TRANSIENT HEAT DIFFUSION PROBLEMS WITH VARIABLES THERMAL PROPERTIES BY GENERALIZED INTEGRAL TRANSFORM TECHNIQUE

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Abstract. *In this work a study of the two-dimensional transient heat diffusion problems in domains of rectangular and elliptical geometries, submitted to boundary conditions of first kind, is carried out. For the problem formulation, the diffusive means were considered with variable thermo physical properties. The differential equation that governs the energy conservation is non linear. In this context, the diffusion equation was linearized by using of the Transformed Integral of Kirchhoff. Transformations of the coordinate systems were realized in order to facilitate the boundary conditions application. The differential equation resulting after these transformations doesn't allow the application of the variable separation techniques. Thus, it was applied the Generalized Integral Transform Technique – GITT to solve the energy equation. As a result of this transformation it was obtained a coupled ordinary differential equation system that can be solved through classic numerical methods. Thus, for the determination of the evolution of the temperature field it was used the inversion formulas of all the transformations realized. Physical parameters of interest were, then, calculated and compared for several cylindrical cross section geometries.*

Keywords*: Kirchhoff Transform, Generalized Integral Transform Technique, Transient heat diffusion.*

1. INTRODUCTION

In recent years, several works dedicated to solve transient diffusive problems can be found in the literature, in particular, those that characterize the heat transfer in a nuclear fuel cell. There are several methods and techniques available to solve this kind of problem, but it is observed that the most accurate and more powerful are, generally, applied to the simplest problems where the results are already known. For fuel cells with a more complex geometry, or for temperature dependent fluid properties, analytic solutions and even approximate solutions are more difficult to be obtained and are not frequently found in the literature.

More recently, a generalization of the Integral Transform Techniques (Cotta, 1998) is being developed to obtain solutions for the most varied and complex diffusion problems, usually those that do not possess a closed form solution by the Classical Integral Transform Techniques – CITT or by the Method of Separation of Variables. This method is being used successfully for solving several kinds of diffusive problems such as those involving irregular domains (Aparecido et al., 1989), diffusive problems with moving boundaries (Diniz et al., 1999) and non linear diffusive problems (Cotta et al., 2003).

Like this, this work explores the GITT potential, presenting the solution of transient diffusive problems with variables sources in fuel cells of the rectangular and elliptical cross section.

The non linear diffusion equation will be conveniently linearized by using a change of variable known as Kirchhoff Transformation (Özisik, 1993). With this procedure are obtained proper conditions to apply the GITT to the energy equation, allowing the solution of the temperature distribution within the cell and others parameters of interest.

2. ANALYSIS

For the proposed cell it will be considered that the source term is proportional to the neutron diffusion flux through the fuel element. The neutron diffusion within a nuclear fuel rod is a complex phenomenon of difficult solution (Maia, 2003). Thus, in this work, this problem will be solved by Fick Diffusion Law. For this analysis, it will be considered that the thermal properties exhibit significative variations due to the wide variation of temperature.

As the temperature gradient along the fuel housing is relatively small when compared to the temperature gradients in the whole fuel cell, it will be assumed that the temperature is constant along of the fuel element boundary. Also has been admitted a uniform initial temperature profile. In this model, the diffusion equation for cylindrical shapes, in a Ω domain and Γ contour, is given by:

$$\nabla \cdot \left[k\left(T\right) \nabla T\left(x,y,t\right) \right] + \bar{\dot{q}}''' \phi\left(x,y\right) = \rho\left(T\right) c_p\left(T\right) \frac{\partial T\left(x,y,t\right)}{\partial t}, \quad \left\{ \left(x,y\right) \in \Omega, \quad t > 0 \right\}; \tag{1}$$

$$T\left(x,y,t\right) = T_w, \quad \left\{ \left(x,y\right) \in \Gamma, \quad t > 0 \right\}; \tag{2}$$

$$T\left(x,y,0\right) = T_i = T_w, \quad \left\{ \left(x,y\right) \in \Omega \right\}; \tag{3}$$

where, $\bar{\dot{q}}'''$ represents the average source term, $\phi\,(x,y)$ the non-dimensional and normalized neutron flux in the fuel cell, T_w the temperature on the fuel element surface and T_i the initial temperature.

2.1. Linearization of diffusion equation

To facilitate the analytical procedure, the Kirchhoff Transformation will be applied on the temperature potential as follows:

$$T^*\left(x,y,t\right) = \frac{1}{k_0} \int_{T_0}^{T(x,y,t)} k\left(T'\right) dT', \tag{4}$$

where $k_0 = k\left(T_0\right)$ and T_0 is a reference temperature. With this new variable T^*, the diffusion equation is transformed:

$$\nabla^2 T^*\left(x,y,t\right) + \frac{\bar{\dot{q}}'''}{k_0} \phi\left(x,y\right) = \frac{1}{\alpha\left(T^*\right)} \frac{\partial T^*\left(x,y,t\right)}{\partial t}, \quad \left\{ \left(x,y\right) \in \Omega, \quad t > 0 \right\}; \tag{5}$$

