PLANE FRAME AND SPACE FRAME: HARMONIC DYNAMIC ANALYSIS BY BOUNDARY ELEMENT METHOD

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Abstract: *This article deals with a suitable strategy for influence matrix assemblage of plane frames and space frames that includes fundamental solutions based on Euler-Bernoulli (bending effects) and Saint Venant (torsion effects) models. Both forced and free vibration of frame structures are performed by BEM where a specific value of frequency must be set for the forced excitation, while only natural frequencies are determined for the modal analysis using a frequency sweep technique. Moreover, numerical results are presented for cases of plane and space frame structures.*

Keywords: *BEM, frames, dynamic analysis*

1. INTRODUCTION

Structural analysis is based on behavior idealization of structural problems using mathematical models. Then, governing equations of these models are usually written as differential equations and/or as integral representations. Analytical and/or numerical techniques are the strategies generally used to find solutions for these equations.

Despite analytical solutions represent exactly the physical fields that hold governing equations, they are usually available only for special and simpler problems. Hence, an alternative way to deal with more general structural problems is to use numerical methods, for instance, Finite Difference Method (FDM), Finite Element Method (FEM), Boundary Element Method (BEM), etc.

There are many reports on numerical analysis of frame structures and the majority of them are associated with finite element analysis. This is a well-known fact because of numerous beginning until recent researches on FEM methodologies that have been developed for structural problems, for instance, Hughes, 1987; Zienkiewicz, 1989; Petyt, 1990; Clough, 1990; Reddy, 1993; Mackerle, 2000; Hutton, 2004. Since the initial investigations on methods of MEC, the main focus on solid mechanics has approached the plate, two-dimensional and three-dimensional problems.

A reasonable amount of textbooks and papers can be found, for instance: Becker, 1992; Dominguez, 1993; Aliabadi, 2002; Katsikadelis, 2002. The boundary element analysis of frame structures has shown a different scenario. A few reports can be found and the majority of them is related to bar or beam problem. Banerjee and Butterfield (1981), Providakis and Beskos (1986) presented respectively BEM solutions for static and harmonic dynamic analysis of single beam using the Euler-Bernoulli model (shear deformation effect for bending problems is neglected). Antes (2003), Antes and Shanz (2001) have established respectively integral equations and fundamental solutions for static and dynamic analysis of single beams from the Timonshenko Model (shear deformation effect and rotatory inertia are taken into account for flexural vibration analysis).

Antes et al. (2004) have presented a symmetric BEM solution for harmonic dynamic analysis of plane frame superposing the beam flexural effect (Timoshenko model) and bar axial stretching effect. To the best author's knowledge, no boundary element dynamic analysis has been done to space frames before.

The aim of this study is to propose a suitable strategy to obtain the algebraic equation system for plane structures and space frame structures under harmonic dynamic excitations. The main focus of the proposed strategy is to create a convenient sequence of transformations (for integral equations of each member) where the local systems of reference are transformed into a unified global system of reference.

2. INTEGRAL AND ALGEBRAIC EQUATIONS OF THE MEMBER

In this section, we discuss integral and algebraic equations for a typical member of plane frames and space frames.

2.1. Plane frame

In order to establish a mathematical representation of plane frame it is necessary a prior study of each structural member behavior. The problem can be split into axial stretching and bending effects, so that their mathematical models require some assumptions (Petyt, 1990), for instance: each structural member is a prismatic bar and made of a homogeneous, isotropic, linear-elastic material; displacements, slopes and strains are represent under small field concept; the kinematics of the member occurs under the planarity conservation of cross-sections; the flexural vibration occurs along the directions of principal moment of inertia.

The bending problem requires an additional assumption for the relative angle between the neutral line and the normal line to the cross-section plane. The Euler-Bernoulli model assumes this angle to remain orthogonal in deformed shape, resulting in the suppression of shear deformation effects to the bending problem.

The governing equation for axial excitation problem under a quiescent initial condition in the Laplace domain is:

$$EA\frac{d^2u(x,s)}{dx^2}+\rho s^2u(x,s)=q_x(x,s) \tag{1}$$

Where s Laplace domain parameter, E is Young modulus; A, the cross-section area; ρ, material density; $u(x,t)$, axial displacement, and $q_x(x,t)$ is distributed axial load along the bar.

Equation (1) can also be rewritten as an equivalent integral equation as Antes et al. (2004):

$$u(\xi,s)+\left[u(x,s)N^*(x-\xi,s)\right]_{-a}^{a}=\left[N(x,s)u^*(x-\xi,s)\right]_{-a}^{a}+\int_{-a}^{a}\left[q_x(x,s)u^*(x-\xi,s)\right]dx \tag{2}$$

where fundamental solutions and symbols are given by Antes et al. (2004).

After collocation process at boundary bar points with the help of Eq. (2) and assuming a harmonic loading (i.e., $s=i\omega$), the algebraic equation for axial vibration (written local reference system, see Fig. 1) can be given by:

$$\begin{Bmatrix}u_i\\u_j\end{Bmatrix}+\frac{1}{2}\begin{bmatrix}-1 & -\cos(\beta_aL)\\-\cos(\beta_aL) & -1\end{bmatrix}\begin{Bmatrix}u_i\\u_j\end{Bmatrix}=\frac{1}{2\beta_aEA}\cdot\begin{bmatrix}0 & -\sin(\beta_aL)\\\sin(\beta_aL) & 0\end{bmatrix}\begin{Bmatrix}N_i\\N_j\end{Bmatrix}+\begin{Bmatrix}f_{ui}\\f_{uj}\end{Bmatrix} \tag{3}$$

Where the nodal displacements are $u_i=u(x=-a)$ and $u_j=u(x=a)$; nodal forces, $N_i=N(x=-a)$ and $N_j=N(x=a)$; the nodal force vector is $f_{ui}=-f_{uj}=q_xL[1-\cos(\beta_aL)]/(2\beta_a^2EA)$; $\beta_a=\sqrt{-s^2E/\rho}$ L, length bar.

