbf{F}^p = \mathbf{R}^p\mathbf{U}^p \qquad \mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e \quad (21)$$

where $\mathbf{R}^p$ and $\mathbf{R}^e \in ort^+$ (the set of all proper orthogonal tensors), and $\mathbf{U}^p$ and $\mathbf{U}^e \in sym^+$ (the set of all positive definite symmetric second order tensor).

Into thermodynamics of non-reversible process context, the Helmholtz free energy potential Ψ can be defined in the general form

$$\Psi = \Psi\left(\varepsilon^e, \alpha_k\right) = \Psi\left(\varepsilon - \varepsilon^p, \alpha_k\right), \tag{22}$$

where α_k is the internal variable set associated to dissipative mechanisms of the non-reversible processes. From energy equation and second law of thermodynamics, it follows the fundamental Clausius-Duhem inequality

$$\sigma : \mathbf{D} - \rho\left(\dot{\Psi} + s\dot{T}\right) - \mathbf{q} \cdot \frac{\nabla T}{T} \geq 0, \tag{23}$$

in that s is specific entropy, $T = T(\mathbf{x}, t)$ is absolute temperature and $\mathbf{q}$ is heat flux vector. Now, supposing the isothermal process, one has

$$\left(\sigma - \rho\frac{\partial\Psi}{\partial\varepsilon^e}\right) : \mathbf{D}^e + \sigma : \mathbf{D}^p - \rho\frac{\partial\Psi}{\partial\alpha_k} \cdot \dot{\alpha}_k \geq 0. \tag{24}$$

Remembering that eq.(24) must be satisfied for any real process (elastic or plastic), one has

$$\sigma = \rho\frac{\partial\Psi}{\partial\varepsilon^e},\ \beta_k = -\rho\frac{\partial\Psi}{\partial\alpha_k}, \tag{25}$$

where β_k is the set associated to internal variables α_k, that's called thermodynamics forces. From this definitions the eq.(24) is written as

$$\sigma : \mathbf{D}^p + \beta_k \cdot \dot{\alpha}_k \geq 0. \tag{26}$$

2.2 The Yield Function

The elastic domain is defined by a yield function (or functional), that depends of the Cauchy stress tensor and internal variables, i. e.

$$\mathcal{F} = \mathcal{F}\left(\sigma, \alpha_k\right). \tag{27}$$

Thus, it is defined the set of plastically admissible stresses (or plastically admissible domain) as

$$\mathcal{E} = \left\{\left(\sigma, \alpha_k\right) \middle| \mathcal{F}\left(\sigma, \alpha_k\right) \leq 0\right\}. \tag{28}$$

The yield function defines the elastic domain as the set $\left\{\left(\sigma, \alpha_k\right) \middle| \mathcal{F}\left(\sigma, \alpha_k\right) < 0\right\}$ and the boundary of elastic domain is the hypersurface (or yield surface) $\left\{\left(\sigma, \alpha_k\right) \middle| \mathcal{F}\left(\sigma, \alpha_k\right) = 0\right\}$.

2.3 The Dissipative Potential and Evolutionary Laws

The concept of dissipative potential is applied to obtain the laws that describe thermodynamic variables system set evolution. Aiming at the evolutionary laws, it is postulate the existence of a proper convex and lower semicontinuous function, and null at origin (internal variables sense) (σ, α_k), called dissipative potential $\Pi\left(\sigma, \alpha_k\right)$. This functional is defined by a indicatrix function regularization of the set $\mathcal{E} = \left\{\left(\sigma, \alpha_k\right) \middle| \mathcal{F}\left(\sigma, \alpha_k\right) \leq 0\right\}$, where $\mathcal{F}$ is yield function

$$I_{\mathcal{E}}\left(\sigma, \alpha_k\right) = \begin{cases} 0, \mathcal{F}\left(\sigma, \alpha_k\right) \leq 0; \\ \infty, \mathcal{F}\left(\sigma, \alpha_k\right) > 0, \end{cases} \tag{29}$$

e. g.

