A BRIEF APPROACH ABOUT THE SHEAR LOCKING PROBLEM IN THE LAMINATED COMPOSITE PLATES FINITE ELEMENTS

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Abstract. *A locking-free plate finite element based on the equivalent lamina assumption and on a first-order shear deformation theory is developed using strain gradient notation. Being physically interpretable is the biggest advantage of the coefficients of the second-order strain gradient displacement polynomials, differently from the standard ones. The coefficients written in the standard notation have an arbitrary and obscure nature, making difficult to understanding the meaning of each term. It is explicated the importance of eliminate the spurious terms, which are found in the polynomials for shear strains and are responsible for shear locking, and how it can be made. At different points through de tickness of the plate, stresses and strains can be correctly calculated as well. In this way, one laminated composite plate problem is solved using a finite element model. The solutions obtained from meshes containing the spurious terms are compared with those which do not have them. Finally, the strain gradient notation validity is done by noting the convergence of numerical results to the analytical solutions.*

Keywords: *Laminated Composite Plates, Strain Gradient Notation, Shear Locking, Parasitic Shear.*

1. INTRODUCTION

Important structural properties such as strength and stiffness can be improved when composites are employed, instead of single materials. This class of material consists in a combination of two or more materials on a macroscopic scale often exhibiting qualities that neither constituent possesses Chiaverini (1986). These advantages, associated to excelents strength-weight ratio and tailoring capability, justifies why they have been specially used in aircraft projects and in other engineering areas Norton (2004). A very important class of composites is that of the laminated fiber-reinforced composites, commonly referred to as simply laminated composites Jones (1975). The laminate consists in the laminae sobreposition oriented in different directions in order to promote increased physical and mechanical characteristics which only one lamina could have.

For the first time, Abdalla and Dow Abdalla and Dow (1994) apud Belo *et al.* (2005) identified qualitative errors in laminated composite plate finite elements analyzes when they were concerned with the deleterious effects of modeling errors in the representation of the behavior of those plates. In this way, this work proposes to show the efficiency of to use strain gradient notation in the development of this finite element cited in order to make it locking-free.

According to Mindlin's theory, the first-order shear deformation theory considers the transverse shear strains and the proposed element is based on it. Further, the equivalent lamina assumption is employed to treat the laminate as one single, orthotropic lamina plate whose constitutive properties are the average of the properties of the constituent laminae Belo *et al.* (2005).

Strain gradient notation is a physically interpretable notation and an alternative for writing finite element polynomials. Its biggest advantage consists in become clear the finite element formulation offering a simple mean to deal with spurious terms, which are responsilbe for shear locking Bonet and Wood (2008); Dow *et al.* (1985); Vinson and Sierakowski (2002); Felippa and Haugen (2005); Hughes (2000); Reddy (2004). Numerical solutions show that the plate finite element provides accurate results and converges quickly after spurious terms are eliminated. This fact happens because those terms increase the element's shear strain energy when bending of the plate occurs, resulting in a model stiffer.

Refining the mesh is another important factor which might be able to remove the false terms and to attain convergence. During analysis, just the computer implementation is necessary to eliminate the locking. Nevertheless, this procedure must be accomplish very carefully mainly in the case of the serendipity element (eight-node rectangular plate), most complex than a four-node one or a six-node one, in order to exclude the legitimate sources of errors.

2. DISPLACEMENT AND DEFORMATION FIELDS

This section intends to give a theoretical basis necessary for the development of the proposed finite element and to start using strain gradient notation. First, it is necessary to make some assumptions, also namely the Mindlin assumptions, from the macromechanical theory for laminated composite plates: 1) The laminate consists in a perfect bond between laminae in order to avoid their relative slipping; 2) Plane sections normal to the middle surface of the plate remain plane, but not necessarily normal after bendind. Thus, the model accounts for transverse shear deformation of the plate; and 3) Normal-to-the-middle-surface components of stress and strain are negligible, and, thus, are not included in the model (Belo *et al.*, 2005).

It is possible for the laminate to displace in all directions and to rotate in the x and y directions. In this way, the plate model kinematic relations can be write by:

$$\begin{cases} u(x,y,z) = u_0(x,y) + z\phi_x(x,y) \\ v(x,y,z) = v_0(x,y) - z\phi_y(x,y) \\ w(x,y,z) = w_0(x,y) \\ \phi_x = \dfrac{\partial u(x,y,z)}{\partial z} \\ \phi_y = \dfrac{\partial v(x,y,z)}{\partial z} \end{cases} \tag{1}$$

where u and v are the in-plane displacements along the x and y directions, respectively, w is the out-of-plane displacement and ϕ_x e ϕ_y are the rotations around the x and y axes, respectively. Similarly, u_0, v_0 and w_0 are middle surface displacements along the axes x, y and z, respectively, and z is the coordinate associated to the thickness of the plate.

