STOCHASTIC ANALYSIS OF A MOORED FLOATING BODY VIA AN ADAPTIVE SPARSE GRID STOCHASTIC COLLOCATION METHOD

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Abstract. *The analysis and design of complex engineering systems usually involves several uncertainties that leads to the need of develop methodologies capable of assessing the unavoidable uncertainties contained in the numerical results. For example, one major issue to be deeper understood is how uncertainties in the input data impacts the reliability of the results obtained through computer simulations. In that sense, the stochastic modeling seems to offer an appropriate framework to handle external forces and uncertainties in the data, like for instance, damping and boundary conditions. Specifically in the present work, the focus relies on predict some statistics of the horizontal motion in a floating moored buoy model represented by a single-degree-of-freedom system. The presence of non-linear restoring forces in the system subjected to random hydrodynamics loads, corresponding to the Morison formula leads to a highly non-linear system, where the velocity and acceleration of the flow are determined using the Pierson-Moskovitz power spectrum. We propose in this work apply an adaptive sparse grid stochastic collocation method to approximate its solution in the stochastic space by polynomial interpolation built through only repeated calls to an existing deterministic solver as in sampling methods like the Monte Carlo. The aim of the adaptive approach is try to improve of the conventional collocation method allowing the identification of discontinuities in the stochastic space, refining the collocation points in that region. Particular emphasis is placed on investigating the uncertainty propagation and the non-linear response of the system under random loads performing a stability analysis of the system. Finally, comparisons are done between the adaptive and conventional collocation methods even as Monte Carlo, taken as reference, to demonstrate the accuracy and efficiency of the method.*

Keywords: *Stochastic collocation method, Adaptive grids, fluid structure interaction*

1. INTRODUCTION

The complexity involved in engineering systems has been, frequently, tackled with the use of sophisticated computational models. That, from the decision makers standpoint, requires the use of robust and reliable numerical simulators. Often, the reliability of those simulations is disrupted by the inexorable presence of uncertainty in the model data, such as inexact knowledge of system forcing, initial and boundary conditions, physical properties of the medium, as well as parameters in constitutive equations. These situations underscore the need for efficient uncertainty quantification (UQ) methods for the establishment of confidence intervals in computed predictions, the assessment of the suitability of model formulations, and/or the support of decision-making analysis.

The traditional statistical tool for uncertainty quantification within the realm of engineering is the Monte Carlo method, (Elishakoff, 2003). This method requires, first, the generation of an ensemble of random realizations associated to the uncertain data, and then it employs deterministic solvers repetitively to obtain the ensemble of results. The ensemble results should be processed to estimate the mean and standard deviation of the final results. The implementation the Monte Carlo is straightforward, but its convergence rate is very slow (proportional to the inverse of the square root of the realization number) and often infeasible due the large CPU time needed to run the model in question. Other technique that has been applied recently is the so called Stochastic Galerkin Method (SG), which employs polynomial chaos expansions to represent the solution and inputs to stochastic differential equations, (Babuska *et al.*, 2004). The galerkin projection minimizes the error of the truncated expansion and the resulting set of coupled equations is solved to obtain the expansion coefficients. SG methods are highly suited to dealing with ordinary and partial differential equations, even in the case of nonlinear dependence on the random data. The main drawback with SG relies on its need of solving a system of coupled equations that requires efficient and robust solvers and, most importantly, the modification of existing deterministic code. This last issue entails difficulties on using commercial or already in use codes. A non-intrusive method, referred to as Stochastic Collocation (SC), (Dongbin and Hesthaven, 2005), arises towards addressing this point. SC methods are built on the combination of interpolation methods and deterministic solvers, likely Monte Carlo. A deterministic problem is solved in each point of an abstract random space. Similarly to SG methods, SC methods achieve fast convergence when the solution possesses sufficient smoothness in random space (Dongbin *et al.*, 2002).

Thus when there are steeps gradients or finite discontinuities in the stochastic space, these methods converge very slowly or even fail to converge. In this work, we present an adaptive sparse grid collocation strategy with the aim of obtaining greater accuracy in nonlinear systems analysis. Specifically in the present work, the focus relies on hydroship dynamics in the context of floating offshore structures. Particular emphasis is placed on investigating uncertainty propagation in the nonlinear response of fluid-structure interaction, (Dongbin *et al.*, 2002). It is important to remind that waves and currents, major agents in the dynamics of the floating structures, are usually modeled as random processes.

Therefore, stochastic modeling seems to offer an appropriate framework to tackle the external forces and uncertainties in the data, like, for instance, damping and boundary conditions. Here, the fluid-structure interaction is modeled in a simple way focusing the assessment of an SC method as an effective tool for uncertainty quantification. The interaction is introduced by means of the Morison's formula, which represents a challenge, despite the simplicity of the model itself, as far as the input is a nonlinear function of the random variables, (Witteveen and Bijl, 2008). Those variables represent the phase angle which inherent to the time series description of the wave induced motion.

2. THEORY

To quantify the uncertainty in a system of differential equations we adopt a probabilistic approach and define a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Where Ω is the event space, $\mathcal{F} \subset 2^{\Omega}$ is the σ-algebra of subsets in Ω and $\mathcal{P} : \mathcal{F} \rightarrow [0, 1]$ is the probability measure. Utilizing this framework, the uncertainty in a model is introduced by representing the model input data as random field.

