SOLVING ODE SYSTEMS IN GITT IMPLEMENTATIONS FOR THERMALLY DEVELOPING FLOW

H. R. de Queiroz, hildsonqueiroz@petrobras.com.br
Unidade de Operações da Bacia de Campos, Petrobras - Petróleo Brasileiro S.A., Av. Elias Agostinho 665, Ed. Caseli, 3 andar, Macaé, RJ, 27913-350, Brazil.
L. A. Sphaier, lasphaier@id.uff.br
Laboratório de Mecânica Teórica e Aplicada, Programa de Pós-Graduação em Engenharia Mecânica, Universidade Federal Fluminense, Rua Passo da Pátria 156, bloco E, sala 216, Niterói, RJ, 24210-240, Brazil

Abstract. *In the realm of simulation studies for heat transfer in duct-flow applications, this paper proposes a comparison between hybrid solution strategies for solving steady heat transfer problems within channels. The Generalized Integral Transform Technique (GITT) is employed as the main solution methodology; however, different solution approaches are investigated in order to determine advantages and drawbacks of each alternative. The employed solution strategies are focused on the solution of the transformed system rather than on the integral transformation of the problme. The presented results can serve as a guidance for choosing an optimum solution methodology for thermally developing heat transfer using GITT implementations.*

Keywords: *Heat Convection, Generalized Integral Transform Technique,*

NOMENCLATURE

$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{M}$	coefficients matrices
H	duct spacing
I	truncation order
$\boldsymbol{I}$	identity matrix
k	thermal conductivity
L	lentgh scale for thermal entry region
N	norm
Nu	Nusselt number
$\boldsymbol{p}$	auxiliary vector in shooting scheme
Pe	Péclet number
$\dot{q}''$	heat flux
T	temperature
u	axial velocity component
x, y	axial and transversal coordinates
Y	eigenfunctions

Greek Symbols

α	thermal diffusivity
γ	eigenvalue in algebraic problem
η, ξ	dimensionless coordinates
μ	eigenvalue in differential problem
ω	generalized boundary condition parameter
θ	dimensionless temperature

Subscripts

$(\ )_{in}$	inlet
$(\ )_m$	mean stream
$(\ )_{\min}$	minimum
$(\ )_{\max}$	maximum
$(\ )_w$	duct wall

Superscripts

$(\ )^*$	filtered quantity
$(\ )^+$	modified quantity
$(\bar{\ })$	transformed quantity

1. INTRODUCTION

The Generalized Integral Transform Technique (GITT) (Cotta, 1993) has been demonstrated to be a powerful tool for solving a variety of convection-diffusion problems. The technique is based on using orthogonal eigenfunctions expansions for expressing the unknown dependent variables; however, different from the Classical Integral Transform Technique (Mikhailov and Özişik, 1984), the transformation of the original problem needs not lead to a decoupled system, making the method applicable to a large number of problems. The resulting transformed system is usually composed of a set of ODEs, which can be readily solved by well-established numerical routines that enable user-prescribed accuracy control. This, together with the analytical nature of this technique allows for better global error control while compared to traditional domain discretization methods. The main drawback usually associated with the GITT is that a notable amount of analytical work can be required; nevertheless this problem can be circumvented by the usage of symbolical computation (Cotta and Mikhailov, 1997, 2006). Some of the most recent applications of the Generalized Integral Transform Technique include, convective heat transfer in flows within wavy walls (Castellões *et al.*, 2010), hyperbolic heat conduction problems (Monteiro *et al.*, 2009), conjugated conduction-convection problems (Naveira *et al.*, 2009), transient diffusion in heterogeneous media (Naveira-Cotta *et al.*, 2009), heat and mass transfer in adsorption (Hirata *et al.*, 2009), atmospheric pollutant dispersion (Almeida *et al.*, 2008) and dispersion in rivers and channels (de Barros and Cotta, 2007), heat transfer in MHD flow (Lima *et al.*, 2007), applications to irregular geometries (Sphaier and Cotta, 2002), solution of the Navier-Stokes equations (de Lima *et al.*, 2007) and the boundary layer equations (Paz *et al.*, 2007), wind-induced vibrations on overhead conductors (Matt, 2009), among others. A very recent study (Sphaier *et al.*, 2011) was aimed at developing a unified solution algorithm for solving a general transient problem, facilitating the implementation of the GITT and allowing a greater number of user to readily employ the technique. The purpose of this paper is to compare the performance of different solution strategies using the GITT. The problem of steady thermally developing flow within parallel plates, with and without axial diffusion, is selected for the comparison analysis. The performance of the differently employed strategies is compared by analyzing the convergence rate with the gradually increasing truncation order. In addition, a comparison of CPU times is presented for demonstrating the amount of computational resources consumed by each strategy.

