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TRIDIMENSIONAL TOPOLOGY OPTIMIZATION UNDER STRESS CONSTRAINT WITH TETRAHEDRICAL ELEMENTS

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Abstract. *The objective of this work is to propose a topology optimization scheme for the optimum material distribution applied to tridimensional structures, which it is competitive and capable of providing high-resolution layouts. The proposal is a formulation to minimization of the mass subject to stress criterion (failure function, the topology is obtained by optimum distribution of the material in the project domain. In order to solve the elasticity problem, the Galerkin Finite Element Method was applied. The process considers tetrahedral elements with four nodes that interpolates not only the displacement fields, but also the relative density field. Furthermore, this work proposes to combine a stabilized element to avoid checkerboard problems. The design variables are given by the nodal relative densities of the finite element mesh using the SIMP material. This formulation uses an integral criterion relaxing an exact condition (local criteria), it allows to work with parametric constraints of efficient way. The topology optimization problem when considerate some stress criterion, can generate feasible topology results like solutions of realistic problems of engineering, but this causes ill-conditioning of the optimization problem. Differently of the classical compliance optimization problem, here the topology structure is obtained at final form, ready to be built. Finally, analyzing the results, the formulation produces good layouts with definition quality proportional to medium size of element used.*

Keywords: *topology, stress, optimization, global criterion, relaxation.*

1. INTRODUCTION

The basic idea behind the concept of topology or layout optimization is the characterization of the body domain by a relative density measure denoted by density function $\rho(\mathbf{x})$, been $\mathbf{x}$ the position vector of body domain. This function characterizes the regions of the body domain which are associated with mass, *i. e.*, with $\rho(\mathbf{x}) \in (0,1]$ and void (without structural material), $\rho(\mathbf{x}) = 0$. The problem is illustrated in Fig. 1. In order to assure the existence of the solution we make use of the composite material with a microstructure approach where the effective homogenized composite material has a variable relative density such that $\rho(\mathbf{x}) \in (0,1]$.

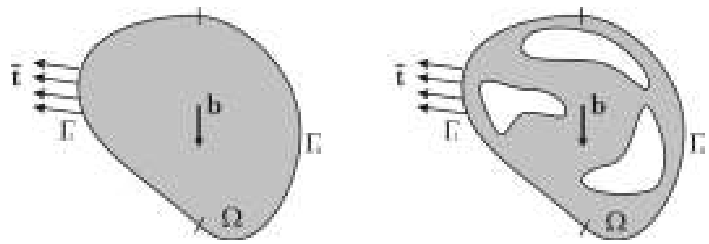
The formulation of the problem is developed to tridimensional models using topology optimization method under stress constraint. The classic finite element called linear tetrahedron since its shape functions are linear polynomials.

This paper is divided at five sections where this introduction is the first one. The second section is dedicated to formulation of the problem, This item explains the considerations of the problem of minimizing the mass subjected to stress constraints, stability constraints (density gradient control) and side constraints. Furthermore, the material type model to be used on the stress optimization (ill-conditioned phenomenon) problem and considerations to the problem of stress singularity. The third section corresponds to study of approach of de problem with formulation of FEM (Finite Element Model). The fourth section presents some numerical examples of the topology optimization problem by using the techniques proposed in the previous sections. Finally, some conclusions are stated in the fifth section.

2. PROBLEM DEFINITION

2.1. Formulation of the Problem

The objective of this work is to determine the optimum layout of the domain. This is obtained by minimizing the mass of the structure subjected to an effective von Mises stress constraints and side constrains. This may be illustrated in Fig. 1, where we consider an initial simply connected domain. As a result of the topology optimization we determine a multi-connected optimum domain.



a) Initial domain (b) Final domain

Figure 1: Definition of the problem

- Ω - domain of the body;
- $\partial\Omega$ - boundary, such that, $\partial\Omega = \Gamma_{\mathbf{u}} \cup \Gamma_{\mathbf{t}}$ and $\Gamma_{\mathbf{u}} \cap \Gamma_{\mathbf{t}} = \varnothing$;
- $\Gamma_{\mathbf{u}}$ - part of the boundary with prescribed displacement, *i. e.*, $\mathbf{u} = \bar{\mathbf{u}}$;
- $\Gamma_{\mathbf{t}}$ - part of the boundary with prescribed traction, *i. e.*, $\mathbf{t} = \bar{\mathbf{t}}$;
- $\mathbf{b}$ - body force.