According to the Mindlin assumptions, the infinitesimal strain field is given by:

$$\begin{cases} \varepsilon_x = u_{0,x} + z\phi_{x,x} \\ \varepsilon_y = v_{0,y} - z\phi_{y,y} \\ \gamma_{xy} = u_{0,y} + v_{0,x} + z\left(\phi_{x,y} - \phi_{y,x}\right) \\ \gamma_{yz} = w_{,y} - \phi_y \\ \gamma_{xz} = w_{,x} + \phi_x \end{cases} \tag{2}$$

The sum of the strain energies of all laminae constitutes the energy of the laminate, wherefore:

$$U = \frac{1}{2}\sum_{k=1}^{n}\int_{\Omega_k} \{\varepsilon\}_k^T \, [Q]_k \, \{\varepsilon\}_k \, d\Omega_k \tag{3}$$

where k is a typical lamina, n is the total number of laminae, $\{\varepsilon\}_k$ is the strain vector of lamina k, $[Q]_k$ is the constitutive properties matrix of lamina k and Ω_k is the volume of lamina k.

The technique's basis of strain gradient notation is to write whatever general distribution of deformation represented by the element as a linear combination of states of well defined deformation. Consequently, it can be obtain the following relations of displacements and strains writen in the proposed notation (Belo, 2006):

$$\{d\} = [\phi]\,\{\varepsilon_{sg}\} \tag{4}$$

$$\{\varepsilon\} = [T_{sg}]\,\{\varepsilon_{sg}\} \tag{5}$$

where $[\phi]$ and $[T_{sg}]$ are the corresponding transformation matrices and $\{\varepsilon_{sg}\}$ is the strain gradients vector. Elimination $\{\varepsilon_{sg}\}$ from Eq. (4) and Eq. (5) and substituing the results into Eq. (3) provides the expression of the strain energy in strain gradient notation:

$$U = \frac{1}{2}\{d\}^T [\phi]^{-T} \left(\sum_{k=1}^{n}\int_{\Omega_k} [T_{sg}]_k^T \, [Q]_k \, \{T_{sg}\}_k \, d\Omega_k\right) [\phi]^{-1} \{d\} \tag{6}$$

The term between parenthesis from Eq. (6) contains the quantities of energy associated to the pure strain modes of the element and it is represented by the following matriz:

$$[U_M] = \int_A \left(\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k} [T_{sg}]_k^T \, [Q]_k \, \{T_{sg}\}_k \, dz_k\right) dA \tag{7}$$

Thus, Eq. (6) can be rewrite in the compact form by:

$$U = \frac{1}{2}\{d\}^T [\phi]^{-T} [U_M] \, [\phi]^{-1} \{d\} \tag{8}$$

Finally, the terms of the transformed reduced stiffness matrix of kth lamina, $Q_{ij}^{(k)}$, could be associated to the extensional, coupled and flexural rigidities terms. Thus, the matrix $[U_M]$ coeficients are given by:

$$U_M^A = A_{ij} \int_A f_1(x,y) dA \tag{9a}$$

$$U_M^B = B_{ij} \int_A f_2(x,y) dA \tag{9b}$$

$$U_M^D = D_{ij} \int_A f_3(x,y) dA \tag{9c}$$

where A_{ij} are called extensional siffnesses, D_{ij} the bending stiffnesses and B_{ij} the bending-extensional coupling stiffnesses, which are defined in terms of the lamina stiffnesses $\bar{Q}_{ij}^{(k)}$ (for $i, j = 1, 2, 6$) as:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}^{(k)} (1, z, z^2)\, dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} (1, z, z^2)\, dz \tag{10}$$

or,

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (z_{k+1} - z_k), \quad B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (z_{k+1}^2 - z_k^2) \quad \text{and} \quad D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} (z_{k+1}^3 - z_k^3) \tag{11}$$

Also, the strain energy of the element can be related to their stiffness matrix, $[K]$, by the expression below:

$$U = \frac{1}{2} \{d\}^T [K] \{d\} \tag{12}$$

The matrix $[T_{sg}]$ values are directly influenced by the presence or no of spurious terms. Therefore, it affects the stiffness matrix writen in strain gradient notation, from Eq. (8) and Eq. (12):

$$[K] = [\phi]^{-T} [U_M] [\phi]^{-1} \tag{13}$$

Differently from the traditional finite element formulation, which makes use of numerical integration to obtain the stiffness matrix, it is noted that strain gradient notation doesn't need it to find the matrix cited. This big advantage allows elements to be modeled precisely and accurately. It is also important to note, especially in thin laminates, the analytical integration of $[K]$ provides to the finite element in strain gradient notation more accurate results. In this case, it is easily to remove the spurious terms during the formulation process. Belo (2006)

3. THREE-NODED TRIANGULAR PLATE ELEMENT

Continuing to use strain gradient notation, the purpose of this section is to develop, in detail, the three-node triangular plate finite element formulation based on the *First-order Shear Deformation Theory*. The element has fifteen degree of freedom in total, i.e. 5 of them at each node.