2.1 Governing Equations

Consider the general differential equation defined on a d-dimensional bounded domain $\mathcal{D} \subset \mathbb{R}^d$, $(d = 1, 2, 3)$ with boundary $\partial\mathcal{D}$. The problem consists on finding a stochastic function, $\mathbf{u}(\mathbf{x}, \omega) : \Omega \times \mathcal{D} \longrightarrow \mathbb{R}$, such that for $\mathcal{P}$-almost everywhere $\omega \in \Omega$, the following equation holds:

$$\mathcal{L}(\mathbf{x}, \omega; \mathbf{u}) = f(\mathbf{x}, \omega) \qquad \mathbf{x} \in \mathcal{D} \tag{1}$$

$$\mathcal{B}(\mathbf{x}, \omega; \mathbf{u}) = g(\mathbf{x}, \omega) \qquad \mathbf{x} \in \partial\mathcal{D} \tag{2}$$

with $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $d \geq 1$, space coordinates in $\mathbb{R}^d$, $\mathcal{L}$ a linear or non linear differential operator and $\mathbf{u}(\omega) = (u_1(\omega), \ldots u_i(\omega)) \in \mathbb{R}^i$, $i \geq 1$, are unknown solutions. Sometimes, to solve the equations (1) and (2) it is necessary reduce the infinite dimensional probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to a finite dimensional one. This can be accomplished by characterizing the probability space by a finite number of random variables. Thus, employing any truncated spectral expansion it is possible characterize the random inputs by a set of N random variables $\mathbf{Y} = (Y_1(\omega), \ldots, Y_N(\omega))$ and rewrite the random inputs as,

$$\mathcal{L}(\mathbf{x}, \omega; \mathbf{u}) = \mathcal{L}(\mathbf{x}, Y_1(\omega), \ldots, Y_N(\omega); \mathbf{u}), \quad f(\mathbf{x}, \omega) = f(\mathbf{x}, Y_1(\omega), \ldots, Y_N(\omega)), \tag{3}$$

Where, following the Dob-Dynkin lemma, (Oskendal, 1998), the solution of (1) and (2) can be represented by the same set of random variables $\{Y_i(\omega)\}_{i=1}^N$, reducing the infinite dimensional probability space to a N-dimensional space, i.e.,

$$\mathbf{u}(\mathbf{x}, \omega) = \mathbf{u}(, \mathbf{x}, Y^1(\omega), \ldots, Y^N(\omega)) \tag{4}$$

Now assuming that $\{Y^i\}_{i=1}^N$ are independent random variables with probability density functions $\rho_i : \Gamma^i \rightarrow \mathbb{R}^+$, and their images $\Gamma^i \equiv Y^i(\Omega)$ bounded intervals in $\mathbb{R}$ for $i = 1, \ldots, N$, the joint probability density of $\mathbf{Y} \equiv (Y^1, \ldots, Y^N)$ hold,

$$\rho(y) = \prod_{i=1}^{N} \rho_i(Y^i) \qquad \forall y \in \Gamma, \tag{5}$$

and the space support,

$$\Gamma \equiv \prod_{i=1}^{N} \Gamma^i \subset \mathbb{R}^N. \tag{6}$$

This allow us to rewrite (1) and (2) as a $(N + d)$ dimensional differential equation as following,

$$\mathcal{L}(\mathbf{x}, \mathbf{Y}; \mathbf{u}) = f(\mathbf{x}, \mathbf{Y}), \qquad (\mathbf{x}, \mathbf{Y}) \in \Gamma \times \mathcal{D} \tag{7}$$

$$\mathcal{B}(\mathbf{x}, \mathbf{Y}; \mathbf{u}) = g(\mathbf{x}, \mathbf{Y}), \qquad (\mathbf{x}, \mathbf{Y}) \in \Gamma \times \partial\mathcal{D} \tag{8}$$

with N dimensionality of the random space Γ and d the dimensionality of the physical space $\mathcal{D}$.

Thus, the original infinite dimensional problem become in a deterministic problem in the physical domain $\mathcal{D}$ and can be solved by a common discretization technique as finite elements for example.

3. STOCHASTIC COLLOCATION METHOD

The idea of this method is approximate the multidimensional stochastic space building a interpolation function on a set of collocation points $\{\mathbf{Y}_i\}_{i=1}^M$ in the stochastic space $\Gamma \subset \mathbb{R}^M$. The method, similarly to Monte Carlo methods, requires only the solution of a set of decoupled equations, allowing the model to be treated as a black box and solved it with existing deterministic solvers. The multidimensional interpolation can be built through either full-tensor product of 1D interpolation rule or by the so called sparse grid interpolation based on the Smolyak algorithm. The Smolyak algorithm provides a way to construct interpolations functions based on minimal number of points in multidimensional space (Bungartz and Griebel, 2004). This method is easily extended from the univariate interpolation to the multivariate case by using tensor products.

Hence, considering a smooth functions $f : [-1,1]^N \rightarrow \mathbb{R}$, for the $1D$ case $(N = 1)$, f can be approximated by the following:

$$\mathcal{U}^i(f)(y) = \sum_{j=1}^{m_i} f(\mathbf{Y}_j^i) a_j^i, \tag{9}$$

with the set of support nodes

$$X^i = \mathbf{Y}_j^i | \mathbf{Y}_j^i \in [0,1] for j = 1, \ldots, m_i \tag{10}$$

where, $i \in \mathbb{N}$, $a_i(\mathbf{Y}_j^i) \in C[0,1]$ are the interpolation nodal basis functions and m_i is the number of elements of the set X^i. Hence, in the multivariate case, the tensor product formula is:

$$(\mathcal{U}^{i_1} \otimes \ldots \otimes \mathcal{U}^{i_N})(f) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} f(Y_{j_1}^{i_1} \ldots Y_{j_N}^{i_N}).(a_{j_1}^{i_1} \otimes \cdots \otimes a_{j_N}^{i_N}) \tag{11}$$

which serve as building blocks for the Smolyak algorithm. So, The Smolyak algorithm build the interpolant $\mathcal{A}_{q,N}(f)$ using products of $1D$ functions as given in (Xiang and Zabaras, 2009).