The displacement field $\mathbf{u}(\rho(\mathbf{x}),\mathbf{x})$ is solution of:

$$a(\mathbf{u},\mathbf{v}) = l(\mathbf{v}), \quad \forall\ \mathbf{v} \in H_o \tag{1}$$

with

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{D}^H(\rho)\boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,d\Omega \tag{2}$$

and

$$a(\mathbf{u},\mathbf{v}) = a_{vol}(\mathbf{u},\mathbf{v}) + a_{dist}(\mathbf{u},\mathbf{v}) \tag{3}$$

been $a_{vol}(\mathbf{u},\mathbf{v})$ are the volumetric terms and $a_{dist}(\mathbf{u},\mathbf{v})$ the distortion terms. And

$$l(\mathbf{v}) = \int_\Omega \mathbf{b}\cdot\mathbf{v}\,d\Omega + \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}\,d\Gamma \tag{4}$$

$H_o(\Omega) = \left\{\mathbf{v}\,\middle|\,\mathbf{v} \in \left[H^1(\Omega)\right]^2,\ \mathbf{v} = \mathbf{0} \text{ at } \mathbf{x} \in \Gamma_{\mathbf{u}}\right\}$ and $H = \{\bar{\mathbf{u}} + H_o\}$ are variation and displacement allowable fields, respectively.

2.2. Model Material Definition

The porous material concept employed here is modeled with the so-called, proportional "fictitious material" model, also name as the solid isotropic material with penalization model (SIMP). So, a continuous variable $\rho,\ 0 \le \rho \le 1$ is introduced. In numerical implementations, a small lower bound, $0 < \rho_{\text{inf}} \le \rho \le 1$, is imposed, in order to avoid a singular FEM problem, when solving for equilibrium conditions equations in the full domain Ω, see Bendsoe and Sigmund (1999). The homogenized constitutive equation of the effective material may be fully expressed in terms of the relative density of the porous material.

$$\boldsymbol{\sigma} = \left[\mathbf{D}^H(\rho)\right]\boldsymbol{\varepsilon}(\mathbf{u}) \tag{5}$$

where

$$\boldsymbol{\sigma}^T = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\} \tag{6}$$

$$\left[\mathbf{D}^{H}(\rho)\right]=\left[\mathbf{D}_{vol}^{H}\right]+\left[\mathbf{D}_{dist}^{H}\right] \tag{7}$$

and

$$\left[\mathbf{D}_{vol}^{H}\right]=\begin{bmatrix}\mathbf{D}_{11}^{H}(\rho) & 0\\ 0 & 0\end{bmatrix}\ ,\ \boldsymbol{\varepsilon}^{\mathrm{T}}=\left\{\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},0,0,0\right\} \tag{8}$$

$$\left[\mathbf{D}_{dist}^{H}\right]=\begin{bmatrix}0 & 0\\ 0 & \mathbf{D}_{22}^{H}(\rho)\end{bmatrix}\ ,\ \boldsymbol{\varepsilon}^{\mathrm{T}}=\left\{0,0,0,\gamma_{xy},\gamma_{yz},\gamma_{zx}\right\} \tag{9}$$

with

$$\left[\mathbf{D}_{11}^{H}(\rho)\right]=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu}\\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu}\\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1\end{bmatrix} \tag{10}$$

$$\left[\mathbf{D}_{22}^{H}(\rho)\right]=\frac{E}{2(1+\nu)}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{11}$$

ν is Poisson's ratio.

$$E(\rho)=\rho^{\eta}E_{o} \tag{12}$$

E_o is the Young's modulus of solid material and η is a penalty parameter of "porous material concept".

Due to the fact to use of stress constraint, it is define a measure to effective stress $\sigma^{*}(\rho)$, according to Duysinx and Sigmund (1998) and Duysinx and Bendsoe (1998).

$$\sigma^{*}(\rho)=\frac{\sigma(\rho)}{\rho^{\eta}} \tag{13}$$

Therefore, the failure criterion of the material defined by von Mises is denoted per:

$$\sigma_{eq}^{*}(\rho)=\frac{\sigma_{eq}(\rho)}{\rho^{\eta}}\leq\sigma_{y} \tag{14}$$

σ_y is the allowable stress.