$$T^*\left(x,y,t\right) = T_w^* = \frac{1}{k_0} \int_{T_0}^{T_w} k\left(T'\right) dT', \quad \left\{ \left(x,y\right) \in \Gamma, \quad t > 0 \right\}; \tag{6}$$

$$T^*\left(x,y,0\right) = T_i^* = T_w^*, \quad \left\{ \left(x,y\right) \in \Omega \right\}; \tag{7}$$

where T_w^* is the Kirchhoff transformed temperature at the boundary, T_i^* is the Kirchhoff transformed initial temperature and $\alpha\left(T^*\right) = k\left(T^*\right) / \rho\left(T^*\right) c_p\left(T^*\right)$ is the thermal diffusivity. In its dimensionless form Eqs. (5), (6) and (7) can be written as:

$$\nabla^2 \theta\left(X,Y,\tau\right) + \phi\left(X,Y\right) = \frac{\partial \theta\left(X,Y,\tau\right)}{\partial \tau}, \quad \left\{ \left(X,Y\right) \in \Omega, \quad \tau > 0 \right\}; \tag{8}$$

$$\theta\left(X,Y,\tau\right) = \theta_w = 0, \quad \left\{ \left(X,Y\right) \in \Gamma, \quad \tau > 0 \right\}; \tag{9}$$

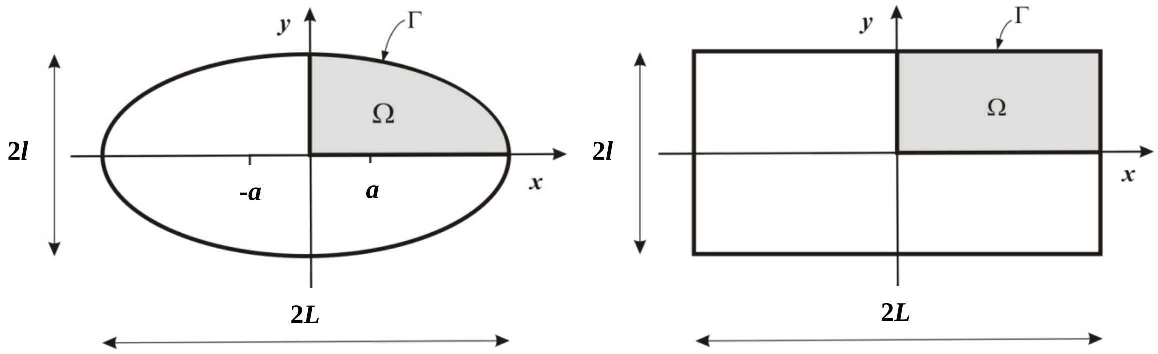
$$\theta\left(X,Y,0\right) = \theta_i = 0, \quad \left\{ \left(X,Y\right) \in \Omega \right\}; \tag{10}$$

with

$$X = \frac{x}{L_{ref}}, \quad Y = \frac{y}{L_{ref}}, \quad \tau = \frac{t\,\alpha\left(T^*\right)}{L_{ref}^2}, \quad \theta\left(X,Y,\tau\right) = \frac{\left[T^*\left(X,Y,\tau\right) - T_w^* \right] k_0}{L_{ref}^2\, \bar{\dot{q}}'''}, \tag{11a, b, c, d}$$

where the parameter L_{ref} is a reference length for each shape considered. In this work, the second order effects due to the local variation of τ with the thermal diffusivity in the time transformation of t to τ was neglected, a procedure adopted in similar investigations (e.g., Alves et al., 2006, Pelegrini, 2005 and Maia, 2003).

All cell shapes present symmetry to the axes X and Y, therefore it is sufficient to consider for solution just the domain indicated by the gray shaded area on Figure 1.



Elliptical cell cross section   Rectangular cell cross section

Figure 1: Cell shapes geometries.

2.2. Changing the coordinate system

For cells with elliptical cross section, it is not easy the representation in a Cartesian coordinate system. Therefore, is adequate to proceed a proper change in the coordinate system in order to facilitate the application of the boundary conditions.

2.3. Cell with elliptical cross section

The orthogonal elliptical coordinate system is used to change the original domain with boundary shaped as an ellipsis on the plane (X,Y) to a new domain with boundary shaped as rectangle defined on the new change plane (u,v):

$$X = a^* \cos(u) \cosh(v), \quad Y = a^* \sin(u) \sinh(v), \tag{12a, b}$$

with

$$a^* = \frac{a}{L_{ref}}, \quad a = \frac{L}{\cosh(v_0)}, \quad v_0 = arc \tanh\left(\frac{l}{L}\right), \quad L_{ref} = \frac{2A_S}{Per}, \tag{13a, b, c, d}$$

where a is the focal distance, A_s is the cross section area, Per is the perimeter and v_0 is the parameter that defines domain boundary on the plane (u,v).