$$\mathcal{A}_{q,N}(f) = \sum_{q-N+1 \leq |i| \leq q} (-1)^{q-|i|} \binom{N-1}{q-|i|} (\mathcal{U}^{i_1} \otimes \ldots \otimes \mathcal{U}^{i_N}) \tag{12}$$

with $q \geq N$, $\mathcal{A}_{N-1,N} = 0$ and where the multi-index $i = (i_1, \ldots, i_N) \in \mathbb{N}^N$ and $|i| = i_1 + \cdots + i_N$. Here $i_k, k = 1, \ldots, N$, is the level of interpolation along the $k-th$ direction. The Smolyak algorithm builds the interpolation function by adding a combination of $1D$ functions of order i_k with the constraint that the sum total $(|i| = i_1 + \ldots + i_N)$ across all dimensions is between $q - N + 1$ and q. Therefore, the Smolyak interpolation $\mathcal{A}_{q,N}$ is given by;

$$\mathcal{A}_{q,N}(f) = \sum_{|i| \leq q} (\triangle^{i_1} \otimes \ldots \otimes \triangle^{i_N}) = \mathcal{A}_{q-1,N}(f) + \sum_{|i|=q} (\triangle^{i_1} \otimes \ldots \otimes \triangle^{i_N}) \tag{13}$$

To compute the interpolant $\mathcal{A}_{q,N}(f)$ is necessary to compute the function at the nodes covered by the sparse grid $\mathcal{H}_{q,N}$:

$$\mathcal{H}_{q,N}(f) = \bigcup_{q-N+1 \leq |i| \leq q} (X^{i_1} \times \cdots \times X^{i_N}) \tag{14}$$

The construction of the algorithm allows to utilizing all the previous results generated to improve the interpolation. By choosing the appropriate points for interpolating the $1D$ function, it is possible ensure that the sets of points are nested $X^i \subset X^{i+1}$. Where to extend the interpolation from level $i-1$ to i, one only has to evaluate the function at grid points that are unique to X^i. Hence, to go from an order $q-1$ to q in N dimensions, one only needs to evaluate the function at the differential nodes:

$$\triangle \mathcal{H}_{q,N}(f) = \bigcup_{|i|=q} (X^{i_1} \otimes \cdots \otimes X^{i_N}) \tag{15}$$

Finally after a choice of collocation points and the nodal basis functions, any function $u \in \Gamma$ can be approximated by;

$$u(x, \mathbf{Y}) = \sum_{|i| \leq q} \sum_{j \in B_i} w_j^i(x) a_j^i(\mathbf{Y}) \tag{16}$$

This equation is a simple weighted sum of the value of the basis functions for all collocations points in the sparse grid, being an approximation to the solution of the equations (7) and (8). From this equation, it is possible calculate easily the useful statistics of the solution for example, the mean of the random solution can be evaluated as follow:

$$\mathbb{E}[u(x)] = \sum_{|i|\leq q} \sum_{j\in B_i} w_j^i(x). \int_\Gamma a_j^i(\mathbf{Y})d\mathbf{Y} \tag{17}$$

where denoting $\int_\Gamma a_j^i(\mathbf{Y})d\mathbf{Y} = I_j^i$ we can write

$$\mathbb{E}[u(x)] = \sum_{|i|\leq q} \sum_{j\in B_i} w_j^i(x).I_j^i \tag{18}$$

the mean is an arithmetic sum of the product of the hierarchical surpluses and the integral weight at each interpolation point. To obtain the variance of the random solution we can be calculate first:

$$u^2(x,\mathbf{Y}) = \sum_{|i|\leq q} \sum_{j\in B_i} v_j^i(x)a_j^i(\mathbf{Y}) \tag{19}$$

and then

$$\mathbb{V}ar[u(x)] = \mathbb{E}[u^2(x)] - (\mathbb{E}[u(x)])^2 = \sum_{|i|\leq q} \sum_{j\in B_i} v_j^i(x).I_j^i - (\sum_{|i|\leq q} \sum_{j\in B_i} w_j^i(x)I_j^i)^2 \tag{20}$$

The method allows us to obtain an approximation of the solution dependent random variables and also easily extract the mean and variance analytically as well its probability density function (PDF) by simple sampling of this function, leaving only the interpolation error.

4. ADAPTIVE SPARSE GRID COLLOCATION METHOD

If the the smoothness condition in the stochastic space is not fulfilled it is possible to use adaptive strategies to improve de interpolation function in the stochastic space. The idea here is to use hierarchical surpluses $w_j^i(x)$ as an error indicator to detect the smoothness of the solution and refine the grid around this region and using less points in the region of smooth variation. This method proposed by (Xiang and Zabaras, 2009), automatically detect the discontinuity region in the stochastic space refining the collocation points. Considering the interpolation level of a grid point Y as the depth of the tree $D(Y)$. After denote the father of a grid point as $F(Y)$, where the father of the root 0.5 is itself. Thus, the conventional sparse grid in the N-dimensional random space Equation 14 can be reconsidered as:

$$\mathcal{H}_{q,N}(f) = \{\mathbf{Y} = \{Y_1 \ldots Y_N\} | \sum_{i=1}^{N} D(Y_i) \leq q\} \tag{21}$$

