AN ALGORITHM FOR THE PLASTIC SHAKEDOWN OF A POROUS MATERIAL

Carlos Cezar de La Plata Ruiz
Mechanical Engineering Department, UERJ, Av. São Francisco Xavier, 524 Maracanã, Sala 5026 Bloco A, delaplataruiz@uol.com.br.

José Luís L. Silveira
Mechanical Engineering Department, COPPE and POLI – UFRJ, Centro de Tecnologia, Sala G-204, jluis@mecanica.ufrj.br.

Summary. *Based on the static theorem of plastic shakedown presented by Polizzoto, static and kinematic variational principles can be established. Such principles are actually, optimization problems whose solutions can be obtained adapting Newton's method to find the optimal points of nonlinear problems formed by equality and inequalities constraints also nonlinear. Therefore, this article aims to present an algorithm based on Newton's iteration to determine the largest amplification of a domain of elastic stress fields in such a way that the body or structure will not undergo a process of alternating plasticity that would lead to its collapse by a low cycle fatigue.*

Keywords: *Porous Material, Plastic Shakedown, Low Cycle Fatigue, Alternating Plasticity, Optimization, Newton's Method.*

1. INTRODUCTION

The static theorem of plastic shakedown, presented by (Polizzotto, 1993), establishes that a body or structure will not undergo a process of alternating plasticity, for the load variations contained in a domain, if and only if a stress distribution T independent of time exists, such that its sum with any stress T^e (which belongs to the domain variations of the corresponding elastic stress) will be plastically admissible.

For porous materials the plastic deformation depends not only on the deviatoric stress but also on the hydrostatic pressure. Doraivelu (1984) proposed a yield function for porous materials which is used in this work, together with the Polizzoto's theorem of plastic shakedown (1993), to develop a Newton based algorithm for the numerical solution for the plastic shakedown problem.

2. A YIELD FUNCTION FOR POROUS MATERIALS

Based on (Doraivelu, 1984) it is possible to demonstrate that a yield function for porous materials can be expressed by:

$$f(\sigma) = A(\sigma)J_2(\sigma) + B(\sigma)I_1^2(\sigma) - \delta(\sigma)Y_o^2 \leq 0 \tag{1}$$

where:

J_2 is the second invariant of the stress deviator tensor;
I_1 is the first invariant of the stress tensor;
Y_o is the yield stress of the base material;
R is the relative density of the porous material;
σ is the stress on the point of study.

$$A = 2 + R^n \tag{2}$$

$$B = \frac{1 - R^n}{3} \tag{3}$$

and where n is an exponential that according to Alves (2006) is given by:

$$n = 4.15R_i - 1.23 \tag{4}$$

In the previous expression, R_i represents the initial relative density of the porous material. Besides that,

$$\delta(R) = \frac{Y^2}{Y_o^2} \tag{5}$$

and Y represents the yield stress of the porous material. Still according to Alves (2006), the parameter δ can be estimated by the following expression:

$$\delta(R) = \frac{0.343R_i^{2.988} \cdot R}{0.343R_i^{2.988} + (1-R)} \tag{6}$$

Based on Heckel (1961), it is possible to express the relative density R as a function of the hydrostatic stress pH :

$$R = 1 - Exp[-(k \cdot pH + c)] \tag{7}$$

where:

$$k \cong \frac{1}{3Y_o} \tag{8}$$

$$c = \ln\left(\frac{1}{1-R_i}\right) \tag{9}$$

$$pH = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{10}$$

3. A MIXED VARIATIONAL PRINCIPLE FOR THE PLASTIC SHAKEDOWN OF A POROUS MATERIAL

Assuming that a domain ΔF exists, where the loads applied to a body or a structure can vary or execute cycles freely and supposing that an amount k of independent mechanical loads acts in the body, then it can be conceived that the load domain ΔF is formed by a convex polyhedron of m vertices, where $m = 2^k$.

The stress fields T^e resulting from the above mentioned loads are obtained with the assumption that the material response is purely elastic to any applied external load and are called unlimitedly elastic stress.