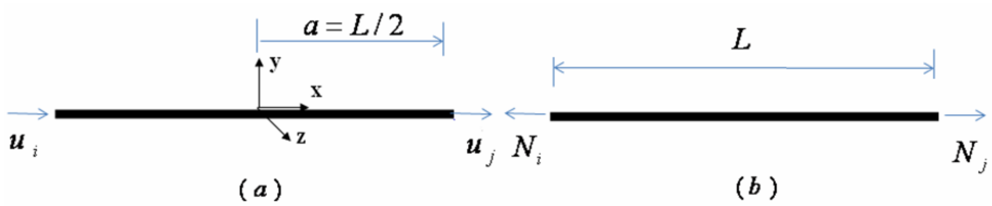


Figure 1 – local reference system of axial problem: a) displacements; b) forces

The Laplace domain governing equation for Euler-Bernoulli beam model under quiescent initial condition is:

$$EI_z\frac{d^4v(x,s)}{dx^4}+\rho s^2v(x,s)=q_y(x,s) \tag{4}$$

Where I_z is the moment of inertia about z-axis; $v(x,s)$, the transverse displacement in y-direction; $q_y(x,s)$, transverse distributed loading.

Equation (6) can be transformed into an equivalent equation for transverse displacement as:

$$\begin{aligned}v(\xi,s)+\left[Q^*_{qy}(x-\xi,s)v(x,s)\right]_{-a}^{a}-\left[M^*_{qz}(x-\xi,s)\varphi_z(x,s)\right]_{-a}^{a}=\left[Q_y(x,s)v^*_q(x-\xi,s)\right]_{-a}^{a}\\-\left[M_z(x,s)\varphi^*_{zq}(x-\xi,s)\right]_{-a}^{a}+\int_{-a}^{a}q_y(x,s)v^*_q(x-\xi,s)dx\end{aligned} \tag{5}$$

Where $\varphi_z(x,s) = dv(x,s)/dx$ and nodal slopes and transverse displacements (at the ends of beam) are respectively $v_i = v(x=-a)$, $v_j = v(x=a)$, $\varphi_{zi} = \varphi_z(x=-a)$ and $\varphi_{zj} = \varphi_z(x=a)$; the nodal efforts (shear force and bending moment) are $Q_{yi} = Q_y(x=-a)$, $Q_{yj} = Q_y(x=a)$, $M_{zi} = M_z(x=-a)$ and $M_{zj} = M_z(x=a)$. The fundamental solutions in Eq. (5) are given buy Providakis and Beskos (1986). All flexural quantities in a local system of reference are shown in Fig. 2.

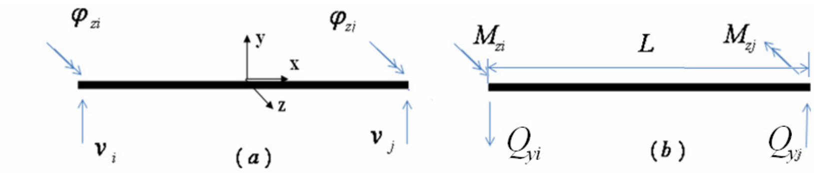


Figure 2 – Local systems of reference: a) displacements; b) forces

Another equation is necessary to solve the flexural vibration problem, so that an integral equation for slopes can be written by differentiation of Eq. (5) at source point. Then, it gives:

$$\begin{aligned}&\varphi_z(\xi,s)+\left[Q^*_{my}(x-\xi,s)v(x,s)\right]_{-a}^{a}-\left[M^*_{mz}(x-\xi,s)\varphi_z(x,s)\right]_{-a}^{a}=\left[Q_y(x,s)v^*_m(x-\xi,s)\right]_{-a}^{a}\\&-\left[M_z(x,s)\varphi^*_{zm}(x-\xi,s)\right]_{-a}^{a}+\int_{-a}^{a}q_y(x,s)v^*_m(x-\xi,s)dx\end{aligned} \tag{6}$$

Where the fundamental solutions for Eq. (6) are given by Providakis and Beskos (1986).

After collocating the source-point at the ends of the bar $(\xi=-a;\xi=a)$ and the calculation of the values given in Eq. (5) and Eq. (6), algebraic representation of the bending problem, referred to the local system of coordinates, is:

$$\begin{Bmatrix} v_i \\ \varphi_{zi} \\ v_j \\ \varphi_{zj} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{2} & \rho_1 & \rho_2 & 0 \\ \rho_3 & -\frac{1}{2} & 0 & \rho_2 \\ \rho_2 & 0 & -\frac{1}{2} & -\rho_1 \\ 0 & \rho_2 & \rho_3 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} v_i \\ \varphi_{zi} \\ v_j \\ \varphi_{zj} \end{Bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 & \alpha_2 \\ 0 & \alpha_3 & \alpha_2 & 0 \\ 0 & \alpha_2 & -\alpha_1 & 0 \\ \alpha_2 & 0 & 0 & -\alpha_3 \end{bmatrix} \begin{Bmatrix} Q_{yi} \\ M_{zi} \\ Q_{yj} \\ M_{zj} \end{Bmatrix} + \begin{Bmatrix} f_{vi} \\ f_{\varphi zi} \\ f_{vj} \\ f_{\varphi zj} \end{Bmatrix} \tag{7}$$

Where: $\lambda_z = \sqrt[4]{-\rho A s^2/(EI_z)}$,

$\alpha_1 = -[\tan(\lambda_z L) - \tanh(\lambda_z L)]/(4\lambda_z{}^3 EI_z)$, $\alpha_2 = [\sec(\lambda_z L) - \operatorname{sec h}(\lambda_z L)]/(4\lambda_z{}^2 EI_z)$,

$\alpha_3 = [\tanh(\lambda_z L) + \tan(\lambda_z L)]/(4\lambda_z EI_z)$, $\rho_1 = [\tanh(\lambda_z L) + \tan(\lambda_z L)]/(4\lambda_z)$,

$\rho_2 = -[\sec(\lambda_z L) + \operatorname{sec h}(\lambda_z L)]/4$, and $\rho_3 = -\lambda_z[\tan(\lambda_z L) - \tanh(\lambda_z L)]/4$.