2. CONVECTION-DIFFUSION PROBLEM

In order to illustrate the proposed methodology, a general problem of flow within parallel plates is considered, which written in dimensionless form is given by:

$$\frac{1}{2}\,u^*\,\frac{\partial\theta}{\partial\xi} = \mathrm{Pe}_H^{-2}\,\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2}, \qquad \text{in} \quad 0 \le \xi \le 1, \quad 0 \le \eta \le 1, \tag{1a}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = 0, \qquad (1-\omega)\,\theta(\xi,1) + \omega\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} = 1, \qquad \theta(0,\eta) = 0, \qquad \left(\frac{\partial\theta}{\partial\xi}\right)_{\xi\to\infty} = 2\,\omega, \tag{1b}$$

where ω is a parameter that indicates the type of boundary condition at the solid wall. For constant wall temperature one finds that $\omega = 0$,whereas for constant wall heat flux $\omega = 1$. The velocity is given by the Hagen-Poiseuille profile:

$$u^* = \frac{3}{2}\left(1-\eta^2\right), \qquad \text{with} \qquad \eta = 2\,y/H, \tag{2}$$

and the employed dimensionless parameters and variables are defined as:

$$\mathrm{Pe}_H = \frac{u_{av}\,H}{\alpha}, \qquad \eta = \frac{y}{H/2}, \qquad \xi = \frac{x}{L}, \qquad L = \frac{H}{2}\,\mathrm{Pe}_H, \qquad \theta = \frac{T - T_{\min}}{T_{\max} - T_{\min}}, \tag{3a}$$

For constant wall heat flux, $T_{\max}$ and $T_{\min}$ are defined as

$$T_{\max} = T_{in} + \frac{(H/2)}{k}\,\dot{q}''_w, \qquad T_{\min} = T_{in}, \tag{3b}$$

whereas for constant wall temperature (isothermal wall), these are defined as:

$$T_{\max} = T_w, \qquad T_{\min} = T_{in}, \tag{3c}$$

Once the dimensionless temperature is calculated, the Nusselt number is computed from:

$$\mathrm{Nu} = \mathrm{Nu}(\xi) = \frac{4}{\theta_w - \theta_m}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}, \qquad \text{with} \qquad \theta_w = \theta(\xi,0), \qquad \theta_m = \int_0^1 u^*\,\theta\,\mathrm{d}\eta. \tag{4}$$

3. Integral transformation

Since the η-direction is non-homogeneous a filter is proposed, based on the following solution separation:

$$\theta(\xi,\eta) = \theta_F(\eta) + \theta^*(\xi,\eta) \tag{5}$$

The selected filter problem that removes the non-homogeneous terms in the transversal direction is given by:

$$\frac{\mathrm{d}^2\theta_F}{\mathrm{d}\eta^2} = \omega, \qquad \theta'_F(0) = 0, \qquad (1-\omega)\,\theta_F(1) + \omega\,\theta'_F(1) = 1, \tag{6a}$$

which yields (generalized solution valid for $\omega = 1$ or $\omega = 0$):

$$\theta_F(\eta) = \frac{1}{2}\,\omega\,\eta^2 + (1-\omega) \tag{7}$$

With the previously proposed filter, the following filtered problem is obtained:

$$\frac{1}{2}\,u^*\,\frac{\partial\theta^*}{\partial\xi} = \mathrm{Pe}_H^{-2}\,\frac{\partial^2\theta^*}{\partial\xi^2} + \frac{\partial^2\theta^*}{\partial\eta^2} + \omega, \qquad \text{in} \quad 0 \le \xi \le 1, \quad 0 \le \eta \le 1, \tag{8a}$$

$$\left(\frac{\partial\theta^*}{\partial\eta}\right)_{\eta=0} = 0, \qquad (1-\omega)\,\theta^*(\xi,1) + \omega\left(\frac{\partial\theta^*}{\partial\eta}\right)_{\eta=1} = 0, \tag{8b}$$

$$\theta^*(0,\eta) = -\theta_F(\eta), \qquad \left(\frac{\partial\theta^*}{\partial\xi}\right)_{\xi\to\infty} = 2\,\omega, \tag{8c}$$

The solution of the filtered problem is obtained by applying the Generalized Integral Transform Technique (Cotta, 1993), based on the following integral-transform pair:

$$\text{Transform} \quad \Longrightarrow \quad \bar{\theta^*}_n(\xi) = \int_0^1 \theta^*(\xi,\,\eta)\,Y_n(\eta)\,\mathrm{d}\eta, \tag{9a}$$

$$\text{Inversion} \quad \Longrightarrow \quad \theta^*(\xi,\,\eta) = \sum_{n=1}^{\infty} \frac{\bar{\theta^*}_n(\xi)\,Y_n(\eta)}{N(\mu_n)}, \tag{9b}$$

where $N(\mu_n)$ is the norm of an eigenfunction Y associated with an eigenvalue μ_n. A commonly adopted solution strategy consists of employing a simple eigenvalue problem based on Helmholtz equation:

$$Y_n''(\eta) + \mu_n^2 Y_n(\eta) = 0 \quad \text{for} \quad 0 \le \eta \le 1, \qquad Y_n'(0) = 0, \qquad (1-\omega)\,Y_n(1) + \omega\,Y_n'(1) = 0. \tag{10}$$