$$\Pi\left(\sigma, \alpha_k\right) = \frac{1}{2\nu} inf_{\kappa,\zeta}\left\{\left\|\left(\sigma - \kappa, \alpha_k - \zeta\right)\right\|^2 \middle| \mathcal{F}\left(\kappa, \zeta\right) \leq 0\right\}. \tag{30}$$

with $\nu \to \infty$. The dissipation associated to purely mechanical problem is given by

$$\mathcal{D} = \sigma : \mathbf{D}^p + \beta_k \cdot \dot{\alpha}_k \geq 0. \tag{31}$$

Defining the generalized stress tensor $\Sigma = (\sigma, \beta_k)$ and the generalized strain tensor $\Delta^p = (\varepsilon^p, \alpha_k)$, where $\Delta = \Delta^e + \Delta^p$ with $\Delta^e = (\varepsilon^e, \mathbf{0})$, then one has

$$\mathcal{D} = \Sigma : \dot{\Delta}^p \geq 0. \tag{32}$$

The classical theory of plasticity postulates (see Lubliner (1972)) where the plastic work rate $\mathcal{D}(\dot{\Delta}^p) = \Sigma : \dot{\Delta}^p$ associated to the plastic strain rate Δ^p must be maximum, i. e.

$$\mathcal{D}(\dot{\Delta}^p) = \Sigma : \dot{\Delta}^p = \max_{\mathbf{T}} \left\{ \mathbf{T} : \dot{\Delta}^p | \mathcal{F}(\mathbf{T}, \alpha_k) \leq 0 \right\}. \tag{33}$$

i. e. one has for a given Σ, where $\mathcal{F}(\Sigma) = 0$, associated to generalized strain measure rate $\dot{\Delta}^p$ such that $\Sigma : \dot{\Delta}^p \geq \mathbf{T} : \dot{\Delta}^p$, $\forall \mathbf{T} \in \{\mathcal{F}(\mathbf{T}, \alpha_k) \leq 0\}$.

It is possible to show that from normal dissipation hypothesis the evolution process is given by

$$\mathbf{D}^p = \dot{\lambda} \mathbf{N} \left(\sigma, \alpha_k\right) = \dot{\lambda} \frac{\partial \mathcal{F}}{\partial \sigma} \ e \ \dot{\beta}_k = \dot{\lambda} \mathbf{H} \left(\sigma, \alpha_k\right) = -\dot{\lambda} \frac{\partial \mathcal{F}}{\partial \alpha_k}, \tag{34}$$

where $\dot{\lambda} \geq 0$ is called plastic multiplier. In associative plasticity model case, one consider that the dissipative potential is equal to yield function ($\mathcal{F}$). It follows from these assumptions that $(\mathbf{N}, \mathbf{H}) \in \breve{\partial}\Pi\left(\sigma, \alpha_k\right)$, where $\breve{\partial}\Pi\left(\sigma, \alpha_k\right)$ is defined as

$$\left\{ \left(\sigma^*, \alpha_k^*\right) | \Pi\left(\sigma, \alpha_k\right) + \sigma^* : (\kappa - \sigma) + \alpha_k^* \cdot (\zeta - \alpha_k) \leq \Pi\left(\kappa, \zeta\right), \forall \left(\kappa, \zeta\right) \right\}, \tag{35}$$

i.e., $(\mathbf{N}, \mathbf{H})$ belongs to subdifferential (denoted by $\breve{\partial}$) of Π or $\mathcal{F}$ em (σ, α_k). Noting that $\mathcal{F} < 0 \Rightarrow \dot{\lambda} = 0$ and $\dot{\lambda} > 0 \Rightarrow \mathcal{F} = 0$, thus $\dot{\lambda}\mathcal{F} = 0$ and consequently $\dot{\lambda}\dot{\mathcal{F}} = 0$. In fact

Definition 1. *It defines normal cone to convex set $M \subset V$-LNS (linear normed space) at $x \in M$, denoted by $N_M(x) \subset V^*$, the set*

$$N_M(x) = \left\{ f \in V^* | \left\langle f, h - x \right\rangle \leq 0, \forall h \in M \right\}. \tag{36}$$