$f_{vi} = f_{vj} = q_y[\sec(\lambda_z L) + \operatorname{sec h}(\lambda_z L) - 2]/(4\lambda_z{}^4 EI_z)$, $f_{\varphi i} = -f_{\varphi j} = q_y[\operatorname{tgh}(\lambda_z L) - \operatorname{tg}(\lambda_z L)]/(4\lambda_z{}^3 EI_z)$

Both algebraic systems, Eq. (3) and Eq. (7), use two distinct local systems of displacements and forces (see Figs 1 and 2). If both local systems for displacement and force are being simultaneously used to influence matrix assemblage of the structure this can require extra calculations to perform different transformation to elemental displacement and force influence matrices.

Hence, in this paper it is proposed an assemblage strategy to obtain BEM influence matrices of frame structures in an elegant and successful way. The first step of the proposed solution is to do the unification of the variable local systems of reference, that is, the force local system must be transformed into equivalent displacement local system of reference. By keeping this sense in mind, Eq. (3) and Eq. (7) can be rewritten as follows:

$$\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} + \frac{1}{2}\begin{bmatrix} -1 & -\cos(\beta_a L) \\ -\cos(\beta_a L) & -1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \frac{1}{2\beta_a EA}\cdot\begin{bmatrix} 0 & -\sin(\beta_a L) \\ -\sin(\beta_a L) & 0 \end{bmatrix}\begin{Bmatrix} \hat{N}_i \\ \hat{N}_j \end{Bmatrix} + \begin{Bmatrix} f_{ui} \\ f_{uj} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} v_i \\ \varphi_{zi} \\ v_j \\ \varphi_{zj} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{2} & \rho_1 & \rho_2 & 0 \\ \rho_3 & -\frac{1}{2} & 0 & \rho_2 \\ \rho_2 & 0 & -\frac{1}{2} & -\rho_1 \\ 0 & \rho_2 & \rho_3 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} v_i \\ \varphi_{zi} \\ v_j \\ \varphi_{zj} \end{Bmatrix} = \begin{bmatrix} -\alpha_1 & 0 & 0 & -\alpha_2 \\ 0 & \alpha_3 & \alpha_2 & 0 \\ 0 & \alpha_2 & -\alpha_1 & 0 \\ -\alpha_2 & 0 & 0 & -\alpha_3 \end{bmatrix} \begin{Bmatrix} \hat{Q}_{yi} \\ \hat{M}_{zi} \\ \hat{Q}_{yj} \\ \hat{M}_{zj} \end{Bmatrix} + \begin{Bmatrix} f_{vi} \\ f_{\varphi zi} \\ f_{vj} \\ f_{\varphi zj} \end{Bmatrix} \tag{9}$$

If axial and flexural algebraic equations, Eq. (8) and Eq. (9), are put conveniently together in order to assemble plane frame problem, its influence matrices, referred to the unified local system of reference (see Fig. 3), can be given as:

$$\{u\} + [H] \cdot \{u\} = [G] \cdot \{p\} + \{b\} \tag{10}$$

Where: $\{u\}^T$, $\{p\}^T$, $\{b\}^T$ are $\{u_i \quad v_i \quad \hat{\varphi}_{zi} \quad u_j \quad v_j \quad \hat{\varphi}_{zj}\}$, $\{\hat{N}_i \quad \hat{Q}_{yi} \quad \hat{M}_{zi} \quad \hat{N}_j \quad \hat{Q}_{yj} \quad \hat{M}_{zj}\}$ and $\{f_{ui} \quad f_{vi} \quad f_{\varphi i} \quad f_{uj} \quad f_{vj} \quad f_{\varphi j}\}$. The influence matrices in Eq. (10) are:

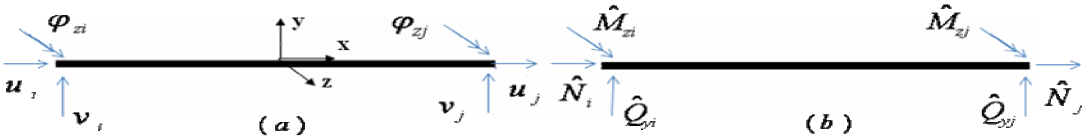


Figure 3 – Unified Local systems of reference: a) displacements; b) forces

$$[H] = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & -\frac{\cos(\beta_a L)}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \rho_1 & 0 & \rho_2 & 0 \\ 0 & \rho_3 & -\frac{1}{2} & 0 & 0 & \rho_2 \\ -\frac{\cos(\beta_a L)}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \rho_2 & 0 & 0 & -\frac{1}{2} & -\rho_1 \\ 0 & 0 & \rho_2 & 0 & \rho_3 & -\frac{1}{2} \end{bmatrix}, \quad [G] = \begin{bmatrix} 0 & 0 & 0 & -\frac{\sin(\beta_a L)}{2\beta_a EA} & 0 & 0 \\ 0 & -\alpha_1 & 0 & 0 & 0 & -\alpha_2 \\ 0 & 0 & \alpha_3 & 0 & \alpha_2 & 0 \\ -\frac{\sin(\beta_a L)}{2\beta_a EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & -\alpha_1 & 0 \\ 0 & -\alpha_2 & 0 & 0 & 0 & \alpha_3 \end{bmatrix} \tag{11}$$