Where we call their sons of a grid point $\mathbf{Y} = (Y_1 \ldots Y_N)$ by:

$$Sons(\mathbf{Y}) = \mathbf{S} = (S_1, S_2, \ldots, S_N) | (F(S_1), S_2, \ldots, S_N) = \mathbf{Y} \tag{22}$$

or

$$(S_1, F(S_2), \ldots, S_N) = \mathbf{Y}, \ldots, or (S_1, S_2, \ldots, F(S_N)) = \mathbf{Y} \tag{23}$$

It is noted here that in general for each grid point there are two sons in each dimension, therefore, for a grid point in a N-dimensional stochastic space, there are 2N sons. Therefore, by adding the neighbor points, we actually add the support nodes from the next interpolation level, so that the magnitude of the hierarchical surplus satisfies $|w_j^i \geq \varepsilon|$. If the criterion is satisfied, one only add the $2N$ neighbor points of the current point to the sparse grid. It is noted that the definition of level of the Smolyak interpolation por the ASGC method is the same as that of the conventional sparse grid even if not all point are included. A more detailed explanation of the method and algorithm can be found in, (**?**).

5. APPLICATION

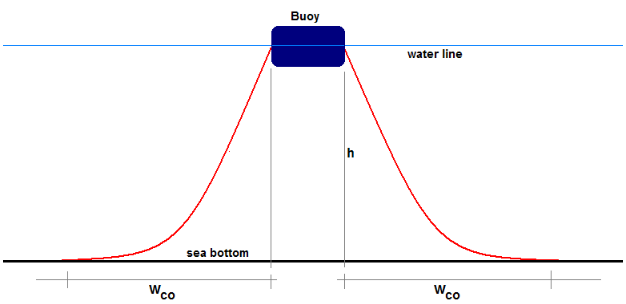
Following, we will illustrate the methods developed in the preceding sections considering the stochastic response of a single-degree-of-freedom structure exited by a random Morison's force with a restoring force expressed by non linear term. This equation can be considered as an idealizing model a floating moored dock system. The numerical results shown in this section were obtained using fully parallels algorithms for high performance computers implemented in C++ and *MPI*. This code was developed by Professor Zabaras's research group at Cornell University who share with the authors as a result of a research partnership. It remained for our staff to adapt it to address the next example. Due to the relative small size of the problem, the analisys were performed using a single desktop with a quad core processor over *Linux* system with*MPICH* library.

6. FLOATING DOCK WITH MOORING LINES

The following physical system represents a prototype model for those structures called catenary anchor leg mooring (CALM), frequently used in offshore engineering (Culla and Carcaterra, 2007), the values used for the model are the same as used in the reference. This model introduces the next approximations, the buoy is considered a point of mass m_b, the pair of cables is approximated by two nonlinear cubic springs, the hydrodynamic force is determined by the Morison equation and the energy dissipation is introduced by a viscous damping (of characteristic constant $\tilde{c}$. With these assumptions, the equation of the dock motion is:

$$m_b\ddot{w} + 2c\dot{w} + F_T(w_c) = q(t) \tag{24}$$

where w_c is the center of mass displacement, $F_T(w_c)$ the nonlinear restoring force, and the $q(t)$ the nonlinear random load.



c_a	mass coefficient	0.2
c_d	drag coefficient	0.9
D	buoy diameter	10 m
h	ocean depth	50 m
L	cable length	150 m
W	weight per unit length	14 kg
ρ_w	water density	1025 Kg/m^3
m_b	buoy mass	12000 Kg
$U_{19.5}$	wind velocity	25 m/s
$\tilde{c}$	viscous damping	1e5 kg/sm^2

Figure 1: Catenary anchor mooring configuration and physical parameters of the simulations

6.0.1 Nonlinear random hydrodynamic load $q(t)$

The modified Morison equation provides the wave load $q(t)$ per unit length on a circular cylinder in terms of the fluid structure relative velocity $(\dot{\xi} - \dot{w})$.

$$q(t) = C_I\ddot{\xi} - m_a\ddot{w} + C_D|\dot{\xi} - \dot{w}|(\dot{\xi} - \dot{w}) \tag{25}$$

where $\xi(x, z, t)$ is the horizontal fluid particle displacement, assumed approximately the same for each point of the wetted surface. To characterize the random nature of $\xi(x, z, t)$ we use the Airy theory of linear waves to approximate the elevation field as,

$$\eta(t) = \sum_{i=1}^{N} cos(\omega_i t - \varphi_i)\sqrt{2S_{\eta\eta}(\omega_i)\Delta\omega_i} \tag{26}$$

where φ_i is a uniformly distributed random number $U(-\pi, \pi)$ and w_i are the discrete sampling frequencies, $\Delta\omega_i = \omega_i - \omega_{i-1}$, N is the number of partitions of the power spectra and $S_{\eta\eta}$ is the Pierson-Moskowitz spectra that defines the distribution of energy with frequency within the ocean. Developed in 1964 the PM spectrum is an empirical relation and one of the simplest descriptions for the ocean energy distribution. It assumes that if the wind blows steadily for a long time over a large area, then the waves would eventually reach a point of equilibrium with the wind. This is known as a fully developed sea.

$$S_{\eta\eta}(\omega) = \frac{A}{\omega^5} e^{(B/\omega^4)} = \frac{\alpha g^2}{\omega_n^2} e^{-\beta(g/U\omega_n)^4} \tag{27}$$