2.3. Stress Singularity Problem

In order to open the degenerated parts of the design space with the possibility of creating or removing holes without violating the effective stress constraint we apply the κ-relaxation technique. In this work, is implemented an automatic and systematic strategy to reduce the initial perturbation parameter κ. The stress relaxation parameter is decremented, as we get closer to the solution. Now, let ρ_0 be the relative density associated with the full material condition. Then, the relaxed admensionalized effective stress constraint may be written as:

$$g(\rho,\mathbf{u}(\rho))=\frac{\sigma_{eq}^{*}(\rho,\mathbf{u}(\rho))}{\sigma_{y}}+\kappa\left(\frac{1}{\rho_{0}}-\frac{1}{\rho}\right)-1\leq 0\,. \tag{15}$$

From this consideration, the relaxed minimization problem may be formatted as:

$$\min \int_{\Omega} \rho d\Omega \tag{16}$$

such that

$$\frac{\sigma_{eq}^*}{\sigma_y} + \kappa\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) - 1 \le 0\,; \tag{17}$$

$$\rho_{\inf} - \rho \le 0\,; \tag{18}$$

$$\rho - \rho_{\sup} \le 0\,, \quad \forall\, \mathbf{x} \in \Omega. \tag{19}$$

3. APPROACH OF THE PROBLEM

3.3. Formulation of the discretized problem

In order to discretize the problem is applied the Galerkin Finite Element Method. It is employed a four nodes tetrahedron finite element that interpolates not only the displacement components but also the relative density ρ. Consequently, the discretization formulation of the problem way be stated as:

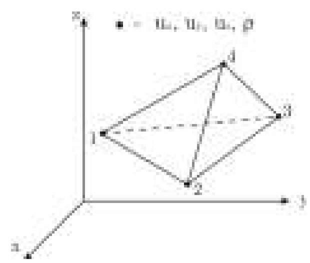


Figura 2: Displacement and density fields.

$$\min \int_{\Omega} \rho d\Omega \tag{20}$$

Subject to:

(i) Stress Constraint:

$$\rho(\mathbf{x})\left(\frac{\sigma_{eq}^*\left(\rho(\mathbf{x}), \mathbf{u}\left(\rho(\mathbf{x}), \mathbf{x}\right)\right)}{\sigma_y} - 1\right) + \rho(\mathbf{x})\kappa\left(\rho_{\sup} - \rho(\mathbf{x})\right) \le 0 \tag{21}$$

at this work was proposed the follow global criteria:

$$\bar{g}\left(\rho, \mathbf{u}(\rho)\right) = \left\{\frac{1}{\Omega}\int_{\Omega}\left\langle \rho\left(\frac{\sigma_{eq}^*}{\sigma_y} - 1\right) + \rho\kappa\left(\rho_{\sup} - \rho\right)\right\rangle^p d\Omega\right\}^{1/p} \le 0 \tag{22}$$

where, $\langle f(\mathbf{x})\rangle = \max\{0, f(\mathbf{x})\}$, for all positive part of $f(\mathbf{x})$.

$$\max_{\mathbf{x}\in\Omega}\left|\left\langle \rho\left(\frac{\sigma_{eq}^*(\rho)}{\sigma_y} - 1\right) - \kappa(1-\rho)\right\rangle\right| \le 0. \tag{23}$$

(ii) Side Constraint:

$$\rho_{\inf} - \rho_i \le 0 \quad \text{and} \quad \rho_i - \rho_{\sup} \le 0\,; \quad i=1,\dots,n \;\; (n \text{ is the number of nodes in the mesh}) \tag{24}$$

(iii) Stability Constraint

$$\bar{h}_j(\rho) = \left\{ \frac{1}{\Omega} \int_{\Omega} \langle h_{ej}(x_e) \rangle^p \, d\Omega \right\}^{1/p} ; \quad j = x, y \text{ and } z \tag{25}$$

with

$$h_{ej}(\rho) = \frac{\sqrt{\left(\frac{\partial \rho}{\partial j} \right)^2}}{c_j} - 1 \;\; ; \;\; j = x, y, z \quad \text{and} \quad e = 1, \dots, n_e \tag{26}$$

The constants c_x, c_y and c_z impose a superior limit to the components of the relative density gradient.