2.2. Space frame

For the mathematical representation of the space frame is required, beyond those considered for plane frames, additional descriptions of the torsion and y-direction bending problems. If the model of uniform torsion of Saint-Venant is assumed some hypotheses are required such as homogeneous, isotropic, linear-elastic material; transverse planes remain mutually parallel, small angle of twist, length and radius of bar remain unchanged. Hence, the governing equation is:

The Laplace domain governing equation under quiescent initial condition is:

$$GI_t \frac{d^2\varphi_x(x,s)}{dx^2} + \rho s^2 \varphi_x(x,s) = t(x,s) \tag{12}$$

Where I_t is Saint-Venant torsion constant; $\varphi_x(x,s)$ is the twist angle; $t(x,s)$ is the distributed torque.

Due to the axial and torsion vibration similarities, see Eq. (1) and Eq. (12), an algebraic representation for the uniform torsion can be written as follows:

$$\begin{Bmatrix} \varphi_{xi} \\ \varphi_{xj} \end{Bmatrix} + \frac{1}{2} \begin{bmatrix} -1 & -\cos(\beta_t L) \\ -\cos(\beta_t L) & -1 \end{bmatrix} \begin{Bmatrix} \varphi_{xi} \\ \varphi_{xj} \end{Bmatrix} = -\frac{1}{2\beta_t GI_t} \cdot \begin{bmatrix} 0 & \sin(\beta_t L) \\ -\sin(\beta_t L) & 0 \end{bmatrix} \begin{Bmatrix} \hat{T}_i \\ \hat{T}_j \end{Bmatrix} + \begin{Bmatrix} f_{\vartheta xi} \\ f_{\vartheta xj} \end{Bmatrix} \tag{13}$$

Where $\beta_t = \sqrt{-s^2 G/\rho}$, the nodal twist angle and twisting moment are: $\varphi_{xi} = \varphi_x(x=-a)$, $\varphi_{xj} = \varphi_x(x=a)$, $T_i = T(x=-a)$, $T_j = T(x=a)$, and loading vector values are: $f_{\theta i} = f_{\theta j} = tL^2/(4GI_t)$.

The flexural vibration about y axis is similar to the direction z case, so that only a few adjustment (sign corrections and I_z by I_y replacement) should be made in Eq. (7). After unification procedures (see Fig. 4), the torsion and flexural vibration (about y) algebraic system referred to unified local system of reference can be written as:

$$\begin{Bmatrix} w_i \\ \varphi_{yi} \\ w_j \\ \varphi_{yj} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{2} & \gamma_1 & \gamma_2 & 0 \\ \rho_3 & -\frac{1}{2} & 0 & \gamma_2 \\ \rho_2 & 0 & -\frac{1}{2} & -\gamma_1 \\ 0 & \gamma_2 & \gamma_3 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} v_i \\ \varphi_{zi} \\ v_j \\ \varphi_{zj} \end{Bmatrix} = \begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_3 & \eta_2 & 0 \\ 0 & \eta_2 & -\eta_1 & 0 \\ \eta_2 & 0 & 0 & -\eta_3 \end{bmatrix} \begin{Bmatrix} \hat{Q}_{yi} \\ \hat{M}_{zi} \\ \hat{Q}_{yj} \\ \hat{M}_{zj} \end{Bmatrix} + \begin{Bmatrix} f_{vi} \\ f_{\varphi zi} \\ f_{vj} \\ f_{\varphi zj} \end{Bmatrix} \tag{14}$$

Where: $\eta_1 = -[\tan(\lambda_y L) - \tanh(\lambda_y L)]/(4\lambda_y^3 EI_y)$, $\eta_2 = [\sec(\lambda_y L) - \sec h(\lambda_y L)]/(4\lambda_y^2 EI_y)$,

$\eta_3 = [\tanh(\lambda_y L) + \tan(\lambda_y L)]/(4\lambda_y EI_y)$, $\gamma_1 = [\tanh(\lambda_y L) + \tan(\lambda_y L)]/(4\lambda_y)$,

$\gamma_2 = -[\sec(\lambda_y L) + \sec h(\lambda_y L)]/4$, and $\gamma_3 = -\lambda_y[\tan(\lambda_y L) - \tanh(\lambda_y L)]/4$,

$f_{\varphi xi} = -f_{\varphi xj} = tL[1-\cos(\beta_t L)]/(2\beta_t^2 GI_t)$, $f_{wi} = f_{wj} = q_z\left[\sec(\lambda_y L) + \sec h(\lambda_y L) - 2\right]/(4\lambda_y^4 EI_y)$,

$f_{\varphi yi} = -f_{\varphi yj} = q_z\left[tgh(\lambda_y L) - tg(\lambda_y L)\right]/(4\lambda_y^3 EI_y)$