Remark 1. *Interesting results: Note that if $f \in N_M(x) \Rightarrow \dot{\lambda} f \in N_M(x), \forall \dot{\lambda} \geq 0$. It is had although that if $x \in int(M) \Rightarrow N_M(x) = \{0\}$. In fact, it takes $x \in int(M), \exists \delta > 0$ s. t. $\tilde{h} = x + \delta' \left(\frac{w}{\|w\|}\right) \in B(x, \delta) \subset M$ in which $0 < \delta' < \delta$ and $\forall w \in V$, with $\|w\| \neq 0$. Then $\left\langle f, \tilde{h} - x \right\rangle \leq 0 \Rightarrow \delta' \left\langle f, \frac{w}{\|w\|} \right\rangle \leq 0, \forall w \in V$, with $\|w\| \neq 0$. Thus $\sup_{\|\hat{w}\|=1} \left\langle f, \hat{w} \right\rangle \leq 0 \Rightarrow \|f\| \leq 0$.*

Theorem 1. *Let $\Pi : X \rightarrow \bar{\mathbb{R}}$, $X - Hilbert$, a non negative and convex functional, with $\Pi(0) = 0$, and let $x \in int(dom(\Pi))$ such that $\Pi(x) > 0$. Defining the set $C = \{z \in X | \Pi(z) \leq \Pi(x)\}$, then $y \in N_C(x) \Leftrightarrow \exists \dot{\lambda} \geq 0$ s. t.*

$$y \in \dot{\lambda}\breve{\partial}\Pi(x) = \left\{ x^* \in X^* | \dot{\lambda}\Pi(h) - \dot{\lambda}\Pi(x) \geq \left\langle x^*, h - x \right\rangle, \forall h \in X \right\}, \tag{37}$$

where $N_C(x) = \{f \in X^* | \left\langle f, h - x \right\rangle \leq 0, \forall h \in C\}$ is the normal cone to set C at $x \in C$.

► ($\Leftarrow$) $\exists \dot{\lambda} \geq 0$ s. t. $y \in \dot{\lambda}\breve{\partial}\Pi(x) \Rightarrow \dot{\lambda}\Pi(z) \geq \left\langle y, z - x \right\rangle + \dot{\lambda}\Pi(x), \forall z \in X$ i. e. $\left\langle y, z - x \right\rangle \leq \dot{\lambda}\left(\Pi(z) - \Pi(x)\right) \leq 0, \forall z \in C \Rightarrow y \in N_C(x)$.($\Rightarrow$) It consider that $\exists \dot{\lambda} \geq 0$ s. t. $y \in \dot{\lambda}\breve{\partial}\Pi(x)$, thus it must have $\dot{\lambda}\Pi(z) \geq \dot{\lambda}\Pi(x) + \left\langle y, z - x \right\rangle, \forall z \in C$. Remembering that C is convex, if $y \neq 0$ and $\left\langle y, z - x \right\rangle = 0 \Rightarrow x \in \partial C$. Analysing for $z \in int(C) \Rightarrow$, it takes $z - x + \delta' \frac{w}{\|w\|} \in B(z, \delta) \in int(C)$, with $0 < \delta' < \delta$ and $0 \neq w \in X$. Then $\left\langle y, z - x + \delta' \frac{w}{\|w\|} \right\rangle = 0 \Rightarrow \|y\| = 0 \Leftrightarrow y = 0 \therefore$ contradiction !. Let $z \in (int(C))^c \Rightarrow \Pi(z) \geq \Pi(x) \therefore \forall \dot{\lambda} \geq 0$ s. t. $y \in \dot{\lambda}\breve{\partial}\Pi(x)$. Note that if $z \in C \therefore \Pi(z) = \Pi(x) \therefore z \in \partial C$ and $x \in \partial C$. Now if $\left\langle y, z - x \right\rangle < 0$, if $z \in (C)^c \Rightarrow \Pi(z) > \Pi(x)$, then $\forall \dot{\lambda} \geq 0$ s. t. $y \in \dot{\lambda}\breve{\partial}\Pi(x)$. If $z \in C$ note that $\Pi(z) - \Pi(x) - \frac{1}{\dot{\lambda}} \left\langle y, z - x \right\rangle = \left(\frac{\Pi(z) - \Pi(x)}{\left\langle y, z - x \right\rangle} - \frac{1}{\dot{\lambda}} \right) \left\langle y, z - x \right\rangle$. Note that if one takes $\frac{1}{\dot{\lambda}} > \max_{z \in C} \left\{ \frac{\Pi(z) - \Pi(x)}{\left\langle y, z - x \right\rangle} \right\} \therefore \Pi(z) - \Pi(x) - \frac{1}{\dot{\lambda}} \left\langle y, z - x \right\rangle \geq 0 \therefore \exists \dot{\lambda} \geq 0$ s. t. $y \in \dot{\lambda}\breve{\partial}\Pi(x)$.■