The Jacobian transformation is obtained by using the following equation:

$$J(u,v) = \frac{\partial(X,Y)}{\partial(u,v)} = a^* \left[\sin^2(u) + \sinh^2(v) \right]. \tag{14}$$

For a domain having just one quadrant represented by $\{ 0 \le u \le u_0, \ \ 0 \le v \le v_0 \}$ with $u_0 = \pi/2$ and v_0 given by Eq. (13c), the diffusion equations and boundary conditions in elliptical coordinate system are given by:

$$\left[\frac{\partial^2 \theta(u,v,\tau)}{\partial u^2} + \frac{\partial^2 \theta(u,v,\tau)}{\partial v^2} \right] + J(u,v)\, g(u,v) = J(u,v) \frac{\partial \theta(u,v,\tau)}{\partial \tau}, \quad \{ 0 \le u \le u_0, \ \ 0 \le v \le v_0 \}; \tag{15}$$

$$\frac{\partial \theta(u,v,\tau)}{\partial u} = 0, \quad \{ u = 0, \ \ u = u_0, \ \ 0 \le v \le v_0 \}; \tag{16}$$

$$\frac{\partial \theta(u,v,\tau)}{\partial v} = 0, \quad \{ 0 \le u \le u_0, \ \ v = 0 \}; \tag{17}$$

$$\theta(u,v,\tau) = 0, \quad \{ 0 \le u \le u_0, \ \ v = v_0 \}. \tag{18}$$

2.4. Cell with rectangular cross-section

For cell with rectangular cross section, the domain boundary matches naturally with the Cartesian coordinate system and for to keep the uniformity in the representation of the space variables and in the identity transformation is applied to this geometry the following transformation:

$$X = u\,,\qquad Y = v\,. \tag{19a, b}$$

Then, for a domain of a quarter, the diffusion equation and its boundary conditions are rewritten as:

$$\left[\frac{\partial^2\theta(u,v,\tau)}{\partial u^2}+\frac{\partial^2\theta(u,v,\tau)}{\partial v^2}\right]+J(u,v)\,g(u,v)=J(u,v)\frac{\partial\theta(u,v,\tau)}{\partial\tau}\,,\qquad \{\,0\le u\le u_0,\quad 0\le v\le v_0\,\}\,; \tag{20}$$

$$\frac{\partial\theta(u,v,\tau)}{\partial u}=0\,,\qquad \{\,u=0,\quad 0\le v\le v_0\,\}\,; \tag{21}$$

$$\theta(u,v,\tau)=0\,,\qquad \{\,u=u_0,\quad 0\le v\le v_0\,\}\,; \tag{22}$$

$$\frac{\partial\theta(u,v,\tau)}{\partial v}=0\,,\qquad \{\,0\le u\le u_0,\quad v=0\,\}\,; \tag{23}$$

$$\theta(u,v,\tau)=0\,,\qquad \{\,0\le u\le u_0,\quad v=v_0\,\}\,; \tag{24}$$

with

$$J(u,v)=1\,,\qquad u_0=\frac{L}{L_{ref}}\,,\qquad v_0=\frac{l}{L_{ref}}\,,\qquad L_{ref}=\frac{A_S}{Per}\,. \tag{25a, b, c, d}$$

2.5. GITT development

To obtain temperature profiles the integral transformation is applied to the diffusion equation. Due to its two-dimensional characteristic, the potential $\theta\,(u,v,\tau)$ is written in terms of expansion in series by using orthonormal eigenfunctions obtained from the solution of auxiliary eigenvalue problems for each space coordinate. In this way, it is done by parts for each one of eigenvalue problems proposed.

2.5.1. *Application of GITT for a cell with elliptical cross section*

Consider the following auxiliary eigenvalue problem:

$$\frac{d^2\psi(u)}{du^2}+\mu^2\psi(u)=0\,,\qquad \{\,0<u<u_0\,\}\,;\qquad \frac{d\psi(0)}{du}=0\,,\qquad \frac{d\psi(u_0)}{du}=0\,. \tag{26a, b, c}$$

The orthogonality properties of the eigenfunctions above allow the development of the following transform-inverse pair:

$$\bar{\theta}_i(v,\tau)=\int_0^{u_0}K_i(u)\,\theta(u,v,\tau)\,du\qquad \text{(transform)}, \tag{27}$$

$$\theta(u,v,\tau)=\sum_{i=1}^{\infty}K_i(u)\,\bar{\theta}_i(v,\tau)\qquad \text{(inverse)}, \tag{28}$$

where $\bar{\theta}_i(v,\tau)$ is the transformed potential related to the axis u and $K_i(u)$ are the normalized eigenfunctions given by:

$$K_i\left(u\right)=\frac{\psi_i\left(u\right)}{N_i^{1/2}}\,,\quad \psi_i=\cos\left(\mu_i\,u\right),\quad \mu_i=\left(i-1\right)\frac{\pi}{u_0}\,,\quad N_i=\int_0^{u_0}\psi_i^2\left(u\right)du=\begin{cases}u_0, & i=1\\ u_0/2, & i>1.\end{cases}\tag{29a, b, c, d}$$