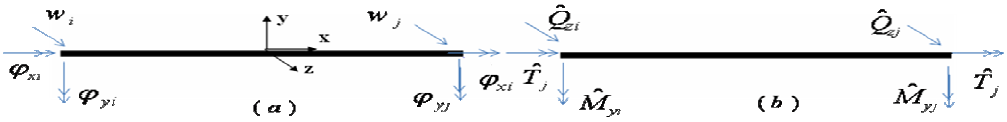


Figure 4 – Unified local systems of reference: a) displacements; b) forces

It should be noted that the rotations φ_y have a opposite direction to the y axis, so that an additional transformation ($\hat{\varphi}_{yi} = -\varphi_{yi}$, $\hat{\varphi}_{yj} = -\varphi_{yj}$) and ($\hat{\hat{M}}_{yi} = -\hat{M}_{yi}$, $\hat{\hat{M}}_{yj} = -\hat{M}_{yj}$) is necessary to Eq.(14), resulting in:

$$\begin{Bmatrix} \hat{w}_i \\ \hat{\varphi}_{yi} \\ \hat{w}_j \\ \hat{\varphi}_{yj} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{2} & \gamma_1 & \gamma_2 & 0 \\ \rho_3 & -\frac{1}{2} & 0 & \gamma_2 \\ \rho_2 & 0 & -\frac{1}{2} & -\gamma_1 \\ 0 & \gamma_2 & \gamma_3 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} \hat{v}_i \\ \hat{\varphi}_{zi} \\ \hat{v}_j \\ \hat{\varphi}_{zj} \end{Bmatrix} = \begin{bmatrix} \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_3 & \eta_2 & 0 \\ 0 & \eta_2 & -\eta_1 & 0 \\ \eta_2 & 0 & 0 & -\eta_3 \end{bmatrix} \begin{Bmatrix} \hat{Q}_{yi} \\ \hat{\hat{M}}_{zi} \\ \hat{Q}_{yj} \\ \hat{\hat{M}}_{zj} \end{Bmatrix} + \begin{Bmatrix} f_{wi} \\ f_{\varphi yi} \\ f_{wj} \\ f_{\varphi yj} \end{Bmatrix} \tag{15}$$

If axial, torsion and two flexural algebraic equations, Eqs. (11-13), and Eq. (15), are put conveniently together in order to assemble space frame problem, its influence matrices, referred to the unified local system of reference (see Fig. 4), can be given in similar form of Eq.(10). It should be noted the following BEM influence matrices for harmonic dynamic for space frame analysis are presented here for first time in literature. Their values are (see Fig.5):

$$\{u\}^T = \{u_i \quad v_i \quad w_i \quad \varphi_{xi} \quad \hat{\varphi}_{yi} \quad \hat{\varphi}_{zi} \quad u_j \quad v_j \quad w_j \quad \varphi_{xj} \quad \hat{\varphi}_{yj} \quad \hat{\varphi}_{zj}\},$$

$$\{p\}^T = \left\{\hat{N}_i \quad \hat{Q}_{yi} \quad \hat{Q}_{zi} \quad \hat{T}_i \quad \hat{\hat{M}}_{yi} \quad \hat{M}_{zi} \quad \hat{N}_j \quad \hat{Q}_{yj} \quad \hat{Q}_{zj} \quad \hat{T}_j \quad \hat{\hat{M}}_{yj} \quad \hat{M}_{zj}\right\},$$

$$\{b\}^T = \{f_{ui} \quad f_{vi} \quad f_{wi} \quad f_{\varphi xi} \quad -f_{\varphi yi} \quad f_{\varphi zi} \quad f_{uj} \quad f_{vj} \quad f_{wj} \quad -f_{\varphi xj} \quad f_{\varphi yj} \quad f_{\varphi zj}\},$$

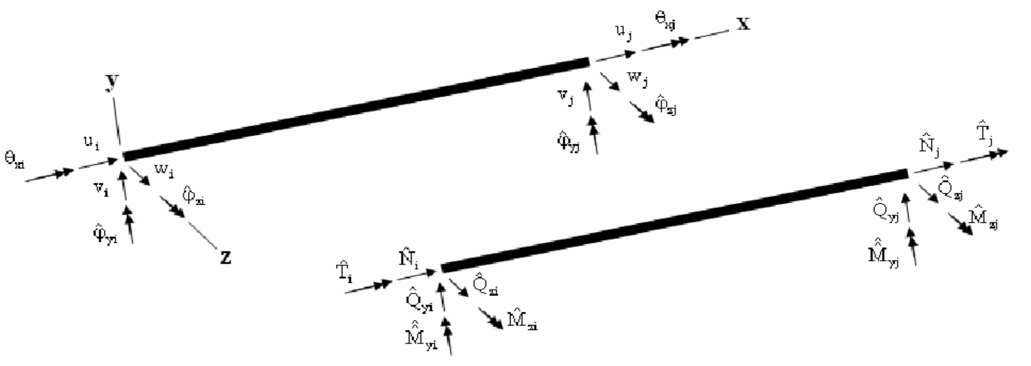


Figure 5 - Unified local reference system of space frame element

$$
[\mathrm{H}]=\begin{bmatrix}
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{\cos\beta_a L}{2} & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{1}{2} & 0 & 0 & 0 & \rho_1 & 0 & \rho_2 & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{1}{2} & 0 & -\gamma_1 & 0 & 0 & 0 & \gamma_2 & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{\cos(\beta_t L)}{2} & 0 & 0\\
0 & 0 & -\gamma_3 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \gamma_2 & 0\\
0 & \rho_3 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \rho_2\\
-\frac{\cos(\beta_a L)}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\rho_1\\
0 & 0 & \gamma_2 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \gamma_1 & 0\\
0 & 0 & 0 & -\frac{\cos(\beta_t L)}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0\\
0 & 0 & 0 & 0 & \gamma_2 & 0 & 0 & 0 & \gamma_3 & 0 & -\frac{1}{2} & 0\\
0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & -\rho_3 & 0 & 0 & 0 & -\frac{1}{2}
\end{bmatrix}
$$

$$
[\mathrm{G}]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sin(\beta_a L)}{2\beta_a EA} & 0 & 0 & 0 & 0 & 0\\
0 & -\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2\\
0 & 0 & -\eta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sin(\beta_t L)}{2\beta_t GI_t} & 0 & 0\\
0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & \alpha_2 & 0 & 0 & 0 & 0\\
-\frac{\sin(\beta_a L)}{2\beta_a EA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & -\alpha_1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\eta_2 & 0 & 0 & 0 & -\eta_1 & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{\sin\beta_t L}{2\beta GI_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_3 & 0\\
0 & -\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_3
\end{bmatrix}
$$

2.3. Elemental global system of reference

One step required to assemble the influence matrices of the frame is re-write unified local representation of bar contributions into global reference system. Hence, local displacements and efforts can transform into respective global counterparts by:

$$\{u\}=[R]\cdot\{U\},\ \{b\}=[R]\cdot\{B\}\ \text{and}\ \{p\}=[R]\cdot\{P\} \tag{16}$$

Where [R] is the transformation matrix and its explicit form is given by Gere and Weaver (1981).