Thus it is possible to present the following mixed variational principle for the plastic shakedown of a porous material (Schatz, 2008):

$$\omega = \sup_{\omega^*,T} \inf_{D^\alpha} \left\{ \omega^* + \sum_\alpha [\chi(D^\alpha) - \int_B \left(\omega^* T^{e\alpha} + T\right) \cdot D^\alpha \, dB] \right\} \alpha = 1 \ldots m \tag{11}$$

where B represents a domain at $\boldsymbol{R^3}$ occupied by the body in question, D^α is the rate of plastic deformation, $\chi(D^\alpha)$ represents a dissipation function and ω the domain amplification factor.

Note that the lagrangian of the previous optimization problem is given by:

$$L(\omega^*, T, D^\alpha) = \omega^* + \sum_\alpha \left[\chi(D^\alpha) - \int_B \left(\omega^* T^{e\alpha} + T\right) \cdot D^\alpha \, dB \right] \tag{12}$$

A local form for the problem of alternating plasticity is exactly the same as the one defined by the variational principle established on Eq. (11) and is given by (Silveira, 1996):

$$\hat{\omega} = \inf_x \omega(x) \tag{13}$$

where

$$\omega(x) = \sup_{\omega^*(x),T(x)} \omega^*(x) \;\Big|\; \omega^*(x)T^{e\alpha}(x) + T(x) \in P \quad \alpha = 1,\dots,m \tag{14}$$

The optimal conditions for the previous local problem are:

$$\sum_{\alpha} \lambda^{\alpha} \nabla f\left(\omega^* T^{e\alpha} + T\right) = 0 \tag{15}$$

$$\sum_{\alpha} \left[T^{e\alpha} \cdot \lambda^{\alpha} \nabla f\left(\omega^* T^{e\alpha} + T\right)\right] - 1 = 0 \tag{16}$$

$$f\left(\omega^* T^{e\alpha} + T\right)\lambda^{\alpha} = 0 \tag{17}$$

$$f\left(\omega^* T^{e\alpha} + T\right) \le 0 \tag{18}$$

$$\lambda^{\alpha} \ge 0 \tag{19}$$

Where all the variables ω^*, λ^{α}, $T^{e\alpha}$ e T are functions of the position x occupied by the point of the body subject to the loads.

4. NEWTON'S METHOD

The formula of Newton's algorithm for the optimization of a function *L(u)*, is given by:

$$\nabla h(u)du = -h(u) \tag{20}$$

where:

$$h(u) = \nabla L(u)$$

and $\nabla h(u)$ is the Hessian of the function *L(u)*.

Initially it is considered only the equality constraints, Eqs. (15), (16) and (17). Thus, it can be written

$$h(u) = \begin{bmatrix} \sum_{\alpha} \lambda^{\alpha} \nabla f T^{e\alpha} \cdot \lambda^{\alpha} \nabla f\left(\omega^* T^{e\alpha} + T\right) \\ \sum_{\alpha} \left\{T^{e\alpha} \cdot \left[\lambda^{\alpha} \nabla f T^{e\alpha} \cdot \lambda^{\alpha} \nabla f\left(\omega^* T^{e\alpha} + T\right)\right]\right\} - 1 \\ F\lambda \end{bmatrix} = 0 \tag{21}$$

where:

$$u = [T \;\; \omega \;\; \lambda \;]^T \tag{22}$$

$$T = [T_{11} \;\; T_{22} \;\; T_{33} \;\; T_{12} \;\; T_{23} \;\; T_{31}]^T \tag{23}$$

$$\lambda = [\lambda^1 \;\; \lambda^2 \dots \lambda^m]^T \tag{24}$$

$$F = \begin{bmatrix} f\left(\omega^* T^{e1} + T\right) & 0 & \dots & 0 \\ 0 & f\left(\omega^* T^{e2} + T\right) & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & f\left(\omega^* T^{em} + T\right) \end{bmatrix} \tag{25}$$