Removing the partial derivation related to the variable u is done through the function dot product between the set of normalized eigenfunctions, $K_i(u)$, and the diffusion equation. Making use of the respective boundary conditions, of the boundary conditions of the eigenvalue problem and of the eigenfunctions orthogonality property, it is achieved the first transformation of the partial differential equation that becomes:

$$\sum_{j=1}^{\infty}A_{ij}\left(v\right)\frac{\partial\bar{\theta}_j\left(v,\tau\right)}{\partial\tau}+\mu_i^2\bar{\theta}_i\left(v,\tau\right)=\frac{\partial^2\bar{\theta}_i\left(v,\tau\right)}{\partial v^2}+C_i\left(v\right),\tag{30}$$

$$A_{ij}\left(v\right)=\int_0^{u_0}K_i\left(u\right)K_j\left(u\right)\;J\left(u,v\right)\;d\,u\,,\qquad C_i\left(v\right)=\int_0^{u_0}K_i\left(u\right)\,J\left(u,v\right)g\left(u,v\right)\;d\,u\,.\tag{31a, b}$$

To proceed the integral transformation related to the coordinate v, consider the following eigenvalue problem:

$$\frac{d^2\varphi\left(v\right)}{dv^2}+\lambda^2\,\varphi\left(v\right)=0\,,\quad \left\{\,0<\varphi<v_0\,\right\};\quad \frac{d\varphi\left(0\right)}{dv}=0\,,\quad \varphi\left(v_0\right)=0\,.\tag{32a, b, c}$$

Eigenfunctions $\varphi\left(v\right)$ are orthogonal and can be used to development of the following transform-inverse pair:

$$\tilde{\bar{\theta}}_{im}\left(\tau\right)=\int_0^{v_0}\int_0^{u_0}K_i\left(u\right)\,Z_m\left(v\right)\,\theta\left(u,v,\tau\right)\,du\;dv\qquad\text{(transform)},\tag{33}$$

$$\theta\left(u,v,\tau\right)=\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}K_i\left(u\right)\,Z_m\left(v\right)\,\tilde{\bar{\theta}}_{im}\left(\tau\right)\qquad\text{(inverse)},\tag{34}$$

where $\tilde{\bar{\theta}}_{im}$ is the transformed temperature and $Z_m(v)$ are the normalized eigenfunctions related to the v axis and given by:

$$Z_m\left(v\right)=\frac{\varphi_m\left(v\right)}{M_m^{1/2}}\,,\quad \varphi_m=\cos\left(\lambda_m\,v\right),\quad \lambda_m=\left(2m-1\right)\frac{\pi}{2v_0}\,,\quad M_m=\int_0^{v_0}\varphi_m^2\left(v\right)\,dv=\frac{v_0}{2}\,.\tag{35a, b, c}$$

Removing of the partial derivation related to the variable v is done through the dot product of the normalized eigenfunctions, $Z_m(v)$, with the one time transformed partial differential equation. Doing use of problem boundary conditions, of the boundary conditions of the second eigenvalue problem, and of the orthogonality property of the respective eigenfunctions, it is reached finally the integral transformation for the diffusion equation that is given by:

$$\sum_{n=1}^{\infty}\;\sum_{j=1}^{\infty}B_{ijmn}\;\frac{d\tilde{\bar{\theta}}_{im}\left(\tau\right)}{d\tau}+\left[\mu_i^2+\lambda_m^2\right]\tilde{\bar{\theta}}_{im}\left(\tau\right)+D_{im}=0\;,\qquad i,m=1,2,3...\tag{36}$$

where:

$$B_{ijmn}=\int_0^{v_0}Z_m\left(v\right)\;Z_n\left(v\right)\;A_{ij}\left(v\right)\;dv=\int_0^{u_0}\int_0^{v_0}K_i\left(u\right)\;K_j\left(u\right)\;Z_m\left(v\right)\;Z_n\left(v\right)\;J\left(u,v\right)\;du\;dv\,,\tag{37}$$

$$D_{im}=-\int_0^{v_0}Z_m\left(v\right)\;C_i\left(v\right)dv=-\int_0^{u_0}\int_0^{v_0}K_i\left(u\right)\;Z_m\left(v\right)\;J\left(u,v\right)\;g\left(u,v\right)\;du\;dv\,.\tag{38}$$

The Equation (38) must to satisfy the transformed initial condition, given by:

$$\tilde{\bar{\theta}}_{im}\left(0\right)=\int_0^{v_0}\int_0^{u_0}K_i\left(u\right)\,Z_m\left(v\right)\;\theta_i\left(u,v,\tau\right)dv\;\;du\quad.\tag{39}$$

The Equation (36) shows the coupled, infinite and linear differential equations system to the transformed potential $\tilde{\bar{\theta}}_{im}(\tau)$, that may be numerically evaluated, just truncating the expansion in series of orthogonal functions, for a given order $i = M$ e $m = N$:

$$\sum_{n=1}^{M}\sum_{j=1}^{N} B_{ijmn}\ \frac{d\tilde{\bar{\theta}}_{im}(\tau)}{d\tau} + \left[\mu_i^2 + \lambda_m^2\right]\tilde{\bar{\theta}}_{im}(\tau) + D_{im} = 0\,. \tag{40}$$

Thus, the potential temperature $\theta(u,v,\tau)$ for fuel cell is obtained through the inversion formula:

$$\theta(u,v,\tau) = \sum_{i=1}^{M}\sum_{m=1}^{N} K_i(u)\ Z_m(v)\,\tilde{\bar{\theta}}_{im}(\tau)\,, \tag{41}$$

where the normalized eigenfunctions $K_i(u)$ e $Z_m(v)$ are defined by specified problem: elliptical cross section fuel cell or rectangular cross section fuel cell.