When the sampling frequencies, ω_i are chosen at equal intervals such $\omega_i = i\omega_1$ the time history will have a period $T = 2\pi/\omega_1$. To avoid this problem, was used the Borgman's method that where the frequencies are chosen so that the area under the spectrum curve, for each interval, is equal, so $\sigma^2 = \sigma_\eta^2/N$ and the time history can be written as,

$$\eta(t) = \sum_{i=1}^{N} \sqrt{\frac{2}{N}} \sigma cos(\omega_i t - k_i \varphi_i) \tag{28}$$

with $k_i = \omega_i/g$ wave number, due to the total variance $\sigma^2 = \int_0^\infty S_{\eta\eta}(\omega)d\omega = \frac{A}{4B}$, the above equation for wave elevation becomes,

$$\eta(t) = \sum_{i=1}^{N} \sqrt{\frac{A}{2NB}} cos(\omega_i t - \varphi_i) \tag{29}$$

where

$$\omega = \left(\frac{B}{ln(N/n) + (B/w_N^4)}\right)^{.25} \tag{30}$$

Finally we arrive to an expression the absolute position for the wave elevation

$$\eta(y,t) = \sum_{i=1}^{N} \sqrt{\frac{A}{2NB}} cos(\omega_i t - k_i y - \varphi_i) \tag{31}$$

and the same way for the absolute velocity of the fluid,

$$\xi(x,y,t) = \sum_{i=1}^{N} \omega \left(\frac{cosh(k_i x)}{senh(k_i d)}\right) \sqrt{\frac{A}{2NB}} cos(\omega_i t - k_i y - \phi_i) \tag{32}$$

where it is easy to immediately calculate the force morrison.

6.0.2 Nonlinear restoring forces model

The nonlinear force displacement relationship for a single cable with one end attached to the sea bottom and the other to the buoying system is determined in this section. With this aim, first we present the nonlinear equations of a uniform inextensible cable suspended among two fixed points. Let s be the curvilinear abscissa, x and z Cartesian coordinates of the cable points, T is the tension along the cable and W its weight per unit length. The static equations of the cable reads,

$$\begin{cases} \frac{\partial}{\partial s}\left(T\frac{\partial x}{\partial s}\right) & = W \\ \frac{\partial}{\partial s}\left(T\frac{\partial z}{\partial s}\right) & = 0 \end{cases} \tag{33}$$

integrating the second equation 33,

$$T\frac{\partial z}{\partial s} = F \tag{34}$$

and applying the inextensibility condition

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = 1 \tag{35}$$

we found the analytical solution of the catenary as,

$$x(z) = \frac{F}{W} cosh\left(\frac{W_z}{F} + \frac{W}{F} c1\right) + c2 \tag{36}$$

where the c_1 e c_2 are determined with the boundary conditions

$$\begin{cases} \frac{\partial x}{\partial s}|_0 & = 0 \\ x|_0 & = 0 \end{cases} \tag{37}$$

then

$$x(z) = \frac{F}{W}\left[cosh\left(\frac{W_z}{F}\right) - 1\right] \quad z(x) = \frac{F}{W} sinh^{-1}\left[\sqrt{\left(\frac{W_x}{F}\right)^2 + 2\left(\frac{W_x}{F}\right)}\right] \tag{38}$$

provides the force displacement relationship at each cable point.

$$s(z) = \int_0^z \sqrt{1 + \left(\frac{\partial x}{\partial z}\right)^2} dz \rightarrow l_s(z) = \frac{F}{W} sinh\left(\frac{Wz}{F}\right) \tag{39}$$

with this expression it is possible to estimate or $Fmax$ that is the maximum expected force for the worst operating condition or $Lmin$ is the minimum length required that warrants a zero slope of the cable line at the anchor point on the sea bottom.

$$l_s(x) = x\sqrt{\frac{2F}{Wx} + 1} \tag{40}$$

$$l_{s_{min}}(x) = x\sqrt{\frac{2F_{max}}{Wh} + 1} \tag{41}$$

The above implies the absence of vertical forces, satisfying an important safety requirement, during the normal operation it is necessary that de $F < Fmax$. Finally, this expression provides the constitutive nonlinear force distance relationship for the cable.

$$d_c = h\sqrt{2\frac{F_{max}}{Wh} + 1} - h\sqrt{2\frac{F}{Wh} + 1} + \frac{F}{W} cosh^{-1}\left(1 + \frac{Wh}{F}\right) \tag{42}$$

where $d_c^{izq} = w_{c0} + w_c \quad d_c^{der} = w_{c0} - w_c$, e w_{c0} is the horizontal distance between the anchor and the fairlead of the mooring line for both the cables in the static reference configuration. Therefore, the mooring actions on the dock can be approximated by a simpler cubic restoring force.

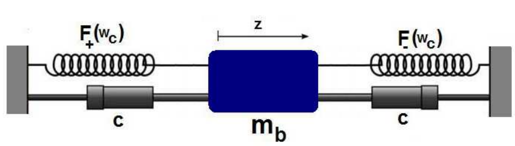


Figure 2: SDOF equivalent model

In offshore structures, often, pairs of cables are anchored to the seabed acting in opposite directions, as shown in Figure 2. In this configuration the total restoring force of the pair of cables is given by

$$F_T(w_c) = F(d_c^{izq}) - F(d_c^{der}) = F_+ - F_-, \tag{43}$$

A Taylor series expansion of FT up to the third-order provides

$$F_T(w_c) = F_T\Big|_0 + \frac{\partial F_T}{\partial w_c}\Big|_0 w_c + \frac{1}{2}\frac{\partial^2 F_T}{\partial w_c^2}\Big|_0 w_c^2 + \frac{1}{6}\frac{\partial^3 F_T}{\partial w_c^3}\Big|_0 w_c^3 \tag{44}$$

where the derivatives are calculated at $w_c = 0$. This equation involves the derivatives of $FT(wc)$, while the force displacement relationship is available.