It is established that:

$$du = [dT \;\; \mathrm{d}\omega \;\; \lambda^o - \lambda \;\;]^T \tag{26}$$

$$\lambda^o = [\lambda^{o1} \quad \lambda^{o2} \ldots \lambda^{om}]^T \tag{27}$$

Therefore, according to Eq. (21), the functional h(u) is formed by three functions:

$$h(u) = \begin{bmatrix} h_1(u) \\ h_2(u) \\ h_3(u) \end{bmatrix} = 0 \tag{28}$$

and for the use of Eq. (20) it is necessary also to know the Hessian $\nabla h(u)$ given by:

$$\nabla h(u) = \begin{bmatrix} \frac{\partial h_1}{\partial T^r} & \frac{\partial h_1}{\partial \omega} & \frac{\partial h_1}{\partial \lambda} \\ \frac{\partial h_2}{\partial T^r} & \frac{\partial h_2}{\partial \omega} & \frac{\partial h_2}{\partial \lambda} \\ \frac{\partial h_3}{\partial T^r} & \frac{\partial h_3}{\partial \omega} & \frac{\partial h_3}{\partial \lambda} \end{bmatrix} \tag{29}$$

Deriving the lines of the functional *h(u)* in relation to the variables *T, ω* and *λ*, it is found $\nabla h(u)$ as shown in Eq. (29). Substituting the expressions of $\nabla h(u)$, *du* and *h(u)* in Eq. (20), it is found:

$$\begin{bmatrix} H & b & G \\ b & \eta & J \\ AG^T & AJ & F \end{bmatrix} \begin{bmatrix} dT^r \\ d\omega \\ \lambda^o - \lambda \end{bmatrix} = \begin{bmatrix} \sum_\alpha \lambda^\alpha \nabla f\left(\omega * T^{e\alpha} + T\right) \\ \sum_\alpha \left\{ T^{e\alpha} \cdot \left[\lambda^\alpha \nabla f\left(\omega * T^{e\alpha} + T\right)\right]\right\} - 1 \\ F\lambda \end{bmatrix} \tag{30}$$

where:

$$H = \sum_\alpha \lambda^\alpha \nabla^2 f\left(\omega * T^{e\alpha} + T\right) \tag{31}$$

and in the previous expression $\nabla^2 f\left(\omega * T^{e\alpha} + T\right)$ represents the Hessian of the yield function.

$$b = \sum_\alpha \lambda^\alpha \nabla^2 f\left(\omega * T^{e\alpha} + T\right) T^{e\alpha} \tag{32}$$

$$G = \begin{bmatrix} \nabla f_1^1 & \nabla f_1^2 & \ldots & \nabla f_1^m \\ \nabla f_2^1 & \nabla f_2^2 & \ldots & \nabla f_2^m \\ \nabla f_6^1 & \nabla f_6^2 & \ldots & \nabla f_6^m \end{bmatrix} \tag{33}$$

and in the previous expression, $\nabla f\left(\omega * T^{e\alpha} + T\right)$ represents the gradient of the yield function.

$$\eta = \sum_{\alpha} \left[(\lambda^{\alpha} \nabla^2 f\left(\omega^* T^{e\alpha} + T\right) \right] T^{e\alpha} \cdot T^{e\alpha}] \tag{34}$$

$$J = \begin{bmatrix} j^1 & j^2 & \ldots & j^m \end{bmatrix}^T \tag{35}$$

where

$$j^{\alpha} = \nabla f\left(\omega^* T^{e\alpha} + T\right) \cdot T^{e\alpha} \tag{36}$$

$$A = \begin{bmatrix} \lambda^1 & 0 & \ldots & 0 \\ 0 & \lambda^2 & \ldots & 0 \\ . & . & . & . \\ 0 & 0 & \ldots & \lambda^m \end{bmatrix} \tag{37}$$

Eq. (30) originates the following linear equation system

$$\begin{cases} H dT + b d\omega + G\lambda^o = 0 \\ b \cdot dT + \eta d\omega + J \cdot \lambda^{o^o} = 1 \\ AG^T dT + AJ d\omega + F\lambda^o = 0 \end{cases} \tag{38}$$

The previous system can be solved by substitution. For instance, from the first equation of the system it can be attained:

$$dT = -H^{-1}(b d\omega + G\lambda^o) \tag{39}$$

From the second equation of the system it is obtained:

$$d\omega = \frac{1}{\beta}(1 - Z^T \lambda^o) \tag{40}$$

where:

$$\beta = \eta - H^{-1} b \cdot b \tag{41}$$

$$Z^T = J^T - b^T H^{-1} G_f \tag{42}$$

Substituting Eq. (39) and Eq. (40) in the last line of Eq. (38), it is found the following linear equation system:

$$K\lambda^o = Z \tag{43}$$

where:

$$K = Z Z^T + \beta W \tag{44}$$

$$W = G^T H^{-1} G - A^{-1} F \tag{45}$$

Solving the linear equation system presented by Eq. (43), it is found λ^o. From Eq. (40), it is found $d\omega$ and finally from Eq. (39), dT.