For $\omega = 0$ (constant wall temperature) the previous problem leads to the following solutions:

$$Y_n(\eta) = \cos(\mu_n\,\eta), \qquad \text{with} \qquad \mu_n = \left(n - \frac{1}{2}\right)\pi, \qquad \text{for} \qquad n = 1, 2, 3, \ldots \tag{11a}$$

and the norms of the eigenfunctions are given by:

$$N(\mu_n) = \int_0^1 Y_n^2(\eta)\,\mathrm{d}\eta = \frac{1}{2}. \tag{11b}$$

Alternatively, for $\omega = 1$ the eigenvalue problem leads to the following solutions:

$$Y_n(\eta) = \cos(\mu_n\,\eta), \qquad \text{with} \qquad \mu_n = n\,\pi, \qquad \text{for} \qquad n = 0, 1, 2, 3, \ldots \tag{12a}$$

and the norms are given by:

$$N(\mu_n) = \int_0^1 Y_n^2(\eta)\,\mathrm{d}\eta = \begin{cases} 1, & \text{for} \quad n = 0, \\ 1/2, & \text{for} \quad n > 0, \end{cases} \tag{12b}$$

The integral transformation of the filtered problem is performed by operating the filtered equations with the integral transform operator, leading to:

$$\mathrm{Pe}_H^{-2}\,\bar{\theta^*}''_n(\xi) - \frac{1}{2}\sum_{m=0}^{\infty} A_{n,m}\,\bar{\theta^*}'_m(\xi) - \mu_n^2\bar{\theta^*}_n(\xi) = -\bar{\omega}_n, \tag{13a}$$

$$\bar{\theta^*}_n(0) = \bar{b}_n = -\int_0^1 \theta_F(\eta)\,Y_n(\eta)\,\mathrm{d}\eta, \qquad \lim_{\xi\to\infty} \bar{\theta^*}'_n(\xi) = 2\,\bar{\omega}_n, \tag{13b}$$

for $n = 0, 1, \ldots, \infty$, in which the $A_{n,m}$ and $\bar{\omega}_n$ coefficients are given by

$$A_{n,m} = \frac{1}{N(\mu_m)} \int_0^1 u^*(\eta)\,Y_n(\eta)\,Y_m(\eta)\,\mathrm{d}\eta, \qquad \bar{\omega}_n = \omega \int_0^1 Y_n(\eta)\,\mathrm{d}\eta. \tag{14}$$

This system is a general form, valid for both constant wall heat flux and isothermal wall; nevertheless, for the isothermal wall situation $Y_0(\eta) = 0$ and there is no need to calculate $\bar{\theta^*}_0(\xi)$, such that the system (13a)-(13b) (and corresponding summations) is modified to start from $i = 1$ instead of $i = 0$. Once the solution of the transformed system (13a)-(13b) is accomplished, the dimensionless temperature profile can be directly obtained using the inversion formula (9b), and the Nusselt number is readily computed from equation (4).

4. SOLUTIONS STRATEGIES FOR TRANSFORMED SYSTEMS

This section presents different solution approaches for the transformed systems obtained in the previous sections.

4.1 Numerical integration of transformed system

The first step towards solving the infinite system of equations given by the integral transformation is to reduce it to a finite representation by truncation to a limited number of terms (denoted the truncation order), I. After truncation, the infinite representation given by equations (13a)-(13b) can be written in vectorial form:

$$\bar{\boldsymbol{\theta}}^{*\prime\prime}(\xi) - \boldsymbol{B}\,\bar{\boldsymbol{\theta}}^{*\prime}(\xi) - \boldsymbol{D}\,\bar{\boldsymbol{\theta}}^{*}(\xi) = -\mathrm{Pe}_H^2\,\bar{\boldsymbol{\omega}}, \qquad \bar{\boldsymbol{\theta}}^{*}(0) = \bar{\boldsymbol{b}}, \qquad \bar{\boldsymbol{\theta}}^{*\prime}(\xi_{\max}) = 2\,\bar{\boldsymbol{\omega}}, \tag{15}$$

in which the coefficients of $\boldsymbol{b}$ are given by eq. (13b) and matrices $\boldsymbol{B}$ and $\boldsymbol{D}$ are given by:

$$B_{n,m} = \frac{1}{2}\,\mathrm{Pe}_H^2\,A_{n,m}, \qquad D_{n,m} = \mathrm{Pe}_H^2\,\mu_n^2\,\delta_{n,m}, \tag{16}$$

where $\delta_{n,m}$ is the Kronecker delta. In addition to truncating the infinite representation given by equation (13a), the boundary condition at $\xi \to \infty$, was also replaced by a finite value $\xi_{\max}$. Numerically, this value must be sufficiently large such that the solution for $\xi \leq \xi_{\max}$ is independent of $\xi_{\max}$.

A traditionally employed approach for solving the presented transformed system is via direct numerical integration using a commercially or publicly available ODE system solver. Naturally, an ODE solver capable of solving boundary value problems is required due to the nature of the given boundary conditions. For this work, the general-purpose ODE solver **NDSolve**, offered by the *Mathematica* system was employed.