2.5.2. *Application of the GITT for cells with rectangular cross section*

The boundary condition of the rectangular cross section cells differs from the previous problem. But, following the same procedure, the application of the GITT leads to a formula similar to the potential:

$$\sum_{n=1}^{M}\sum_{j=1}^{N} B_{ijmn}\ \frac{d\tilde{\bar{\theta}}_{im}(\tau)}{d\tau} + \left[\mu_i^2 + \lambda_m^2\right]\tilde{\bar{\theta}}_{im}(\tau) + D_{im} = 0\,, \tag{42}$$

with, B_{ijmn} e D_{im} is given by Eqs.(37) e (38) and

$$K_i(u) = \frac{\psi_i(u)}{N_i^{1/2}}\,, \quad \psi_i = \cos(\mu_i\, u)\,, \quad \mu_i = (2i-1)\frac{\pi}{2u_0}\,, \quad N_i = \int_0^{u_0} \psi_i^2(u)\, du = \frac{u_0}{2}\,, \tag{43a, b, c, d}$$

$$Z_m(v) = \frac{\varphi_m(v)}{M_m^{1/2}}\,, \quad \varphi_m = \cos(\lambda_m\, v)\,, \quad \lambda_m = (2m-1)\,\frac{\pi}{2v_0}\,, \quad M_m = \int_0^{v_0} \varphi_m^2(v)\, dv = \frac{v_0}{2}\,. \tag{44a, b, c, d}$$

2.6. Neutron Diffusion Equation

How mentioned before, the source term of the energy equation is proportional to the thermal neutrons flux within the fuel cell. This particularity may be represented, in a first approximation, by Neutron Diffusion Equation, obtained by Fick's Law and this equation and the boundary conditions are:

$$\nabla^2\phi - \frac{1}{L_d^2}\phi = 0\,, \quad L_d^2 = \frac{D}{\Sigma_a}\,, \quad \phi = \phi_w\,, \quad \{(x,y)\in\Gamma\}\,; \tag{45a, b,c}$$

where ϕ is the neutron flux, L_d^2 is the length diffusion, D is the diffusion coefficient and the Σ_a is the absorption cross section to thermal neutrons in a fuel cell element.

Making use of the same procedure described above, the non-dimensional and homogenized diffusion equation and the boundary conditions for the fuel cell with elliptical cross section are given by:

$$\frac{\partial^2\phi(u,v)}{\partial u^2} + \frac{\partial^2\phi(u,v)}{\partial v^2} + J(u,v)\left[\phi(u,v)+1\right] = 0\,, \quad \{0\le u\le u_0,\ \ 0\le v\le v_0\}\,; \tag{46}$$

$$\frac{\partial\phi(u,v)}{\partial u} = 0\,, \quad \{u=0,\ \ u=u_0,\ \ 0\le v\le v_0\}\,; \tag{47}$$

$$\frac{\partial\phi(u,v)}{\partial v} = 0\,, \quad \{0\le u\le u_0,\ \ v=0\}\,; \tag{48}$$

$$\phi(u,v)=0\,,\quad \{\,0\le u\le u_0,\quad v=v_0\,\}\,; \tag{49}$$

where $\phi\,(u,v)$ is the non-dimensional and normalized neutrons flux, and $J(u,v)$ is the Jacobian of the transformation, $u_0=\pi/2$ and v_0 is given by Eq. (13c).

For the fuel cell with rectangular cross section the diffusion equation and the boundary conditions are presented in Eq. (20) to Eq. (24).

$$\frac{\partial^2\phi(u,v)}{\partial u^2}+\frac{\partial^2\phi(u,v)}{\partial v^2}+J(u,v)\left[\,\phi(u,v)+1\,\right]=0\,,\quad \{\,0\le u\le u_0,\quad 0\le v\le v_0\}\,; \tag{50}$$

$$\frac{\partial\phi(u,v)}{\partial u}=0\,,\quad \{\,u=0,\quad 0\le v\le v_0\,\}\,; \tag{51}$$

$$\phi(u,v)=0\,,\quad \{\,u=u_0,\quad 0\le v\le v_0\,\}\,; \tag{52}$$

$$\frac{\partial\phi(u,v)}{\partial v}=0\,,\quad \{\,0\le u\le u_0,\quad v=0\,\}\,; \tag{53}$$

$$\phi(u,v)=0\,,\quad \{\,0\le u\le u_0,\quad v=v_0\,\}\,. \tag{54}$$

The neutron diffusion problem in a nuclear fuel cell, when represented by Fick's Law, may be calculate by GITT. The GITT is applied in the *u* e *v* coordinates by internal product of $K_i(u)$, $Z_m(v)$ and $\phi\,(u,v)$. Making use of the boundary conditions, the following algebraic system can be obtained:

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}B_{ijmn}\,\tilde{\bar{\phi}}_{jn}+D_{im}=\left(\mu_i^2+\lambda_m^2\right)\tilde{\bar{\phi}}_{im}\,,\qquad i,\ m=1,2,3\ ... \tag{55}$$

Despite all involved terms in the problem to be transformed, the equation system is not coupled due to absorption term (D_{im}) in the resultant equation, which makes the solution matrix of this problem sparse. Finally, the infinite equation system above may be calculate truncating the expansion for a given order $i = N$ and $j = M$ sufficiently larger to get the required accuracy.