5. STRESS CONTRACTION AND INCREMENT REDUCTION

In this stage, it is verified if the increments obtained comply with inequality constraints. If they don't, then a contraction (factor p) of the obtained stress, combined with eventual reductions (factor s) of the increments will take place. To do this, for each vertex α, the values of p^α are calculated, in such a way that the equality below is satisfied:

$$f\{p^\alpha[\omega T^{e\alpha} + T + s(d\omega T^{e\alpha} + dT)]\} = \gamma\, f(\omega T^{e\alpha} + T) \tag{46}$$

where:

γ is a real number, taken initially as being equal to a γ^o , where γ^o is arbitrarily chosen in the interval [0, 1].
s is a factor initially taken as being equal to one.

After the first step, the factor γ can be determined in the following way:

$$\gamma = \min\left[\gamma^{o,}, \frac{d\omega}{\omega}\right] \tag{47}$$

After the values of p^α are obtained, the smaller one is assumed. Thus,

$$p = \min[p^\alpha] \tag{48}$$

Next, ϖ *is* calculated by the formula:

$$\varpi = p \cdot (\omega + s d\omega) \tag{49}$$

If $\varpi < \omega$, then:

$$s = s \cdot s_o \tag{50}$$

where s_o is an arbitrary parameter chosen in the interval [0, 1].

Having the values of s and γ, Eq. (46) is once more used to calculate new values of p^α. Next, Eq. (48) is used to determine the smaller of its values. ϖ is calculated through Eq. (49) and the procedure is repeated until $\varpi \geq \omega$. ω is updated through the expression:

$$\omega \leftarrow \varpi = p \cdot (\omega + s d\omega) \tag{51}$$

The updating of the stress is done through the following recursive formula:

$$T = p \cdot (T^r + s \cdot dT) \tag{52}$$

In the updating of λ it is imposed that all the λ^α be strictly positive, therefore λ^α are taken by the rule:

$$\lambda^\alpha = \max[\lambda^o, \gamma_\lambda \cdot \|\lambda^o\|_\infty] \tag{53}$$

where

$$\gamma_\lambda = \min\left[\gamma^o, \frac{\|dT\|}{\|T\|}\right] \tag{54}$$

6. ALGORITHM FOR THE LOCAL PROBLEM OF ALTERNATE PLASTICITY

1. Initialization and Data Input

$\gamma^o = 0.0001$ $\gamma_L^o = 0.0001$ $s_o = 0.999$

$T^{e\alpha}$

For all α do:

$$\lambda^\alpha = -\frac{1}{f(0)}$$

$$H^\alpha = \lambda^\alpha \nabla^2 f(0)$$

$$b^\alpha = \lambda^\alpha \nabla^2 f(0) T^{e\alpha}$$

End

$$H^{-1} = \left[\sum_\alpha H^\alpha\right]^{-1}$$

$$b = \sum_\alpha b^\alpha$$

For all α find p^α such as:

$$f[p^\alpha (T^{e\alpha} - H^{-1} b)] = \gamma^o f(0)$$

End

$p = \min[p^\alpha]$

$\omega = p$

$T = -pH^{-1} b$

For all α do:

$$\sigma^\alpha = \omega * T^{e\alpha} + T$$

$$\lambda^\alpha = -\frac{1}{f(\sigma^\alpha)}$$

End

2. Increment Estimation

For all α do:

$$H^\alpha = \lambda^\alpha \nabla^2 f(\sigma^\alpha)$$

$$b^\alpha = \lambda^\alpha \nabla^2 f(\sigma^\alpha) T^{e\alpha}$$

$$j^\alpha = \nabla f(\sigma^\alpha) \cdot T^{e\alpha}$$

End

$$H^{-1} = \left[\sum_\alpha H^\alpha\right]^{-1}$$

$$b = \sum_\alpha b^\alpha$$

$$\eta = \sum_\alpha b^\alpha \cdot T^{e\alpha}$$

$$\beta = \eta - H^{-1} b \cdot b$$

$$G = [\nabla f(\sigma^\alpha)]$$

$$A = diag[\lambda^\alpha]$$

$$F = diag[f(\sigma^\alpha)]$$

$$W = G^T H^{-1} G - A^{-1} F$$

$$Z = J - G^T H^{-1} b$$

$$K = Z Z^T + \beta W$$

Solve the system:

$$K\lambda^o = Z$$

$$d\omega = \frac{1}{\beta}(1 - Z^T \lambda^o)$$

$$dT = -H^{-1}(bd\omega + G\lambda^o)$$

3. Admissibility Verification

$s = 1.0$

$\gamma = \min\left[\gamma^{o,}, \dfrac{d\omega}{\omega}\right]$

Repeat until $\varpi \geq \omega$

For all α find p^{α} *such as:*

$$f\{p^{\alpha}[\omega T^{e\alpha} + T + s(d\omega T^{e\alpha} + dT)]\} = \gamma f(\omega T^{e\alpha} + T)$$

End

$p = \min\left[p_f^{\alpha}\right] \forall \alpha$

$\omega = p$

$T = -pH^{-1}b$

For all α do:

$\sigma^{\alpha} = \omega * T^{e\alpha} + T$

$\lambda^{\alpha} = -\dfrac{1}{f(\sigma^{\alpha})}$

End

$p = \min[p^{\alpha}]$

$\varpi = p \cdot (\omega + sd\omega)$

$s = s \cdot s_o$

End

4. Updating

$\omega = \varpi$

$T = p \cdot (T + s \cdot dT)$

$\gamma_L = \min[\gamma_L^o, \dfrac{\|dT\|}{\|T\|}]$

For all α do:

$\lambda^{\alpha} = \max[\lambda^{o\alpha}, \gamma_L \cdot \|\lambda^o\|_{\infty}]$

End

5. Convergence Check

If

$\|G\lambda\| > eps$

$J \cdot \lambda - 1 > eps$

$\|F\lambda\|_{\infty} > eps$

Then

Return to step 2

Else

End

7. EXAMPLES

7.1 Analysis of a tube of dense material subjected to variable pressure and temperature

As a first example, a tube made of a dense metal and subjected to variable pressure and temperature is studied. Let:

$$k = \frac{R_i}{R_o} \tag{55}$$

$$r = \frac{R}{R_o} \tag{56}$$

In the former expression, R represents the radial distance from a point to the axis of the tube.

$$p = \frac{k^2}{1-k^2} \cdot p_i \tag{57}$$

Given that:

E is the Young modulus;
α is the thermal expansion coefficient;
ν is the Poisson ratio;
Θ is the temperature.

and defining:

$$q = \frac{k^2 \cdot E \cdot \alpha}{2(1-k^2)(1-\nu)}\Theta \tag{58}$$

$$\delta = \frac{1-k^2}{2k^2 \cdot \ln k} \tag{59}$$

Then, elastic stresses, according to Gokhfeld and Cherniavsky (1980), can be found through:

$$\sigma_r^e = p \cdot (1-r^2) + -q \cdot (1-r^{-2} + 2\delta \cdot \ln r) \tag{60}$$

$$\sigma_\theta^e = p \cdot (1+r^2) + -q \cdot [1 + r^{-2} + 2\delta \cdot (1+\ln r)] \tag{61}$$

$$\sigma_z^e = p + -2q \cdot [1 + \delta \cdot (1 + 2\ln r)] \tag{62}$$

Varying simultaneously the pressure from 0 to $\overline{p}_i$ and the temperature from 0 to $\overline{\Theta}$, the graph shown in Figure 1 can be found.