The complexity of the problem is significantly reduced once the effects of axial diffusion are unimportant, which is generally considered for large Péclet numbers. When this consideration comes into play, a simplified first-order form of system (13a)-(13b) is obtained:

$$\frac{1}{2}\sum_{m=0}^{\infty} A_{n,m}\,\bar{\theta^*}'_m(\xi) + \mu_n^2\bar{\theta^*}_n(\xi) = \bar{\omega}_n, \qquad \bar{\theta^*}_n(0) = \bar{b}_n. \tag{17}$$

for $n = 0, 1, 2 \ldots$. After truncation to a finite order I, equations (17) can be written in the following vector form:

$$\boldsymbol{A}\,\bar{\boldsymbol{\theta}}^{*\prime}(\xi) + \boldsymbol{D}^+\,\bar{\boldsymbol{\theta}}^*(\xi) = 2\,\bar{\boldsymbol{\omega}}, \qquad \bar{\boldsymbol{\theta}}^*(0) = \bar{\boldsymbol{b}}, \tag{18}$$

in which $\boldsymbol{A}$ is given by the integral coefficients in equation (14) and $\boldsymbol{D}^+$ is given by:

$$D^+_{n,m} = 2\,\mu_n^2\,\delta_{n,m}. \tag{19}$$

A traditional way of solving system (18) is, again, by directly employing an ODE solver, as **NDSolve**. Nevertheless, the coupling in the derivatives term, will require an implicit solving routine, which may be more involved from a numerical standpoint. As an attempt to circumvent this obstacle, an alternative form of this system is proposed by employing the inversion of matrix $\boldsymbol{A}$:

$$\bar{\boldsymbol{\theta}}^{*\prime}(\xi) = \boldsymbol{M}\,\bar{\boldsymbol{\theta}}^*(\xi) + \boldsymbol{g}, \qquad \text{where} \qquad \boldsymbol{g} = 2\,\boldsymbol{A}^{-1}\,\bar{\boldsymbol{\omega}}, \qquad \boldsymbol{M} = -\boldsymbol{A}^{-1}\,\boldsymbol{D}^+. \tag{20}$$

The numerical solution of (20) will generally require less computational resources than solving (18). On the other hand, one needs to numerically invert matrix $\boldsymbol{A}$ for arriving at equation (20), which could also be time-consuming. In order to determine the best alternative, both numerical solutions will be analyzed and properly compared.

4.2 Analytical solutions for no axial diffusion

After discussing purely numerical solutions for transformed systems, this and the following section present analytical and semi-analytical approaches for solving the coupled system of equations. Due to the linear character of the problem solutions involving analytical integration of the ODE system can be achieved. The solution of the simplified case without axial diffusion is first presented, because of its simpler form.

4.2.1 Isothermal wall

For constant wall temperature, $\boldsymbol{g} = \bar{\boldsymbol{\omega}} = \boldsymbol{0}$ and hence equation (20) leads to a homogeneous form, which, together with the inlet condition in eq. (18), admits the simple closed-form analytical solution:

$$\bar{\boldsymbol{\theta}}^*(\xi) = \boldsymbol{C}\,\bar{\boldsymbol{b}}, \qquad \text{with} \qquad \boldsymbol{C} = \exp\left(\boldsymbol{M}\,\xi\right), \tag{21}$$

where $\boldsymbol{C}$ is a matrix exponential (Greenberg, 1998). Although a closed-form analytical solution is obtained, this solution approach requires the numerical inversion of matrix $\boldsymbol{A}$ and a numerical evaluation of a matrix exponential, which might be more time-consuming than the direct numerical solution of the ODE system.

4.2.2 Constant wall heat flux

For the constant wall heat-flux solution, at first, one could attempt a modified solution by including a particular solution to equation (21), in the form

$$\bar{\boldsymbol{\theta}}^*(\xi) = \boldsymbol{C}\,(\bar{\boldsymbol{b}} - \boldsymbol{x}) + \boldsymbol{x}, \qquad \text{com} \qquad \boldsymbol{x} = \boldsymbol{M}^{-1}\,\boldsymbol{g}; \tag{22}$$

however, this formula cannot be applied because the $\boldsymbol{M}$ matrix is not invertible, since it has the following structure

$$\boldsymbol{M} = \begin{pmatrix} 0 & M_{0,1} & \dots & M_{0,I} \\ 0 & M_{1,1} & \dots & M_{1,I} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & M_{I,1} & \dots & M_{I,I} \end{pmatrix} \tag{23}$$

due to the zero eigenvalue present in the matrix $\boldsymbol{D}^+$. Another way of viewing this problem is that the constant vector $\boldsymbol{x}$ cannot be used as a particular solution to equation (20) because $\lambda = 0$ already produces a constant vector as a solution.