2.7. Parameters of Physical Interest

2.7.1. *Average Temperature and Normalized Temperature*

The average temperature of this problem in a τ instant is given by:

$$\theta_{av}(\tau)=\frac{4}{A_s}\int_0^{v_0}\int_0^{u_0}\theta(u,v,\tau)\ J(u,v)\ du\ dv\ . \tag{56}$$

2.7.2. *Time Constant*

For a better problem's analysis, is convenient to define an appropriate parameter able to verify the heat diffusion transient behavior in function of the aspect ratio for elliptical and rectangular cross section cylinders. Thus, the maximum normalized potential that occurs in the domain, for any τ instant, is given by:

$$\theta_{N\max}(\tau)=\frac{\theta_{\max}(\tau)}{\theta_{\max}(\infty)}\,. \tag{57}$$

In Equation (57), may be observed that the potential $\theta_{N\max}(\tau_{\max})$ will be in [0,1] interval. Thus, is defined the time constant τ_c as the parameter that determines the necessary time for the temperature $\theta_{\max}(\tau)$ stay to $1/e$ of your value in steady state $\theta_{\max}(\infty)$ (Maia, 2003), that can be represented by:

$$\frac{\theta_{\max}(\tau_{\max})}{\theta_{\max}(\infty)} = \left[1 - \frac{1}{e}\right] = 0.63212 \,. \tag{58}$$

3. RESULTS AND DISCUSSION

In order to determine the coefficients $\tilde{\tilde{\theta}}_{im}(\tau)$, the series expansion has been truncated to several choices of values for M and N. The parameters B_{ijmn} and D_{im} have been numerically calculated by a Gauss quadrature method (36 points of quadrature) (Pelegrini, 2005) and the equation system resultant has been solved by using the routine DIVPAG of the IMSL Library (IMSL Library, 1994).

For cells with rectangular cross section, it was observed that the series convergence to compute temperature distribution becomes slower when the aspect ratio l/L is small ($l/L < 0.1$), being necessary a high number of terms to get stable results over four or five decimal numeric places ($N = M > 20$). For cells with elliptical cross section, this fact happens when the focal distance tends to zero ($a \to 0$), for example, when an ellipse tends toward a circular shape ($l/L \to 1.0$). For all cases, it was verified that the series converge to 4 or 5 decimal places when it is truncated to an order of approximately $M = N = 15$. Anyway, even considering a high number of terms in the series, the computer processing time is small.

In Figures 2 and 3 are shown, respectively, the variation of maximum temperature with non dimensional time in elliptical cross section cylinders, for several l/L and length diffusion L_d^2.

In Figures 4 and 5 are presented the behavior of the maximum temperature for rectangular cross section cylinders in function of the l/L and length diffusion L_d^2, respectively. Particularly, in the Fig. 4 should be noted the major influence of the l/L in a maximum temperature profiles.

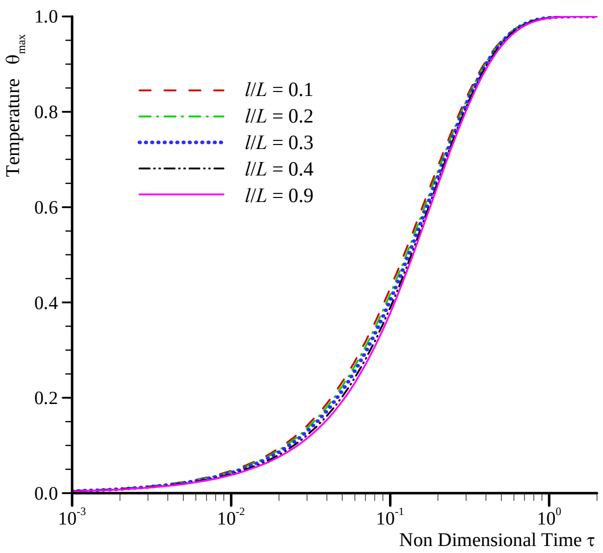


Figure 2: Variation of maximum temperature with non dimensional time, in elliptical cross sections cylinders for L_d^2 = 1.0 and several l/L.