$$\frac{\partial w_c}{\partial F}\frac{\partial F}{\partial w_c} = 1 \tag{45}$$

The terms of order zero and two are void and, under the hypothesis of small displacements, a cubic dependence of the horizontal force on the displacement of the cable end is determined

$$F_T(w_c) = \gamma_1 w_c + \gamma_2 w_c^3 \tag{46}$$

where

$$\gamma_1 = \frac{2}{\frac{\partial w_c}{\partial F}}\Big|_{F_0} \quad \gamma_3 = -\frac{3\left(\frac{\partial^2 w_c}{\partial F^2}/\frac{\partial w_c}{\partial F}\right)^2 - \frac{\partial^3 w_c}{\partial F^3}/\frac{\partial w_c}{\partial F}}{3\left(\frac{\partial w_c}{\partial F}\right)^3}\Big|_{F_0} \tag{47}$$

$$\frac{\partial F}{\partial w_c} = \frac{1}{\frac{\partial w_c}{\partial F}}, \quad \frac{\partial^2 F}{\partial w_c^2} = -\frac{\frac{\partial^2 w_c}{\partial F^2}}{(\frac{\partial w_c}{\partial F})^3}, \quad \frac{\partial^3 F}{\partial w_c^3} = -\frac{3(\frac{\partial^2 w_c}{\partial F^2})^2 - (\frac{\partial^3 w_c}{\partial F^3})(\frac{\partial w_c}{\partial F})}{(\frac{\partial w_c}{\partial F})^5} \tag{48}$$

Therefore, writing the equation 24 an explicit form for the forces $F_T(w_c)$ and $q(t)$ we obtain,

$$(m_b + m_a)\ddot{w} + 2c\dot{w}\gamma_1 w + [C_D|\dot{\xi} - \dot{w}|(\dot{\xi} - \dot{w}) + \gamma_3 w^3] = C_I\ddot{\xi} \tag{49}$$

where $(\dot{\xi} - \dot{w})$, C_I and C_D are the inertia and the drag coefficient, respectively, m_a the added mass and γ_1,γ_3 coefficients to be calculated. Using,

$$m = m_a + m_b, \quad a_1 = \frac{2c}{m}, \quad a_2 = \frac{\gamma_1}{m}, \quad a_3 = \frac{C_D}{m}, \quad a_4 = \frac{\gamma_3}{m}, \quad a_5 = \frac{C_I}{m}$$

These coefficients depend on the dimension of the dock, the water density and some nondimensional quantity, namely:

$$C_I = (1 + c_a)\frac{\rho_w \pi D^2}{4}, \quad C_D = \frac{1}{2}c_d\rho_w D, \quad m_a = c_a\frac{\rho_w \pi D^2}{4}$$

where ca is the non-dimensional added mass coefficient (caffi1), ρ_w is the seawater density, D is the dock diameter and cd is the non-dimensional drag coefficient $(0.6 \leq c_d \leq 1.2)$. Finally can we rewrite our equations system as,

$$\ddot{\upsilon} + a_1\dot{\upsilon} + (a_2 + 3a_4\xi^2)\upsilon + [a_3|\dot{\upsilon}|\dot{\upsilon} - 3a_4\xi^2 + a_4\upsilon^3] = f(\xi) \tag{50}$$

with the new force term,

$$f(\xi) = (1 - a_5)\ddot{\xi} + a_1\dot{\xi} + a_4\xi^3 + a_2\xi \tag{51}$$

The effect of the cable drag force is not explicitly included, its inclusion does not alter substantially the mathematical nature of the problem considered.