In oder to obtain a correct solution for the constant wall heat flux case, the following change of variable is proposed:

$$\bar{\boldsymbol{\theta}}^*(\xi) = \boldsymbol{Q}\,\boldsymbol{z}(\xi). \tag{24}$$

Substituting equation (24) in equations (20) and rearranging yields:

$$\boldsymbol{z}'(\xi) = (\boldsymbol{Q}^{-1}\,\boldsymbol{M}\,\boldsymbol{Q})\,\boldsymbol{z}(\xi) + \boldsymbol{h}, \qquad \boldsymbol{z}(0) = \boldsymbol{a}, \qquad \text{where} \qquad \boldsymbol{h} = \boldsymbol{Q}^{-1}\,\boldsymbol{g}, \qquad \boldsymbol{a} = \boldsymbol{Q}^{-1}\,\bar{\boldsymbol{b}}. \tag{25}$$

Once the matrix $\boldsymbol{Q}$ is constructed by using the eigenvectors of $\boldsymbol{M}$ as columns, the following decoupled system is obtained:

$$z_i'(\xi) = \gamma_i\,z_i(\xi) + h_i, \tag{26}$$

where γ_is are the eigenvalues of $\boldsymbol{M}$. Since the only null eigenvalue is γ_0, the following solutions are obtained:

$$z_i(\xi) = (a_i - x_i)\,\exp(\gamma_i\,\xi) + y_i\,\xi + x_i, \qquad \text{for} \qquad i \geq 0, \tag{27}$$

in which the coefficients x_i and y_i are given by:

$$y_0 = h_0 \qquad x_0 = 0, \qquad y_i = 0, \qquad x_i = -\frac{h_i}{\gamma_i}, \qquad \text{for} \qquad i > 0. \tag{28}$$

Finally, by employing equation (24), the following solution for the transformed potentials is obtained:

$$\theta_i^* = \sum_{j=0}^{I} Q_{i,j}\,[(a_j - x_j)\,\exp(\gamma_j\,\xi) + y_j\,\xi + x_j] \tag{29}$$

Although another closed-form analytical solution is obtained, its computational implementation involves the numerical evaluation of an inverse matrix and the numerical calculation of eigenvalues and eigenvectors.

4.3 Solution with analytical integration and numerical shooting scheme

A solution for the complete problem, as given by system (15), using analytical integration can also be obtained. The proposed solution is based on converting this system to a first order initial-value problem by replacing the boundary condition at $\xi_{\max}$ by an initial condition and introducing a new dependent variable vector $\bar{\boldsymbol{\phi}}$:

$$\bar{\boldsymbol{\theta}}^{*\prime}(0) = \boldsymbol{p}, \qquad \bar{\boldsymbol{\theta}}^{*\prime}(\xi) = \bar{\boldsymbol{\phi}}(\xi), \tag{30}$$

which substituted in equation (15) yields:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\frac{\bar{\boldsymbol{\theta}}^*}{\bar{\boldsymbol{\phi}}}\right\} = \left(\begin{array}{c|c} \boldsymbol{0} & \boldsymbol{I} \\ \hline \boldsymbol{D} & \boldsymbol{B} \end{array}\right)\left\{\frac{\bar{\boldsymbol{\theta}}^*}{\bar{\boldsymbol{\phi}}}\right\} - \mathrm{Pe}_H^2\left\{\frac{\boldsymbol{0}}{\bar{\boldsymbol{\omega}}}\right\} \tag{31}$$

where $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{0}$ is a zero matrix. In this form analytical solutions for the transformed potentials can be obtained as previously described for the simpler cases without axial diffusion.

4.3.1 Isothermal wall

For a constant wall temperature, equation (31) is simplified since $\bar{\boldsymbol{\omega}} = \boldsymbol{0}$, leading to the following solution

$$\left\{\frac{\bar{\boldsymbol{\theta}}^*}{\bar{\boldsymbol{\phi}}}\right\} = \boldsymbol{C}^+\left\{\frac{\bar{\boldsymbol{b}}}{\boldsymbol{p}}\right\}, \tag{32}$$

in which

$$\boldsymbol{C}^+ = \exp\left(\boldsymbol{M}^+\,\xi\right), \qquad \text{with} \qquad \boldsymbol{M}^+ = \left(\begin{array}{c|c} \boldsymbol{0} & \boldsymbol{I} \\ \hline \boldsymbol{D} & \boldsymbol{B} \end{array}\right). \tag{33}$$

where, naturally, all matrices are generated from $m = 1$ and $n = 1$ since there is no non-trivial solution for the zero eigenvalue. Although the obtained solution is obtained via analytical integration, the calculation of the vector $\boldsymbol{b}$ requires the employment of a numerical shooting method. In this work, the **FindRoot** function was used for iteratively calculating the appropriate value of $\boldsymbol{p}$ that will satisfy the boundary condition at $\xi = \xi_{\max}$ given in equations (15).

4.3.2 Constant wall heat flux

When constant heat flux is considered, the same problem generated by the $\mu = 0$ eigenvalue for the case without axial diffusion occurs. As a result, a solution a procedure similar to that described in section 4.2.2must be employed. The solution is initiated by defining a vector $\boldsymbol{w}$:

$$\boldsymbol{w} = \left\{ \frac{\bar{\boldsymbol{\theta}}*}{\bar{\boldsymbol{\phi}}} \right\}, \tag{34}$$

allowing equation (31), and the associated boundary conditions, to be written in the following form:

$$\boldsymbol{w}'(\xi) = \boldsymbol{M}^+ \boldsymbol{w}(\xi) + \boldsymbol{g}^+, \tag{35a}$$

$$\boldsymbol{w}(0) = \boldsymbol{b}^+ \tag{35b}$$

where

$$\boldsymbol{g}^+ = -\mathrm{Pe}_H^2 \left\{ \frac{\boldsymbol{0}}{\bar{\boldsymbol{\omega}}} \right\}, \qquad \boldsymbol{b}^+ = \left\{ \frac{\bar{\boldsymbol{b}}}{\boldsymbol{p}} \right\}, \tag{36}$$