From this consideration it has the Kuhn-Tucker conditions

$$\mathcal{F} \leq 0, \dot{\lambda} \geq 0, \dot{\lambda}\mathcal{F} = 0, \tag{38}$$

and the consistence condition in $\mathcal{F} = 0$ ($\dot{\lambda}\dot{\mathcal{F}} = 0 \Rightarrow \dot{\mathcal{F}} = 0$, because $\dot{\lambda} > 0$). For additional details over the simplified elastoplastic constitutive model with isotropic hardening and damage, see Lubliner (1990) and Lemaitre (1996).

Remark 2. *It can be shown that $\Sigma \in \breve{\partial}\mathcal{D}$ and the Legendre-Fenchel conjugate $\mathcal{D}^* = \mathbf{I}_{\mathcal{E}}$, i. e. it is the indicatrix function of admissible generalized stresses. In this sense $\mathcal{D}$ is a pseudopotential, and not a potential, for Σ.*

3. The Main Problem - Some Formalities

It is assumed that any mechanical system is associated with a Hilbert space $Y = X \bigcap range(A)$ the elements of which are called generalized deformations of the system. It can be proved that the dual space to Y will be denoted by $Z = X \bigcap range(H)$, such that its elements are to be regarded as generalized stresses. Let $S = [0, t_f]$ be a fixed finite time interval. In this work mappings from S into a Hilbert space will be called processes(for example a mapping from S into Y is called a deformation process). The behaviour of a system is governed by a constitutive relation, i. e. by a relation between its deformation processes and its stress processes. It shall call a mechanical system plastic if its deformation processes $\Delta \in Y$ and its stress processes $\Sigma \in Z$ are related in the following way:

$$\dot{\Delta}^p(t) \in \breve{\partial} I_{\mathcal{E}} \left(\Sigma(t)\right), \; a.\, e.\, t \in S, \tag{39}$$

where $\mathcal{E}$ is a convex closed nonempty subset of Z. From classical convex analysis results, if $\dot{\Delta}^p \in \breve{\partial} I_{\mathcal{E}} \;\therefore\; \dot{\lambda}\dot{\Delta}^p \in \breve{\partial} I_{\mathcal{E}}, \forall \dot{\lambda} \geq 0$, then this constitutive relationship is rate independent.

Generalizing the procedure it shall consider constitutive relations in this mechanical system the deformation processes Δ and its stress processes Σ of the form:

$$\dot{\Delta}^p(t) \in \breve{\partial}\Pi \left(\Sigma(t)\right), \; a.\, e.\, t \in S \tag{40}$$

where $\Pi : Z \rightarrow]-\infty, +\infty]$ denotes the proper convex and lower semicontinuous Lipschitzian dissipative potential.

In this paper it shall deal with constitutive relations describing the serial superposition of an elasticity law and a plastic law. Such constitutive relations can be written as follows

$$\Delta(t) = \Delta^p(t) + \Delta^e(t); \tag{41}$$

$$\Sigma(t) = \mathbf{C}\Delta^e(t) \; in\; elastic\; domain; \tag{42}$$

$$\dot{\Delta}^p(t) \in \breve{\partial}\Pi(t) \; a.\, e.\, t \in S \; in\; plastic\; domain, \tag{43}$$

where $\Sigma = (\sigma, \beta_k)$, $\Delta^p = (\varepsilon^p, \alpha_k)$ in plastic domain, $\Sigma = (C\varepsilon^e, 0)$, $\Delta^e = (\varepsilon^e, 0)$ in elastic domain and C $(C \in \mathcal{L}\left(Y; Z\right)$ symmetric, positive, linear continuous mapping an equivalently $\mathbf{C}$) represents the linear elasticity law.