By substitution of Eq. (16) into Eq. (10), a global algebraic representation of element can be written as follows:

$$\{U\}+\lfloor H_g\rfloor\cdot\{U\}=\lfloor G_g\rfloor\cdot\{P\}+\{B\} \tag{17}$$

Where: $\left[H_g\right]=[R]^T\cdot[H]\cdot[R]$, $\left[G_g\right]=[R]^T\cdot[G]\cdot[R]$, $[B]=[R]^T\cdot[b]$.

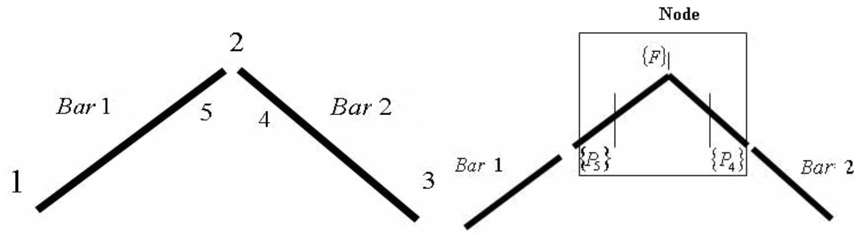
3. ALGEBRAIC SYSTEM OF THE STRUCTURE

The algebraic representation of the structure (plane frame or space frame) requires that the contributions coming from the bars are properly accumulated in order to describe the behavior of the structure as a whole.

In the sequel we discuss the second step of proposed strategy to establish BEM solution for plane and space. For sake of conciseness, a small frame is taken as a guiding example in order to obtain the final algebraic system of the structure.

Considering two or more bars sharing the same node, the unknowns are generally associated with both displacement and force fields. Hence, additional relations should be built in order to become the algebraic system solvable. These extra set of equation can be obtained by applying displacement continuity and force equilibrium conditions at all sharing nodes. For instance, if the bars (1) and (2) are converging to the node 2, then the displacement continuity conditions can be stated as, see Fig. 6a:

$$\{U_5\}=\{U_4\}=\{U_2\} \tag{18}$$



(a) (b)

Figure 6 – Node frame bars: a) compatibility b) equilibrium relations

The equilibrium conditions at node 2 can be given by (see Figure 6b):

$$\{P_5\}+\{P_4\}+\{F\}=\{0\} \tag{19}$$

Where {F} is the vector that stores forces and moments applied directly at node 2; $\{P_5\}$and $\{P_4\}$ are the vectors that store efforts acting at right and left sides at node 2, respectively.

The elemental global algebraic system of the Bar (1) is given by:

$$\begin{cases}[H_{11}]\{U_1\}+[H_{12}]\{U_5\}=[G_{11}]\{P_1\}+[G_{12}]\{P_5\}+\{B_1\}\\ [H_{21}]\{U_1\}+[H_{22}]\{U_5\}=[G_{21}]\{P_1\}+[G_{22}]\{P_5\}+\{B_2\}\end{cases} \tag{20}$$

Bar (2) algebraic representation is:

$$\begin{cases}\left[\hat{H}_{22}\right]\{U_4\}+\left[\hat{H}_{23}\right]\{U_3\}=\left[\hat{G}_{22}\right]\{P_4\}+\left[\hat{G}_{23}\right]\{P_3\}+\left\{\hat{B}_2\right\}\\ [H_{32}]\{U_4\}+[H_{33}]\{U_3\}=[G_{32}]\{P_4\}+[G_{33}]\{P_3\}+\{B_3\}\end{cases} \tag{21}$$

When Eq. (20) and Eq. (21) are inserted into Eq. (18) and Eq. (19), a system of equations for the frame can be written as follows:

$$\begin{bmatrix}[H_{11}] & [H_{12}] & [0] & [0] & [-G_{12}]\\ [H_{21}] & [H_{22}] & [0] & [0] & [-G_{22}]\\ [0] & [H_{32}] & [H_{33}] & [-G_{32}] & [0]\\ [0] & [\hat{H}_{22}] & [\hat{H}_{23}] & [-\hat{G}_{22}] & [0]\\ [0] & [0] & [0] & [I] & [I]\end{bmatrix}\begin{Bmatrix}\{U_1\}\\ \{U_2\}\\ \{U_3\}\\ \{P_4\}\\ \{P_5\}\end{Bmatrix}=\begin{bmatrix}[G_{11}] & [0] & [0] & [0] & [0]\\ [G_{21}] & [0] & [0] & [0] & [0]\\ [0] & [0] & [G_{33}] & [0] & [0]\\ [0] & [0] & [\hat{G}_{23}] & [0] & [0]\\ [0] & [0] & [0] & [0] & [0]\end{bmatrix}\begin{Bmatrix}\{P_1\}\\ \{0\}\\ \{P_3\}\\ \{0\}\\ \{0\}\end{Bmatrix}+\begin{Bmatrix}\{B_1\}\\ \{B_2\}\\ \{B_3\}\\ \{B_4\}\\ -\{F\}\end{Bmatrix} \tag{22}$$

4. NUMERICAL RESULTS

Examples for two frame structures are used to demonstrate the proposed formulation in this paper.