6.0.3 Numerical results 2D

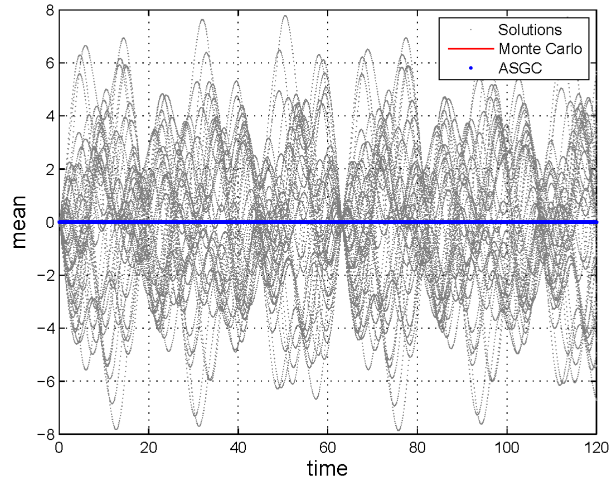


Figure 3: Mean of total outputs with 2 stochastic dimensions

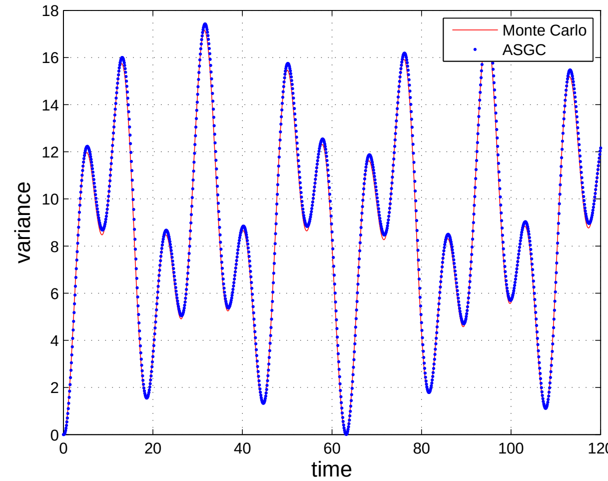


Figure 4: Variance of total outputs with 2 stochastic dimensions

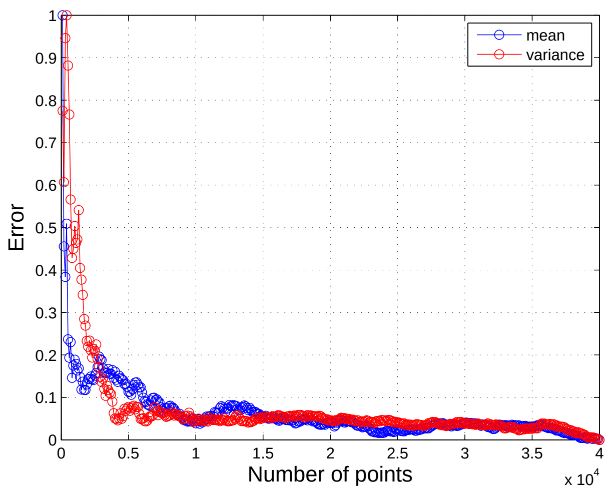


Figure 5: Convergence of Monte-carlo method

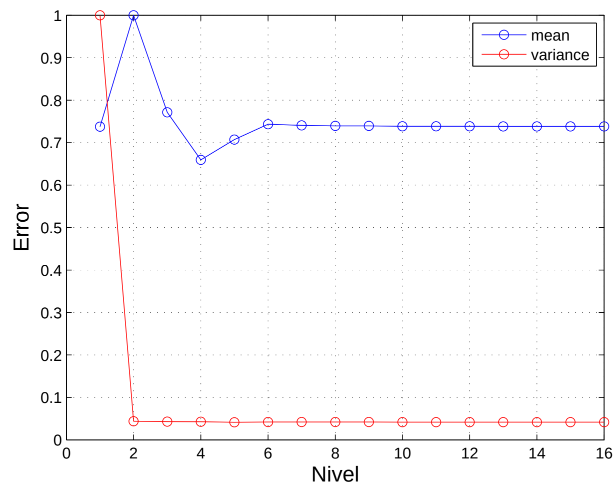
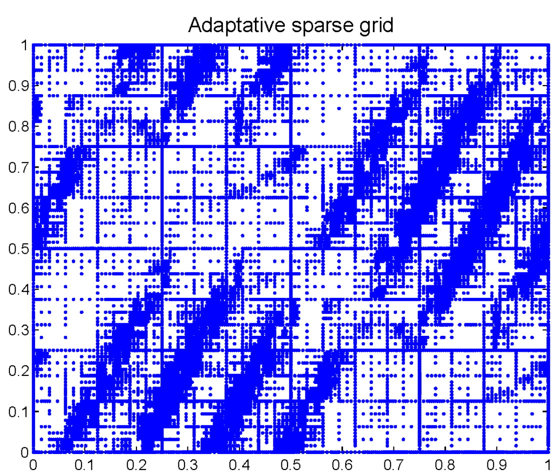


Figure 6: Convergence of ASGC method



Level	No Points ASGC	No Points CSGC
1	4	4
2	8	8
3	16	16
4	36	36
5	80	80
6	176	176
7	382	384
8	776	1792
9	1234	3840
10	1796	8192
11	2036	17408
12	2338	.
13	2532	.
14	2668	.
15	3560	.
16	4449	.

Figure 7: Adaptive sparse grid for maximum level and table with number of points with ASGC and SGC methods