Let suppose that the system is associated with a Hilbert space U of displacements and that the deformation process $\Delta : S \rightarrow Y$ corresponding to a displacement process $u : S \rightarrow U$ is given by

$$\Delta(t) = \mathbf{K}u(t) \; a.\, e.\, t \in S, \tag{44}$$

where $\mathbf{K} \in \mathcal{L}\left(U; Y\right)$ and $\exists k_o > 0 \; s.\, t.\; ||\mathbf{K}u||_Y \geq k_o ||u||_U, \forall u \in U$ (considering small deformations only).

The space U^* dual to U is to be regarded as a space of forces. In other way, supposing that a force $\mathbb{k} \in U^*$ acts on a system the generalized stress of which is $\Sigma \in Z$ then the work corresponding to a virtual displacement $h \in U$ is given by

$$\mathcal{W} = \langle \Sigma, \mathbf{K}h \rangle - \langle \mathbb{k}, h \rangle = \langle \mathbf{K}^*\Sigma - \mathbb{k}, h \rangle \,. \tag{45}$$

Therefore the principle of virtual work yields that the condition of quasi-static equilibrium may be written as follows:

$$\mathbf{K}^*\Sigma(t) = \mathbb{k}(t) \; for\; a.\, e.\, t \in S, \tag{46}$$

and the corresponding condition of dynamic equilibrium is

$$\rho\ddot{u}(t) + \mathbf{K}^*\Sigma(t) = \mathbb{k}(t) \; for\; a.\, e.\, t \in S. \tag{47}$$

In order that these equations make sense, suppose that U is densely and continuously imbedded into U^*. Besides U and U^* one shall need later the interpolation space $\tilde{H} = [U; U^*]_{\frac{1}{2}}$ (see Brudnyi and Krugljak (1991)) which elements may be considered as "generalized" displacements.

In order to obtain precise formulations of the problems, it follows from some regularity conditions on the data and on the functions one is looking for (setting the reference configuration $(._o)$):

$$\mathbb{k} \in W^{1,1}\left(S; \tilde{H}\right), \dot{u}_o \in U, \Delta_o^p \in Y, \Sigma_o \in D(\breve{\partial}\Pi)\ s.\ t. \tag{48}$$

$$\mathbf{K}^* \Sigma_o \in \tilde{H}. \tag{49}$$

It is looking for processes $\dot{u}, \Sigma$ s. t.

$$\dot{u} \in L^2(S; U), (\dot{u}, \Sigma) \in W^{1,\infty}\left(S; \tilde{H} \times Z\right), (\dot{u}, \Sigma)(0) = (\dot{u}_o, \Sigma_o), \tag{50}$$

$$\rho \ddot{u} + \mathbf{K}^* \Sigma = \mathbb{k}, \mathbf{K}\dot{u} \in \mathbf{C}^{-1}\dot{\Sigma} + \breve{\partial}\Pi. \tag{51}$$

Defining $\hat{U} = U \times Z$, $\hat{H} = \tilde{H} \times Z$, $\hat{U}^* = U^* \times Z$ and defining the inner product on Z by

$$\langle \Sigma_1, \Sigma_2 \rangle_Z = \frac{1}{\rho}\left\langle \Sigma_1, \mathbf{C}^{-1}\Sigma_2 \right\rangle, \forall \Sigma_1, \Sigma_2 \in Z. \tag{52}$$

and assuming that

$$w + \mathbf{K}\dot{u} \in \breve{\partial}\Pi(\Sigma) \Rightarrow \exists c > 0\ s.\ t.\ \|\dot{u}\|_U \leq c\left(\|w\|_Y + \|\Sigma\|_Z + 1\right), \tag{53}$$

one has the following results

Lemma 2. *The operator $M_{\tilde{H}} \subset \hat{H} \times \hat{H}$ defined by*

$$M_{\tilde{H}}(\dot{u}, \Sigma) = \left\{\left(\frac{1}{\rho}\mathbf{K}^*\Sigma, \mathbf{C}(\eta - \mathbf{K}\dot{u})\right) | \dot{u} \in U, \mathbf{K}^*\Sigma \in \tilde{H}, \eta \in \breve{\partial}\Pi(\Sigma)\right\}, \tag{54}$$