Once the system is expressed in the above form, the same solution method described for the no axial diffusion case is employed, such that the solution to equations (35a)-(35b) is given by the following expression:

$$w_i = \sum_{j=0}^{\infty} Q_{i,j}^+ \left[\left(a_j^+ - x_j^+ \right) \exp(\gamma_j^+ \xi) + y_j^+ \xi + x_j^+ \right], \tag{37}$$

in which $Q_{i,j}^+$ are the coefficients of a matrix having the eigenvectors of $\boldsymbol{M}^+$ as columns and γ_i^+ are the associated eigenvalues. In addition, the coefficients x_i^+ and y_i^+ are given by:

$$y_0^+ = h_0^+ \qquad x_0^+ = 0, \qquad y_i^+ = 0, \qquad x_i^+ = -\frac{h_i^+}{\gamma_i^+}, \qquad \text{for} \qquad i > 0. \tag{38}$$

where the coefficients h_i and a_i are components of the vectors below:

$$\boldsymbol{h}^+ = (\boldsymbol{Q}^+)^{-1} \boldsymbol{g}^+, \qquad \boldsymbol{a}^+ = (\boldsymbol{Q}^+)^{-1} \boldsymbol{b}^+. \tag{39}$$

5. RESULTS AND DISCUSSION

Table 1 presents the results for the Nusselt number convergence with the isothermal wall condition and no axial diffusion. The calculated values of Nu are the same, regardless the employed strategy for solving the transformed system;

Table 1. Nusselt number for isothermal wall and no axial diffusion.

I	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
10	25.2425	12.0299	7.63289	7.54128
20	24.7301	12.0165	7.63224	7.54077
30	24.7006	12.0151	7.63218	7.54072
40	24.6934	12.0147	7.63216	7.54071
50	24.6909	12.0146	7.63216	7.54071
60	24.6898	12.0146	7.63215	7.54070
100	24.6885	12.0145	7.63215	7.54070
200	24.6883	12.0145	7.63215	7.54070
300	24.6883	12.0145	7.63215	7.54070
400	24.6882	12.0145	7.63215	7.54070
500	24.6882	12.0145	7.63215	7.54070

hence, a single convergence table for this condition is presented. The results show that the convergence rate is worse for positions near the channel entrance, as expected for GITT solutions due to the proximity to the discontinuous inlet condition. For $\xi = 10^{-3}$ as most as 400 terms are required for six-digit convergence, whereas in the vicinity of the fully developed region ($\xi = 1$) 60 terms are sufficient for the same precision.

Table 2 presents a comparison between the amount of CPU time spent for calculating the solution (in seconds) for the differently employed solution strategies. As can be seen, the direct numerical solution without matrix inversion (as given by eq. (18)) requires much more time than the other approaches, to point of being prohibitive with for more than 10 terms. For relatively lower truncation orders (lower than 20 terms) the analytical integration option (eq. (21)) is clearly the best option. This solution is competitive up to about 50 terms; over this limit the numerical solution given by equation (20) is the best option. The last column in this table provides the CPU time spent for a fully analytical solution of a slug-flow (uniform velocity profile) version of the problem, for comparison purposes.

Table 2. CPU time for no axial diffusion with isothermal wall.

	Numerical Integration		Analytical Integration	
I	eq. (18)	eq. (20)	eq. (21)	slug-flow
2	4.5×10^{-3}	2.3×10^{-3}	1.4×10^{-3}	7.0×10^{-4}
4	2.8×10^{-2}	5.7×10^{-3}	2.4×10^{-3}	1.0×10^{-3}
6	1.5×10^{-1}	1.0×10^{-2}	4.0×10^{-3}	1.3×10^{-3}
8	80.2	1.5×10^{-2}	6.2×10^{-3}	1.8×10^{-3}
10	–	2.0×10^{-2}	1.2×10^{-2}	2.1×10^{-3}
20	–	5.5×10^{-2}	5.9×10^{-2}	3.7×10^{-3}
40	–	2.0×10^{-1}	5.7×10^{-1}	7.2×10^{-3}
60	–	5.8×10^{-1}	2.1	1.1×10^{-2}
80	–	1.8	7.0	1.6×10^{-2}
100	–	2.9	15.8	1.8×10^{-2}

As similarly presented for the isothermal wall heating condition, tables 3 and 4 illustrat the convergence behavior and the CPU time for the constant wall-flux solution with no axial diffusion, respectively. As seen for the isothermal wall case, better convergence rates are obtained for positions upstream, with 80 terms providing a six-digit converged solution for $\xi = 1$ and 300 terms providing the same precision at $\xi = 10^{-3}$.

Table 3. Nusselt number for uniform wall flux and no axial diffusion.