6.0.4 Numerical results 6D

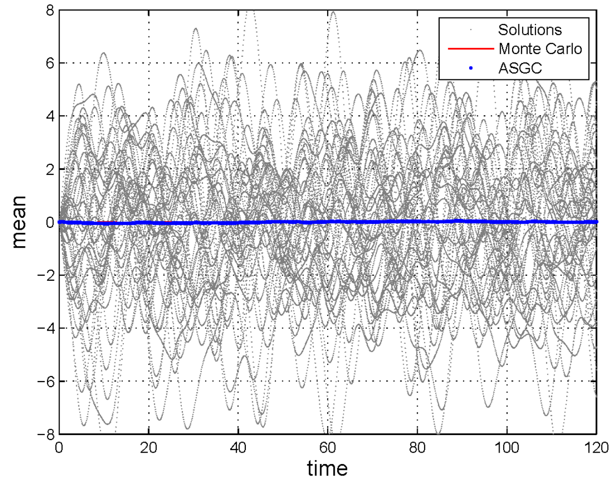


Figure 8: Mean of total outputs with 6 stochastic dimensions

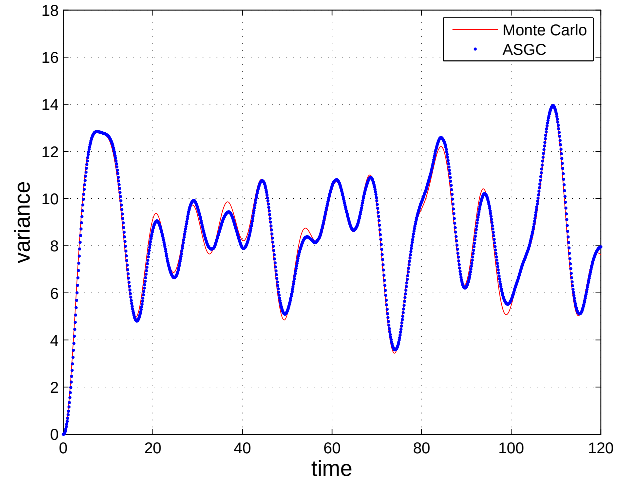


Figure 9: Variances of total outputs with 6 stochastic dimensions

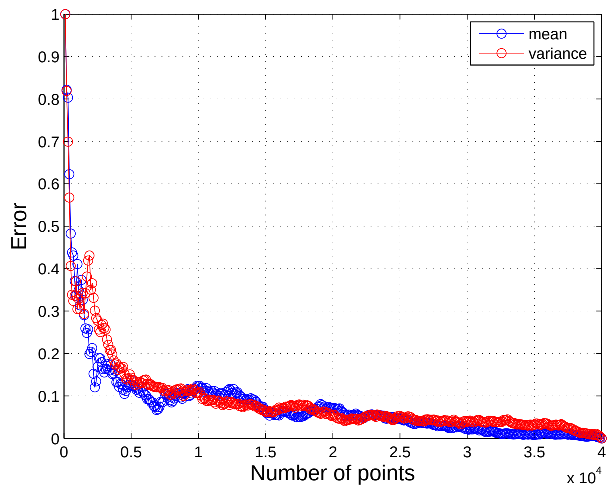


Figure 10: Convergence of Monte-carlo method

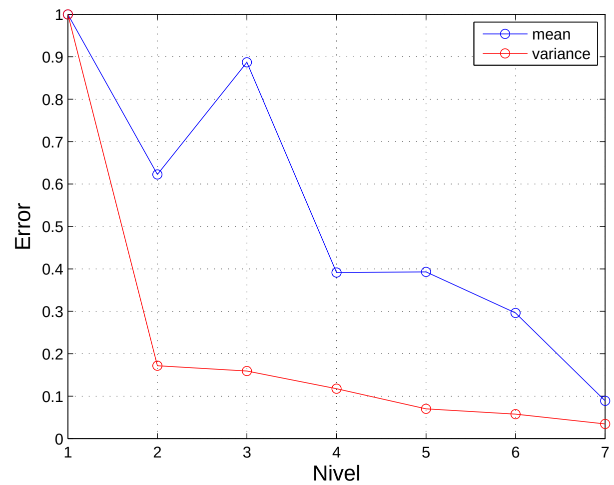
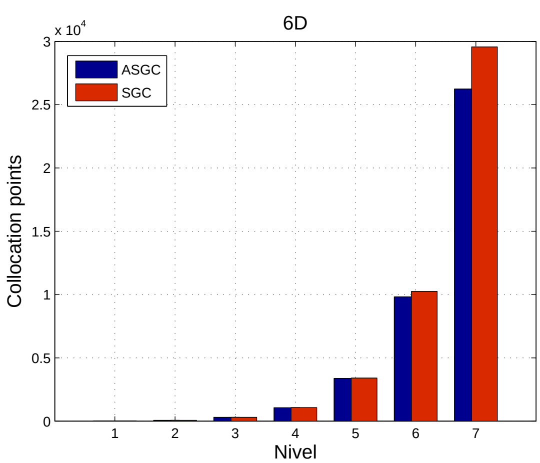


Figure 11: Convergence of ASGC method



Level	No Points ASGC	No Points CSGC
1	12	12
2	72	72
3	304	304
4	1066	1068
5	3380	3408
6	9827	10256
7	26240	29568

Figure 12: Adaptive sparse grid for maximum level and table with number of points with ASGC and SGC methods

To analise the convergence of the method, the relative error was estimated by calculating the L2 norm between consecutive means using a reference Monte Carlo solution with 40000 points and likewise with the variance. Figures 3and 4 shown the mean and variance for 2 dimensional stochastic space for ASGC method and Monte Carlo method, Figures 5and 6 shown the evolution of the error. In the Figure 5 one can observe that the relative error is high because the exist gap between the average calculated between methods but like the criterion of convergence used was the surplus of variances the solutions obtained are very satisfactory. Figure 7 show the effect od the adaptative strategy on the sparse grid, identifying automatically non smooth regions in the stochastic space. It is also possible to see in the table a improve in computational cost comparing, the conventional sparse grid collocation method (CSGC) with the adaptative sparse grid collocation method (ASGC) using less collocation point than (CSGC) to perform the analisys. The same considerations can be done about the analisys of the model with six dimensional stochastic space, where only less levels of interpolation were used to avoid the grow of number of collocation points, reaching a good accuracy in the result as is show in Figures 8,9 and 10,11.

7. CONCLUSIONS

Like the Monte Carlo method, the Adaptive Sparse Grid Stochastic Collocation (ASGC) method leads to the solution of uncoupled deterministic problems and, as such, it is simple to implement and parallelize. These non-intrusive methods, allow convert any deterministic code into a code that solves the corresponding stochastic problem. Compared with the Monte Carlo Simulation method, the (ASGC) shown a significative reduction in the number of experiments required to achieve the same level of accuracy. On the other hand, the results obtained, comparing the Conventional Sparse Grid Collocation method and an adaptive strategy, show that it is possible refine the grid locally identifying automatically non smooth regions in the stochastic space achieving the same accuracy and reducing significatively the cost by the use of less collocations points in smooth regions of the stochastic space. Due to that the majority of engineering problems varying rapidly in only some dimensions, remaining much smoother in other dimensions and in general it have more stochastic dimensions. Future work of this research will include the study high-dimensional methods mixed with Adaptative Sparse Grid Stochastic Collocation methods, in high performance computer environment, aiming to obtain tools to solve real problems of interest in Engineering.

8. ACKNOWLEDGEMENTS

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