$\forall (\dot{u}, \Sigma) \in \hat{H}$ *is maximal monotone.*

► Note that $M_{\tilde{H}} = M \cap \left(\hat{H} \times \hat{H}\right)$, with $M \cap \hat{U} \times \hat{U}^*$ and $M = M_I + M_{II}$ where $M_I(\dot{u}, \Sigma) = \left(\frac{1}{\rho}\mathbf{K}^*\Sigma, -\mathbf{C}\mathbf{K}\dot{u}\right)$ and $M_{II}(\dot{u}, \Sigma) = \left\{(0, \mathbf{C}\eta) | \eta \in \breve{\partial}\Pi(\Sigma)\right\}$, thus due to the choice of inner product on Z, it is easy to verify that M_I and M_{II} are monotone. As the subdifferential mappings are maximal monotone so M_{II} is maximal monotone and since M_I is continuous it is maximal. Then M maximal monotonicity follows from convex analysis classical results (see Brezis (1973), cor. 2.7 and Bröndsted and Rockafellar (1965); Burachik and Svaiter (2003); Rockafellar (1970)) therefore $M_{\tilde{H}}$ is monotone. To show the maximality of $M_{\tilde{H}}$, first it must be prove that $\left(M + id_{\hat{U}}\right)^{-1}$ is bounded ($id_{\hat{U}}$ represents the identity oprator on $\hat{U}$). Taking $\left(M + id_{\hat{U}}\right)(\dot{u}, \Sigma) = \left(\tilde{\dot{u}}, \tilde{\Sigma}\right) \in \hat{U}^*$, i. e., $\frac{1}{\rho}\mathbf{K}^*\Sigma + \dot{u} = \tilde{\dot{u}}$ and $\mathbf{C}(\eta - \mathbf{K}\dot{u}) + \Sigma = \tilde{\Sigma}$, $\eta \in \breve{\partial}\Pi(\Sigma)$. Let η_o an element from Σ_o, it follows $\left\langle \mathbf{K}\dot{u} - \mathbf{C}^{-1}\left(\Sigma - \tilde{\Sigma}\right) - \eta_o, \Sigma - \Sigma_o \right\rangle \geq 0$. Hence

$$\begin{aligned}
\left\langle \mathbf{C}^{-1}\Sigma, \Sigma \right\rangle &\leq c_1\left(\|\Sigma\|_Z \|\tilde{\Sigma}\|_Z + \|\Sigma\|_Z + 1\right) + \langle \dot{u}, \mathbf{K}^*\Sigma - \mathbf{K}^*\Sigma_o \rangle; \\
&\leq c_1\left(\|\Sigma\|_Z \|\tilde{\Sigma}\|_Z + \|\Sigma\|_Z + 1\right) + \left\langle \dot{u}, \rho\left(\tilde{\dot{u}} - \dot{u}\right) - \mathbf{K}^*\Sigma_o \right\rangle; \\
&\leq c_2\left(\|\Sigma\|_Z \|\tilde{\Sigma}\|_Z + \|\Sigma\|_Z + \|\dot{u}\|_U \|\tilde{\dot{u}}\|_U + \|\dot{u}\|_U + 1\right),
\end{aligned} \tag{55}$$

with c_1 and c_2 independent on $\tilde{\Sigma}$ and $\tilde{\dot{u}}$. Thus it can be written

$$\|\dot{u}\|_U \leq c\left(\|\Sigma\|_Z + \|\tilde{\Sigma}\|_Z + 1\right). \tag{56}$$

These inequalities show that $\|(\dot{u}, \Sigma)\|_{\hat{U}}$ has a bound depending only on $\|\left(\tilde{\dot{u}}, \tilde{\Sigma}\right)\|_{\hat{U}}$. From these results it follows that $\left(M + id_{\hat{U}}\right)^{-1}$ is bounded and $range\left(M + id_{\hat{U}}\right) = \hat{U}$ (see Brezis (1973), th. 2.3). However if $\left(\tilde{\dot{u}}, \tilde{\Sigma}\right) \in \hat{H} \Rightarrow \mathbf{K}^*\Sigma \in H \Rightarrow (\dot{u}, \Sigma) \in D\left(M_{\tilde{H}}\right)$, then $M_{\tilde{H}} + id_{\hat{H}}$ is surjective hence this implies the maximality of $M_{\tilde{H}} \subset \hat{H} \times \hat{H}$. ■