I	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
10	29.9970	14.4399	8.57417	8.23559
20	29.9085	14.4327	8.57375	8.23533
40	29.8881	14.4317	8.57370	8.23530
60	29.8860	14.4316	8.57369	8.23530
80	29.8854	14.4316	8.57369	8.23529
100	29.8852	14.4316	8.57369	8.23529
200	29.8851	14.4316	8.57369	8.23529
300	29.8850	14.4316	8.57369	8.23529
400	29.8850	14.4316	8.57369	8.23529

Table 4. CPU time for no axial diffusion with uniform wall flux.

	Numerical Integration		Analytical Integration	
I	eq. (18)	eq. (20)	eq. (29)	slug-flow
2	1.4×10^{-2}	3.4×10^{-3}	1.9×10^{-3}	8.6×10^{-4}
4	4.7×10^{-2}	7.3×10^{-3}	3.0×10^{-3}	1.2×10^{-3}
6	1.2	1.0×10^{-2}	4.5×10^{-3}	1.7×10^{-3}
8	78	1.4×10^{-2}	6.7×10^{-3}	2.0×10^{-3}
10	–	2.0×10^{-2}	1.1×10^{-2}	2.5×10^{-3}
20	–	5.4×10^{-2}	5.6×10^{-2}	4.7×10^{-3}
40	–	1.8×10^{-1}	5.5×10^{-1}	8.5×10^{-3}
60	–	3.6×10^{-1}	2.3	1.3×10^{-2}
80	–	7.3×10^{-1}	6.9	1.7×10^{-2}
100	–	1.9	16.3	2.2×10^{-2}

The next tables present the results of computational simulations including the axial diffusion effect. The first portion of table 5 presents the results calculated with the numerical solution of the system given by equations (15) for an isothermal wall and $\mathrm{Pe}_H = 10$. This solution was only calculated up to 10 terms since higher truncation orders became prohibitive using the *Mathematica* routine **NDSolve** due to an extremely high computational cost. An important factor that contributes to the high computational cost is the elevated working precision (denoted as WP) required. This is the actual number of digits used in floating-point operations necessary to ensure that the employed ODE solver completed the solution algorithm in a stable way. This occurs due to the more complex nature of the problem, which now is of second order, and is a boundary value problem (whose solution is more involved than an initial value one). In addition, the case with $\mathrm{Pe}_H = 10$ possess a significant stiffness.

Table 5. Nusselt number for isothermal wall with $\mathrm{Pe} = 10$.

	I	WP	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
numerical integration	1	33	8.11742	8.11742	8.11742	8.11742
	2	42	15.8436	14.2828	8.38243	7.75794
	3	59	23.3481	18.5103	8.19601	7.74381
	4	85	30.6041	21.4518	8.16546	7.74123
	5	110	37.6244	23.5640	8.15701	7.74044
	6	140	44.4207	25.1411	8.15349	7.74014
	7	190	51.0030	26.6099	8.15174	7.74000
	8	240	57.3797	28.0810	8.15077	7.73993
	9	290	63.5599	30.1564	8.15019	7.73989
analytical integration	10	200	69.5288	27.7774	8.14983	7.73986
	20	300	120.003	28.7805	8.14897	7.73982
	40	600	183.733	28.8168	8.14887	7.73982
	60	800	217.714	28.8162	8.14886	7.73982
	80	1100	235.837	28.8160	8.14886	7.73982
	100	1400	245.504	28.8160	8.14886	7.73982

In order to circumvent the problem encountered with the numerical solution, the alternative strategy that uses analytical integration and a numerical shooting scheme for calculating the additional inlet condition (as given by equation (32)). The results are presented in the second portion of table 5. As can be seen, with this solution algorithm a smaller floating-point precision is required when compared to the previous solution scheme. However, when compared to a traditional 16-digit precision, a large number of digits are still necessary. This occurs because the matrix exponential calculation is quite elaborate from a computational standpoint, especially for this stiff case with $\mathrm{Pe}_H = 10$.

When looking into the convergence behavior with the axial position, one notices that the same behavior seen for the no axial-diffusion case is repeated here, with the solution convergence rate being much worse for positions near the channel entrance. Twenty terms yield a six-digit converged solution at $\xi = 1$, whereas 80 are required for the same convergence at $\xi = 10^{-2}$. At $\xi = 10^{-3}$, 100 terms are hardly enough for a one-digit convergence.

The next table (tab. 6) presents similar results calculated for $\mathrm{Pe}_H = 1$ also with an isothermal wall condition. Again,

Table 6. Nusselt number for isothermal wall with $\mathrm{Pe} = 1$.

	I	WP	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
numerical integration	1	16	8.11742	8.11742	8.11742	8.11742
	2	16	16.0111	15.7825	13.8134	8.45040
	3	16	23.9401	23.2625	17.9676	8.45242
	4	16	31.8560	30.5201	20.9937	8.45088
	5	16	39.7508	37.5555	23.1994	8.45041
	6	60	47.6223	44.3740	24.8078	8.45025
	7	70	55.4698	50.9819	25.9811	8.45018
	8	110	63.2931	57.3857	26.8370	8.45015
	9	180	71.0920	63.5919	27.4615	8.45013
analytical integration	10	100	78.8666	69.6035	27.9173	8.45012
	20	100	155.286	120.397	29.0950	8.45011
	40	100	301.107	184.589	29.1472	8.45010
	60	100	438.048	218.833	29.1472	8.45010
	80	200	566.649	237.101	29.1472	8.45010
	100	200	687.418	246.847	29.1472	8.45010
	200	300	1189.50	257.510	29.1472	8.45010
	300	500	1556.23	257.970	29.1472	8.45010