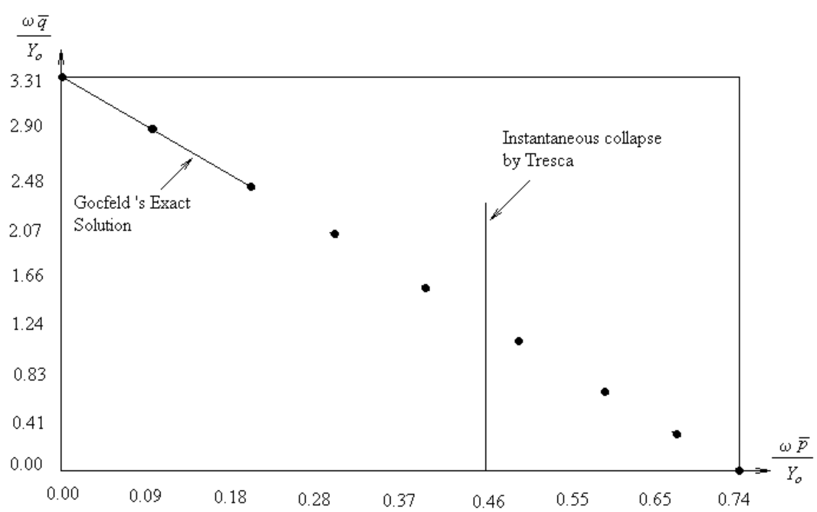

Figure 1: Bree's diagram for the alternating plasticity of the tube.

7.2 Analysis of a bar of porous material subjected to a variable axial compressive load

A bar with a square cross section with side b equal to 10 mm was used in this example simulation. The yield stress of the base material is 156 MPa and the upper value of the axial load N is 5kN. The initial relative density R_i is 0.707. and the yield function used was proposed by Doiravelu (1984).

Four different domains were simulated in the bar analysis as shown in Table 1.

Table 1 - Domain variation for the axial load.

Domain	**Lower Axial Load**	**Upper Axial Load**
I	0.500 *N*	*N*
II	0.625 *N*	*N*
III	0.750 *N*	*N*
IV	0.875 *N*	*N*

The results of the simulations for each domain are presented on the following table:

Table 2: Results of the simulation for the porous Material bar.

Domain	**Lower Axial Load**	**Upper Axial Load**	**Amplification Factor**	**Relative Density**
I	0.500 *N*	*N*	$\omega = 6.23$	$R = 0.75$
II	0.625 *N*	*N*	$\omega = 8.65$	$R = 0.77$
III	0.750 *N*	*N*	$\omega = 14.13$	$R = 0.81$
IV	0.875 *N*	*N*	$\omega = 36.69$	$R = 0.92$

7. CONCLUSIONS

Departing from the static theorem of plastic shakedown presented by Polizzoto, it was developed an algorithm based on Newton's method to find the largest amplification of a load domain applied to a porous material in such a way that it will not suffer a process of alternating plasticity that leads to a collapse by low cycle fatigue.

In a test with a tube made of a dense material and subjected to variable pressure and temperature, the solution found based on proposed algorithm had a perfect coincidence with the exact one presented by Gokhfeld and Cherniavsky.

8. REFERENCES

Polizzotto, C. On the Conditiions to Prevent Plastic Shakedown of Structures: Part I Theory. *Journal of Applied Mechanics*, ASME, vol. 60 15 – 19, 1993.

Polizzotto, C. On the Conditions to Prevent Plastic Shakedown of Structures: Part II The Plastic Shakedown Limit Load. *Journal of Applied Mechanics*, ASME volume 60 20 – 25, 1993.

Doraivelu, S. M. A New Yield Function for Compressible P/M Materials. *International Journal of Mechanical Science*, vol. 26, n.9/10, 527-535, 1984.

Alves, M. M.; Martins, P. A. F.; Rodrigues, J. M., C. A New Yield Function for Porous Materials, *Journal of Materials Processing Technology*, vol.179, p.36-43, 2006.

Heckel, R, W., An Analysis of Powder Compaction Phenomena. Transaction of The Metallurgical Society of AIME, volume 221, 1001-1008, 1961.

Gokhfeld, D.A, and Cherniavsky, O.F., *Limit analysis of structures at thermal cycling*, Sijthoff & Noordhoff, 1980.

Silveira, J. L., Sobre Adaptação Elástica, Adaptação Plástica e Adaptação Incremental. *Tese de Doutorado COPPE UFRJ*, 1996.

Schatz, F. O., Análise de Fadiga de Baixo Ciclo para Materiais com Função de Escoamento dependente da Pressão Hidrostática e da Densidade. *Tese de Mestrado COPPE UFRJ* , 2008.

9. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.