Consider a generic element, according to Fig. 3, with $\mathbf{x}_n = (x_i, y_i, z_i)$; $i = 1,\dots,4$ vertices coordinates and $\mathbf{x}_m = (x_m, y_m, z_m)$ as barycenter coordinates of tetrahedral element.

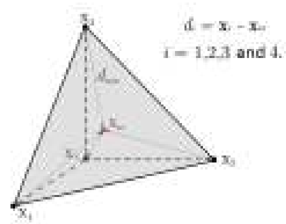


Figura 3: Four nodes tetrahedral coordinates.

$$d_{\min} = \min_j \left\| \mathbf{d}_j \right\| \;\; \text{p/} \;\; i = 1,\dots,4 \tag{27}$$

$$c_x^e = c_y^e = c_z^e = \frac{1}{d_{\min}}\,. \tag{28}$$

3.4. Augmented Lagrangian Method

Being $\boldsymbol{\rho} \in \mathbf{X}$ with $\mathbf{X} = \left\{ \boldsymbol{\rho} \in R^n \,\middle|\, \rho_{\inf} \le \rho_i \le \rho_{\sup}\,, i = 1,\dots,n \right\}$ and n_e is the number of elements in the mesh.

Step – 1. Initial Conditions: $k = 0$, $\lambda^k = 0$, $\mu^k = \mathbf{0}$, $erro = 1{,}0$, ζ, ω^k and tol.

Step – 2. While $erro > tol$, to do:

(i) Solution of the minimization problem with side constraint:

$$\min \Pi(\boldsymbol{\rho}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \zeta, \boldsymbol{\omega}), \quad \forall\, \boldsymbol{\rho} \in \mathbf{X} \tag{29}$$

where,

$$\Pi(\rho, \lambda, \mu; \zeta, \omega) = f(\rho) + \frac{1}{\zeta} \sum_{e=1}^{n_e} \Lambda_e(g_e, \zeta\lambda_e) + \sum_{j=1}^{3} \left[\frac{1}{\omega_j} \sum_{e=1}^{n_e} \Psi_e^j \left(h_e^j, \omega_j \mu_{ej} \right) \right] \tag{30}$$

with

$$\Lambda_e\left(g_e,\zeta\lambda_e\right)=\begin{cases} g_e\left(g_e+\zeta\lambda_e\right) & , \text{ if } g_e \geq -\dfrac{\zeta\lambda_e}{2} \\ -\left(\dfrac{\zeta\lambda_e}{2}\right)^2 & , \text{ if } g_e < -\dfrac{\zeta\lambda_e}{2} \end{cases}, \tag{31}$$

and

$$\Psi_e^j\left(h_e^j,\omega_j\mu_{ej}\right)=\begin{cases} h_e^j\left(h_e^j+\omega_j\mu_{ej}\right) & , \text{ if } h_e^j \geq -\dfrac{\omega_j\mu_{ej}}{2} \\ -\left(\dfrac{\omega_j\mu_{ej}}{2}\right)^2 & , \text{ if } h_e^j < -\dfrac{\omega_j\mu_{ej}}{2} \end{cases} \quad ; \ j=1,\ldots,3. \tag{32}$$

(ii) Update of Lagrange multipliers

$$\lambda_e^{k+1}=\max\left\{0,\lambda_e^k+\frac{2}{\zeta}g_e\left(\mathbf{x}^k\right)\right\} \tag{33}$$

and

$$\mu_{ej}^{k+1}=\max\left\{0,\mu_{ej}^k+\frac{2}{\omega_j}h_e\left(\mathbf{x}^k\right)\right\} \ ; \ j=1,\ldots,3. \tag{34}$$

(iii) Update of penalty parameters

$$\zeta^{k+1}=\begin{cases}\gamma_1\zeta^k \ \text{ with } \ \gamma_1\in(0,1), \ \text{ if } \ \gamma_1\zeta^k>\zeta^{crit} \\ \zeta^{crit}\end{cases} \tag{35}$$

and

$$\omega_j^{k+1}=\begin{cases}\beta_j\omega_j^k \ \text{ with } \ \beta_j\in(0,1), \ \text{ if } \ \beta_j\omega_j^k>\omega_j^{crit} \\ \omega_j^{crit}\end{cases} \quad ; \ j=1,\ldots,3. \tag{36}$$

(iv) Error

$$a=\max_e\left|\lambda_e^{k+1}-\lambda_e\right|, \ b=\max_e\left|\mu_{e1}^{k+1}-\mu_{e1}\right|, \ c=\max_e\left|\mu_{e2}^{k+1}-\mu_{e2}\right| \ \text{ and } \ d=\max_e\left|\mu_{e3}^{k+1}-\mu_{e3}\right| \tag{37}$$

so, $erro=\max\{a,b,c,d\}$.