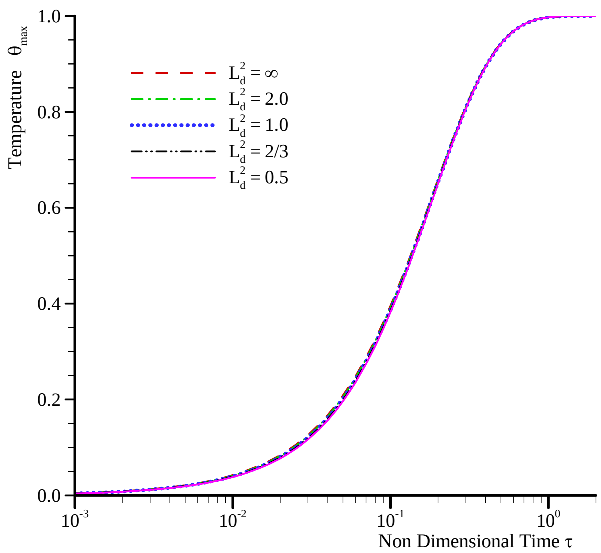


Figure 3: Variation of maximum temperature with non dimensional time, in elliptical cross sections cylinders for l/L = 0.5 and several L_d^2.

In Tables 1 and 2 are shown, respectively, the maximum temperature in steady state and maximum time constant for several l/L and L_d^2 in elliptical and rectangular cross-sections cylinders, where can be noted that, when the smaller length diffusion, major the maximum time constant value.

Finally, the Figures 6 and 7 show the maximum time constant along l/L-axis for several L_d^2 in elliptical and rectangular cross-sections cylinders, where may be noted that, when the smaller length diffusion, major the maximum time constant value.

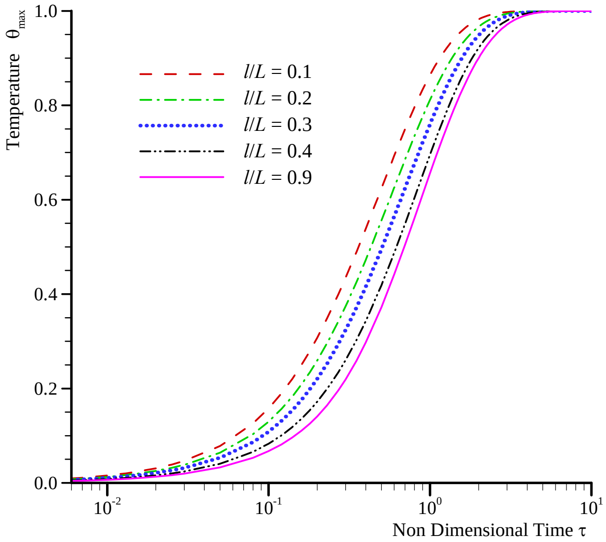


Figure 4: Variation of maximum temperature with non dimensional time, in rectangular cross sections cylinders for L_d^2 = 1.0 and several *l/L*.

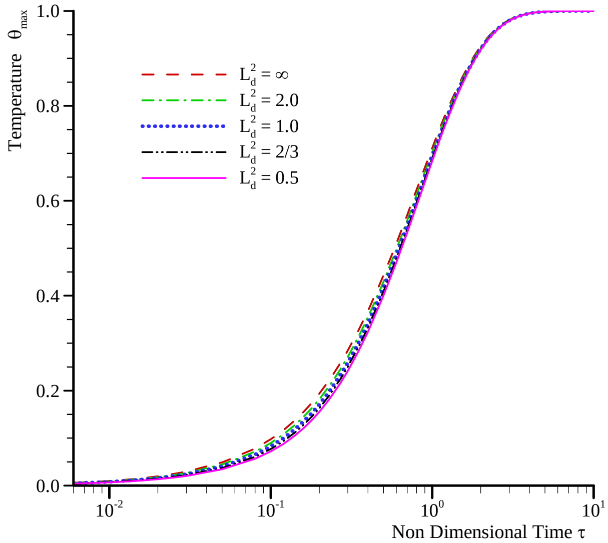


Figure 5: Variation of maximum temperature with non dimensional time, in rectangular cross sections cylinders for *l/L* = 0.5 and several L_d^2.

Table 1: Maximum temperature in steady state and maximum time constant for several *l/L* and $L^2{}_d$ in elliptical cross sections cylinders.

l/L	$L_d^2 = \infty$		$L_d^2 = 2.0$		$L_d^2 = 1.0$		$L_d^2 = 0.5$	
	θ_{max}	τ_{max}	θ_{max}	τ_{max}	θ_{max}	τ_{max}	θ_{max}	τ_{max}
0.10	0.2071	0.1725	0.2004	0.1730	0.1944	0.1734	0.1838	0.1743
0.20	0.2150	0.1771	0.2082	0.1779	0.2019	0.1786	0.1908	0.1799
0.30	0.2235	0.1812	0.2165	0.1822	0.2100	0.1832	0.1984	0.1849
0.40	0.2313	0.1843	0.2241	0.1855	0.2175	0.1866	0.2055	0.1886
0.50	0.2378	0.1865	0.2305	0.1877	0.2237	0.1889	0.2114	0.1911
0.60	0.2427	0.1881	0.2353	0.1894	0.2284	0.1906	0.2160	0.1928
0.70	0.2463	0.1892	0.2388	0.1905	0.2318	0.1917	0.2192	0.1940
0.80	0.2485	0.1900	0.2410	0.1912	0.2340	0.1924	0.2212	0.1947
0.90	0.2497	0.1903	0.2421	0.1916	0.2351	0.1928	0.2223	0.1950
1.00	0.2509	0.1904	0.2425	0.1916	0.2354	0.1928	0.2226	0.1951