the same behavior in which the convergence rate improves as the position is further upstream is seen here. However, when compared to the $\mathrm{Pe}_H = 10$ case, a worse convergence rate is seen; that is, more terms are necessary for obtaining the same precision at the same positions. On the other hand, the problem stiffness is reduced for $\mathrm{Pe}_H = 1$ and a smaller precision is required for solving the transformed system, both with numerical or analytical integration. Nevertheless, the numerical integration – as seen for the larger Péclet solution – requires a greater numerical precision (WP) than its analytical counterpart, such that for a system of more than 10 equations the numerical integration routine becomes extremely time-consuming.

The last two tables present the calculated Nusselt number for the uniform wall heat flux condition. Table 7 presents results of the analytical and numerical integration routines for $\mathrm{Pe}_H = 10$. As one can observe is this table, the same trends previously seen for the isothermal wall condition are seen again here. This is corroborated by table 8, which shows

Table 7. Nusselt number for constant wall flux with $\mathrm{Pe} = 10$.

	I	WP	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
numerical integration	1	50	28.2280	20.9677	9.52822	8.32780
	2	60	43.8673	23.7975	9.29037	8.25660
	3	60	56.8196	24.6976	9.26217	8.24322
	4	80	67.5323	25.0673	9.25414	8.23906
	5	100	76.4172	25.2381	9.25092	8.23737
	6	150	83.8209	25.3502	9.24938	8.23655
	7	150	90.0227	25.3841	9.24855	8.23612
	8	180	95.2505	25.4257	9.24807	8.23586
	9	220	99.6870	25.5359	9.24777	8.23570
analytical integration	10	160	103.448	25.3899	9.24758	8.23559
	20	290	122.225	25.3933	9.24710	8.23533
	30	430	128.001	25.3919	9.24705	8.23531
	40	570	130.266	25.3915	9.24703	8.23530
	50	700	131.287	25.3913	9.24703	8.23530

the Nusselt results using both integration routines for $\mathrm{Pe}_H = 1$. Again, a higher working precision is required by the stiffer large Péclet case, whereas a slower convergence rate is seen for the smaller Péclet case. Comparing the results with

Table 8. Nusselt number for constant wall flux with $\mathrm{Pe} = 1$.

	I	WP	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
numerical integration	1	16	29.2474	27.4616	17.9558	8.97248
	2	16	48.4083	42.1910	20.3740	8.88656
	3	16	67.1320	54.2479	21.2754	8.87046
	4	18	85.3885	64.0991	21.6648	8.86552
	5	18	103.169	72.1785	21.8503	8.86352
	6	22	120.473	78.8449	21.9450	8.86256
	7	100	137.302	84.3836	21.9957	8.86205
	8	100	153.662	89.0177	22.0239	8.86175
	9	200	169.559	92.9226	22.0400	8.86156
analytical integration	10	16	185.003	96.2312	22.0494	8.86144
	20	32	316.899	112.460	22.0632	8.86114
	30	42	415.600	117.372	22.0631	8.86111
	40	60	490.490	119.288	22.0630	8.86110
	50	70	548.298	120.150	22.0630	8.86110

the two different heating conditions, a small difference in the convergence rate near the fully developed region ($\xi = 1$) is seen, with the uniform temperature case presenting a slightly better convergence rate.

6. CONCLUSIONS

This work presented a comparative analysis of different solution strategies for steady thermally-developing flow using the Generalized Integral Transform Technique (GITT). Both an isothermal wall and a constant heat flux case were considered, as wall heating conditions and different values of the Péclet number were analyzed.

The simpler case without axial diffusion was shown to result in first-order ODE systems after transformation, behaving initial-value problems, whose solution is simpler than that for boundary value problems (which are encountered in the presence of axial diffusion). Three different solutions were implemented for the case without axial diffusion using closed-form analytical solutions in terms of matrix exponentials and fully numerical algorithms. The numerical algorithm that removed the coupling in the first-order term prior to integration resulted, in overall, in a smaller CPU time. The analytical solution was only faster than this numerical algorithm for very low truncation orders.

For the cases with axial diffusion, boundary value ODE systems were obtained after integral transformation. These were solved by two different solution approaches: a fully numerical solution and a solution with analytical integration and a numerical shooting scheme required to convert the BVP into an IVP form. The fully numerical scheme was demonstrated to be ineffective for more than 10 terms due to a high CPU time and the large working precision required for the solution. For these cases, the analytical integration solution with numerical shooting was more effective. Nevertheless, as the truncation order increases a higher numerical precision is always required, for both computational algorithms.

The results herein presented demonstrate that for cases where axial diffusion must be considered, the solution strategy with analytical integration is superior to the direct numerical solution using the **NDSolve** function. As final commentaries, one must mention that the data presented in this work not only serve to illustrate which solution strategy is superior for each in case, but also serve as a motivation for the development of different solutions approaches for a variety of problems, using the Generalized Integral Transform Technique.

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