Remark 3. *Note that if it takes multivalued mapping* $M_{\tilde{H}S}, \hat{M}_{\tilde{H}S} : L^2\left(S;\hat{H}\right) \to L^2\left(S,\hat{H}\right)$ *defined by*

$$M_{\tilde{H}S}\left(\dot{u},\Sigma\right) = \left\{\left(\overline{\dot{u}},\overline{\Sigma}\right) \in L^2\left(S;\hat{H}\right) \mid \left(\overline{\dot{u}},\overline{\Sigma}\right)(t) \in M_{\tilde{H}}\left(\dot{u},\Sigma\right)\ a.\ e.\ t \in S\right\}; \tag{57}$$

$$\hat{M}_{\tilde{H}S}\left(\dot{u},\Sigma\right) = \left\{\left(\frac{1}{\rho}\mathbf{K}^*\Sigma, \mathbf{C}\left(\eta - \mathbf{K}\dot{u}\right)\right) \in L^2\left(S;\hat{H}\right) | \dot{u} \in L^2\left(S;U\right), \right.$$
$$\left. \eta(t) \in \breve{\partial}\Pi\left(\Sigma(t)\right)\ a.\ e.\ t \in S\right\}, \tag{58}$$

then $M_{\tilde{H}S} = \hat{M}_{\tilde{H}S}$. *In fact, in the same way that was proved the maximal monotonicity of* $M_{\tilde{H}}$ *it can shown that* $M_{\tilde{H}S}$ *and* $\hat{M}_{\tilde{H}S}$ *are maximal monotony operators and since* $M_{\tilde{H}S}$ *is an extension of* $\hat{M}_{\tilde{H}S}$ *therefore both operators are equal.*

Theorem 2. *Considering the problem 50-51 under assumptions 48, 49, 53, then there exists a unique solution* $(\dot{u},\Sigma)$ *for this problem and the mapping* $(\mathbb{k},\dot{u}_o,\Sigma_o) \to (\dot{u},\Sigma)$ *is Lipsehitzian from the set of data satisfying 48-49 (equipped with the metric of* $L^1\left(S;\tilde{H}\right) \times \tilde{H} \times Z$*) into* $C\left(S;\tilde{H} \times Z\right)$.

► Noting that the problem 50-51 can be rewritten as

$$\left(\frac{1}{\rho}\mathbb{k}, 0\right) \in \left(\hat{u},\hat{\Sigma}\right) + M_{\tilde{H}}\left(\dot{u},\Sigma\right), \left(\dot{u},\Sigma\right)(0) = \left(\dot{u}_o,\Sigma_o\right), \left(\dot{u},\Sigma\right) \in W^{1,\infty}\left(S;\hat{H}\right). \tag{59}$$

Noting that if $(\dot{u},\Sigma)$ is solution of 59 thus $\dot{u} \in L^2\left(S;\hat{H}\right)$ by lemma above. Since $(\dot{u}_o,\Sigma_o) \in D\left(M_{\tilde{H}}\right)$ the assertion follows from classical results on evolutionary equations with monotone operators (for more details see Brezis (1973), prop. 3.3/lemma 3.1). ■

4. CONCLUSION

The above results demonstrate that practical surface load approach should be made with the consideration of two factors. On the one hand, the approximating functions should sufficiently approximate the real load surfaces obtained from the experimental data, on the other hand, they should satisfy the conditions whereby one-valued solvability of the boundary-value problem is ensured.

In this work was explained/demonstrated some impotant mathematical formalities about the formulation (existence and unicity of solution) of the dynamic elastoplastic problem. In this sense the dynamic elastoplastic problem has characterized with inherent mathematical formalities. This development makes possible a clear understanding of the mathematical elastoplastic problem.

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