Table 2: Maximum temperature in steady state and maximum time constant for several *l/L* and $L^2{}_d$ in rectangular cross-sections cylinders.

l/L	$L_d^2 = \infty$		$L_d^2 = 2.0$		$L_d^2 = 1.0$		$L_d^2 = 0.5$	
	θ_{max}	τ_{max}	θ_{max}	τ_{max}	θ_{max}	τ_{max}	θ_{max}	τ_{max}
0.10	0.6061	0.5049	0.5712	0.5080	0.5415	0.5109	0.4928	0.5163
0.20	0.7198	0.6009	0.6631	0.6058	0.6177	0.6103	0.5482	0.6182
0.30	0.8359	0.6936	0.7538	0.7033	0.6907	0.7113	0.5986	0.7243
0.40	0.9403	0.7670	0.8345	0.7832	0.7548	0.7964	0.6416	0.8172
0.50	1.0250	0.8183	0.8999	0.8393	0.8066	0.8568	0.6759	0.8843
0.60	1.0884	0.8526	0.9489	0.8765	0.8453	0.8967	0.7014	0.9289
0.70	1.1325	0.8749	0.9829	0.9004	0.8721	0.9221	0.7191	0.9572
0.80	1.1601	0.8884	1.0042	0.9147	0.8890	0.9373	0.7301	0.9742
0.90	1.1746	0.8953	1.0153	0.9220	0.8977	0.9450	0.7359	0.9826
1.00	1.1788	0.8973	1.0186	0.9241	0.9003	0.9472	0.7376	0.9851

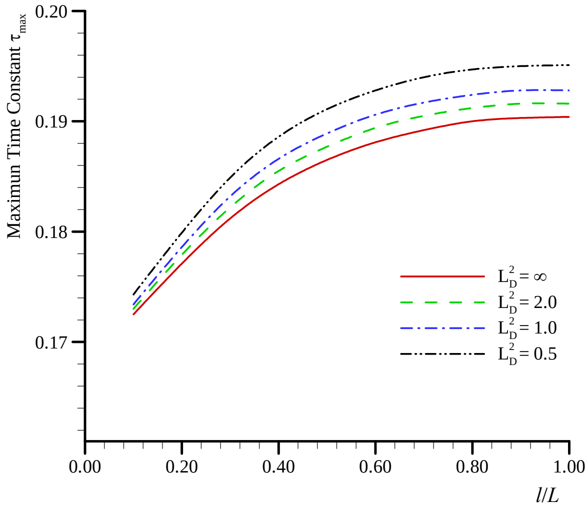


Figure 6: Maximum time constant along *l*/*L*-axis for several L_d^2 in elliptical cross-sections cylinders.

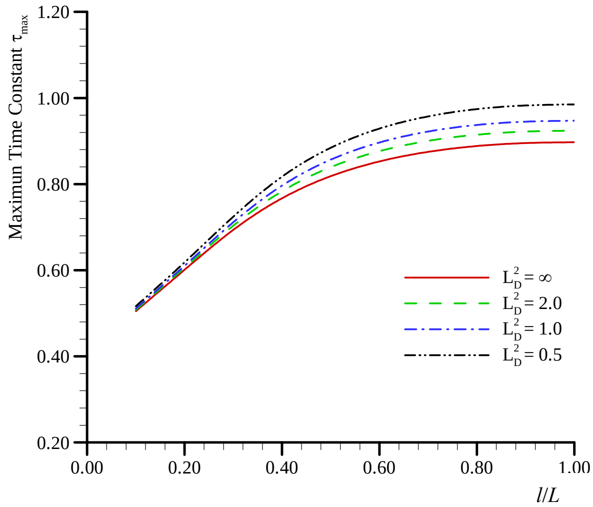


Figure 7: Maximum time constant along *l*/*L*-axis for several L_d^2 in rectangular cross-sections cylinders.

4. CONCLUSION

In this work it was analyzed a class of diffusion problems that characterizes cylindrical fuel cells with rectangular and elliptical cross section. The diffusive problem studied presents variables sources in its domain that are proportionals to the thermal neutrons flux within the fuel cell. Assuming a general thermal dependency on the physical properties, the diffusion equation was linearized through the use of Kirchhoff transformation and, to facilitate the application of boundary conditions, the coordinate system was changed from cartesian to elliptical, according to each case.

Analytical solutions were obtained, by applying the Generalized Integral Transform Technique to the diffusion equation, resulting in a decoupled system of linear equations for the transformed potential. The expansion that determines the temperature distribution presented a slow convergence for rectangular cells with aspect ratio (*l*/*L*) tending to zero and for elliptical cells with focal distance (*a*) tending to zero, being necessary to consider a high quantity of terms to achieve accurate results.

Finally, the results presented are interesting, since that was possible to demonstrate the efficiency of GITT to obtain analytical solution for diffusive complex problems, which does not have solution through classical techniques, such as separation of variables, as is the case for the problem of fuel cells with elliptical cross section.

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6. RESPONSIBILITY NOTICE

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