4.1 Pined cross-shaped plane frame

In this example the BEM solution is used to detect the five lowest natural frequencies of the cross-shaped frame structure shown in Fig. 7. The material Young's modulus and mass density are $E = 200GPa$ and $\rho = 8000kg/m^3$. A square cross-section with side 0.175m is assumed to all members.

For BEM modeling only a single boundary element per member is required and horizontal and vertical time-harmonic forces with amplitude F=10kN are applied at the node of cross intersection. In addition, BEM results require an incremental procedure for frequency values from 7.958 Hz to 48.0 Hz with 0.001592 Hz for frequency increment. In Figure 8 are shown the plotted values of vertical displacement amplitudes versus frequencies at the loaded node. The natural frequencies given by BEM solution can be identified at discontinuities on the result plots.

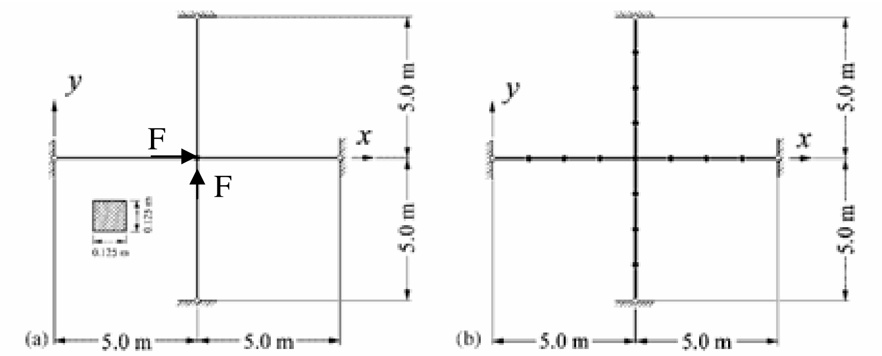


Figure 7 - (a) pined frame b) Standard FEM discretization

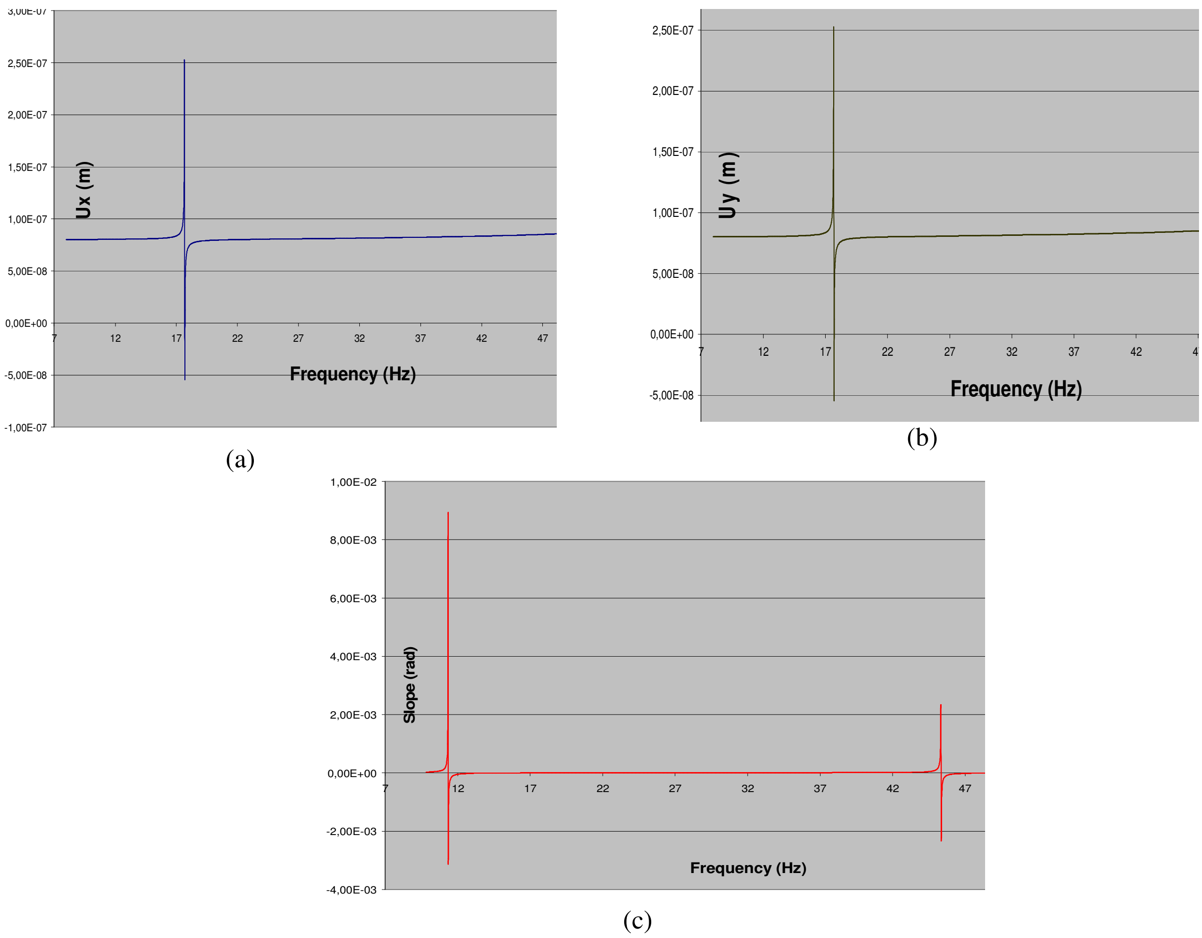


Figure 8 - Amplitude vs frequency: a) Ux , b) Uy displacements and c) slope φ

In Table 1 the comparison for frequencies (at discontinuities) taken from Fig. 8 and results for two different FEM approaches given in Abbassian et al. (1987) and Ma (2008) are shown. Abbassian's group results were obtained using standard (polynomial interpolation) finite elements where each member of the frame is modeled with four elements of equal length (see Fig. 7b). The second FEM formulation uses special (exponential interpolation) finite elements where each member is discretized with one finite element