Step – 3. End

The problem can be formulated as: $\lambda,\mu_1,\mu_2,\mu_3\in R^{n_e}$ and $\zeta,\ \omega_1,\omega_2,\omega_3\in R$, determinate $\boldsymbol{\rho}^*\in R^n$, such that:

$\boldsymbol{\rho}^*=\arg\min\Pi\left(\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\mu};\zeta,\boldsymbol{\omega}\right)$, $\forall\ \boldsymbol{\rho}\in\mathbf{X}$.

4. NUMERICAL EXAMPLES

4.1. Problem #01

Consider a problem according to Fig. 4. The case consists in one block of dimensions: $a=5.0\,m$, $b=20.0\,m$ and $c=4.0\,m$. The properties material are: Young's modulus, $E=215\times10^9$ N/m^2 , Poisson's ratio $\nu=0.3$ and Tensile Strength, Yield of $S_y=260\times10^6$ N/m^2 . On block is applied a load of $\mathbf{W}=150\times10^6$ N . It was analyzed the structure with 11,133 elements 2,337 nodes.

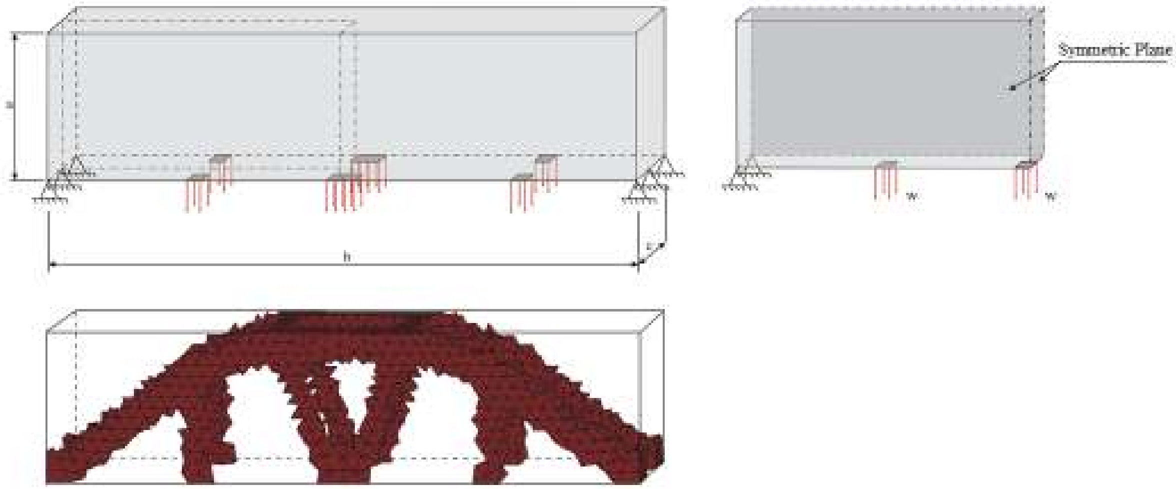


Figure 4: Problem #1 Topology Result.

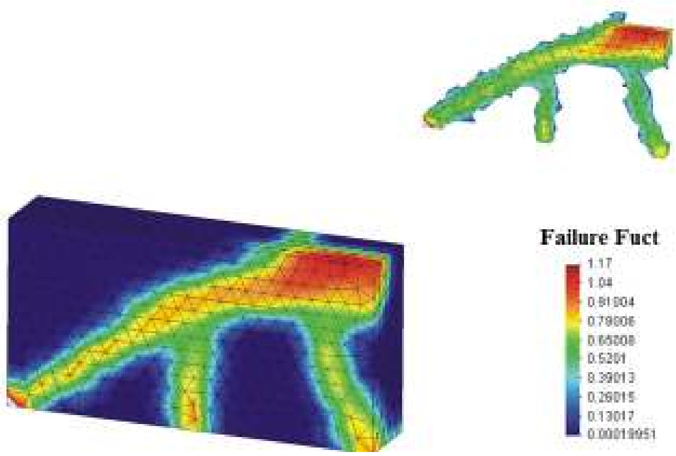


Figure 5: Problem #1 Failure Function Result.

4.3. Problem #02

Consider a problem according to Fig. 5. The block dimensions are: $a = 4.0\,m$, $b = 6.0\,m$ and $c = 2.0\,m$. On block is applied a load of W=150x10^{6}N. The block present a prescribed displacement of $[0,0,0]$ on $\overline{\Omega}_1 = \overline{\Omega}_2 = \overline{\Omega}_3 = \overline{\Omega}_4$ **.** The properties material are: Young Module, $E = 215 \times 10^9$ N/m^2 , Poisson's Ratio $\nu = 0.3$ and Tensile Strength, Yield of $S_y = 260 \times 10^6$ N/m^2 .
It was analyzed the ¼ of block structure with 10,651 elements and 2,261 nodes.

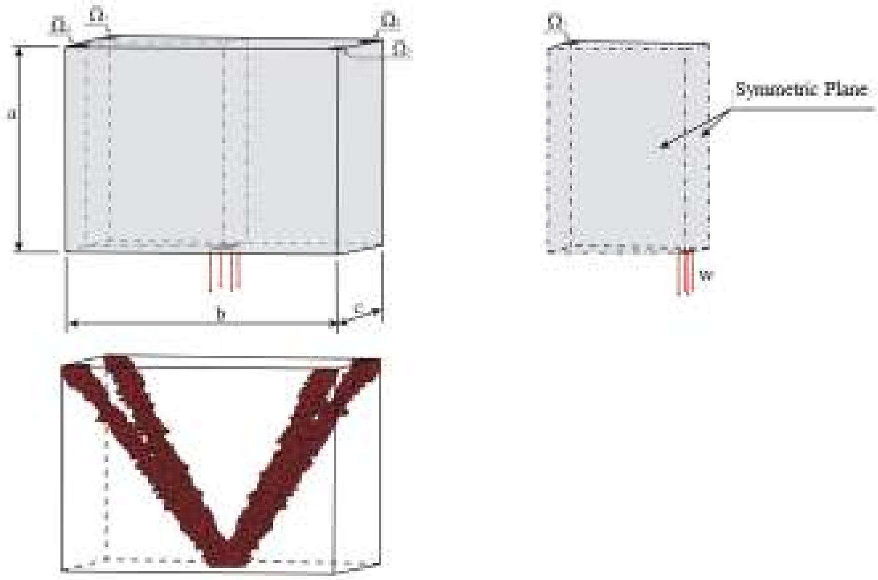


Figure 6: Problem #2 Result.

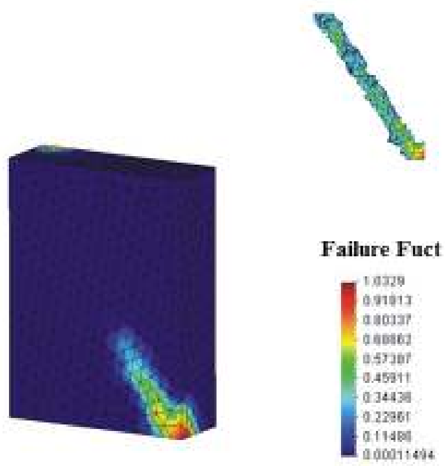


Figure 6: Problem #2 Failure Function Result.

5. CONCLUSIONS

The method considered showed to be effective and robust in the generation of excellent structural layout for 3D structural problems. The final resolution of the contour material is directly dependent of the average size of the finite element employee, representing the direct relation with the computational cost. Note that, the stress optimization problems, differently of the optimization problems without stress constraints (like classical compliance formulation), are sufficient to project structures. The optimum stress problems should be applied to get an ultimate conception of topology structure, ready for construction.

The formulation revealed promising for the implementation of *h*-adaptive process, that is, the implementation of an intelligent process of refinement of the mesh with information of the topology gotten in the original mesh. It is detailed in the works of Costa Jr (2003), Costa Jr and Alves (2003a-b). For the solution optimization problem, according to described in the main body of this paper, the Augmented Lagrangian Method is applied. The solver presented excellent performance, in a posterior version will compare with computational structure TANGO of Andreani *et al*. (2004), Andreani *et al.* (2005) and Birgin and Martinez (2002).

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7. RESPONSIBILITY NOTICE

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