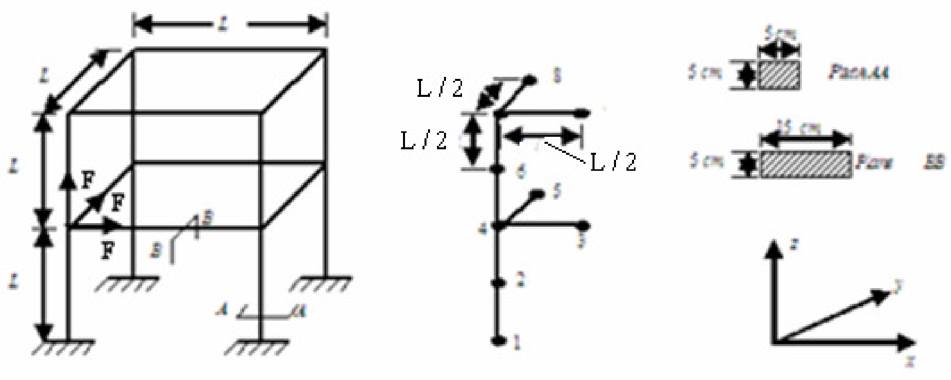
Mode	**FEM**	**BEM**	**FEM**
	Abbassian et al. (1987)	Current Paper	Ma (2008)
1	11.336	11.331842	11.33626
2	17.709	17.682129	17.68079
3	17.709	17.682129	17.68079
4	17.709	17.698045	17.70940
5	45.345	45.343282	45.34504

Table 1- Lower natural frequencies (Hz) of the pined cross-shaped frame

4.2 Clamped space frame

In this example lower axisymmetric natural frequencies of a clamped two-storey space frame (see Figure 9a) are aimed to be determined by BEM. The material Young's modulus and mass density are $E = 219.9GPa$ and $\rho = 7900kg/m^3$. The dimensions of the cross-section members are shown in Figure 9c. Components of time-harmonic force with amplitude F=1kN are applied at both x-, y-, and z-directions, see Figure 9a. For sake of symmetry only a quarter structure analyses is done. BEM discretization requires only one boundary element per member. For the determination of the BEM results an incremental procedure for frequency values from 4.775 Hz to 50.0 Hz with 0.159Hz for frequency increment is used.

Petyt (2004) suggests the following boundary conditions for mid-nodes of the horizontal members: nodes 5 and 8, displacement $U_z = 0$ and rotations $\theta_x = \theta_z = 0$; nodes 3 and 7, displacements $U_y = U_z = 0$, see Figure 9b. Figure 10 shows the values of horizontal displacement amplitudes versus frequencies at the loaded node. The natural frequencies given by BEM solution can be identified at discontinuities on the result plots.



(a) (b) (c)

Figure 9 - Space frame and its discretization

Table 2 shows the comparison for frequencies (at discontinuities) taken from Figure 10 and FEM results for a standard FEM approach using two finite element per member given in Petyt(1990), see Figure 9b.

Mode	Frequency (Hz)	
	BEM (current solution)	FEM (Petyt, 2004)
1	11.777476	11.8
2	34.059187	34.1

Table 2 – Lower axisymmetric natural frequencies of clamped space frame

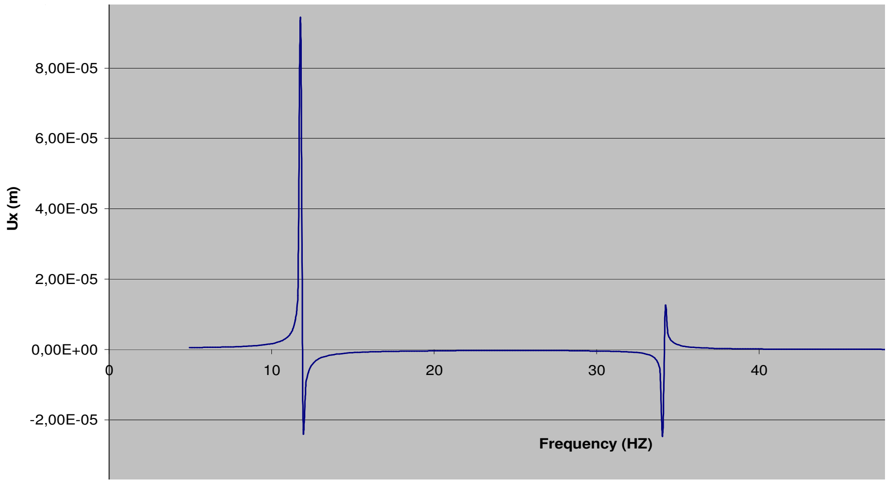


Figure 10 - Amplitude versus frequency: Horizontal displacement

5. CONCLUSIONS

In this article was presented a convenient approach to assemble the influence matrices of BEM for plane and space frame problems. In this model, shear deformation effect and rotary inertia are neglected (Euler-Bernoulli flexural theory). Both forced and free vibration problems can be determined by BEM, since a frequency sweep technique is applied. To best authors' knowledge, this is the first time that vibration analysis of space frame is modeled by BEM.

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