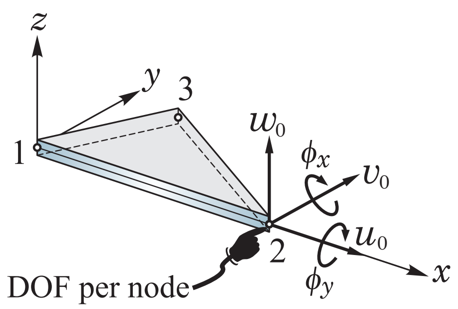


Figure 1. Degrees of freedom of the triangular plate element.

Another way to represent the displacement field polynomials expressed in Eq. (1), but in this time making use of strain gradient notation is as follows:

$$u(x,y,z) = [u]_0 + [\varepsilon_x]_0\, x + \left[\frac{\gamma_{xy}}{2} - \phi_z\right]_0 y + [\varepsilon_{x,y}]_0\, xy + \left[\frac{\gamma_{xy}}{2} + \phi_x\right]_0 z + \left[\frac{\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y}}{2}\right]_0 yz \tag{14}$$

$$v(x,y,z) = [v]_0 + \left[\frac{\gamma_{xy}}{2} + \phi_z\right]_0 x + [\varepsilon_y]_0 y + [\varepsilon_{y,x}]_0 xy + \left[\frac{\gamma_{xy}}{2} - \phi_y\right]_0 z + \left[\frac{\gamma_{xy,z} + \gamma_{yz,x} - \gamma_{xz,y}}{2}\right]_0 xz \quad (15)$$

$$w(x,y) = [w]_0 + \left[\frac{\gamma_{xz}}{2} - \phi_x\right]_0 x + \left[\frac{\gamma_{yz}}{2} + \phi_y\right]_0 y \quad (16)$$

$$\phi_x(x,y) = \left[\frac{\gamma_{xz}}{2} + \phi_x\right]_0 + [\varepsilon_{x,z}]_0 x + \left[\frac{\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y}}{2}\right]_0 y + [\varepsilon_{x,yz}]_0 xy \quad (17)$$

$$\phi_y(x,y) = \left[\phi_y - \frac{\gamma_{yz}}{2}\right]_0 + \left[\frac{-\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y}}{2}\right]_0 x + [\varepsilon_{y,z}]_0 y + [\varepsilon_{y,xz}]_0 xy \quad (18)$$

According to inspection of these expressions, it is found a link between the displacements and the rigid body modes, i.e. a function which related them.

The deformation polynomial expansions obtained from the application of theory of elasticity and write in strain gradient notation are:

$$\varepsilon_x = [\varepsilon_x]_0 + [\varepsilon_{x,y}]_0 x + [\varepsilon_{x,z}]_0 z + [\varepsilon_{x,yz}]_0 yz \quad (19)$$

$$\varepsilon_y = [\varepsilon_y]_0 + [\varepsilon_{y,x}]_0 y + [\varepsilon_{y,z}]_0 z + [\varepsilon_{y,xz}]_0 xz \quad (20)$$

$$\gamma_{xy} = [\gamma_{xy}]_0 + [\varepsilon_{x,y}]_0 x + [\varepsilon_{y,x}]_0 y + [\gamma_{xy,z}]_0 z + [\varepsilon_{y,xz}]_0 yz + [\varepsilon_{x,yz}]_0 xz \quad (21)$$

$$\gamma_{yz} = [\gamma_{yz}]_0 + [\gamma_{yz,x}]_0 x + [\varepsilon_{y,z}]_0 y + [\varepsilon_{y,xz}]_0 xy \quad (22)$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\gamma_{xz,y}]_0 y + [\varepsilon_{x,z}]_0 x + [\varepsilon_{x,yz}]_0 xy \quad (23)$$

Being physically interpretable is the characteristic of strain gradient notation which permits to make a prior evaluation of the finite element model. Inspection of the normal strain expansions in Eq. (19) and Eq. (20) show the only presence of strain states associated to the corresponding normal strains. However, analysing Eq. (21), Eq. (22) and Eq. (23), it is noted that the expansions for shear strains contains spurious terms, which increase the shear strain energy of the element unduly. This phenom is know by parasitic shear and causes locking. These terms with first and second derivatives of the normal deformation arise in the formulation due to the incompatibility between the polynomials used to approximate the displacement field of the element (Belo, 2006). Thus, removing the spurious terms from the equations cited in order to correct the element results in:

$$\gamma_{xy} = [\gamma_{xy}]_0 + [\gamma_{xy,z}]_0 z \quad (24)$$

$$\gamma_{yz} = [\gamma_{yz}]_0 + [\gamma_{yz,x}]_0 x \quad (25)$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\gamma_{xz,y}]_0 y \quad (26)$$

It is noteworthy that the presence of such terms is inherent to the element formulation process, but its proper identification and removal is ensured by the strain gradient notation employing. Clearly, therefore, the relevant advantage of this notation: the possibility of to make an early analysis of the modeling capacity of each element.

4. NUMERICAL APPLICATIONS

Starting from the expansion, based on the definitions of the theory of elasticity, already made, this section shows how the spurious terms can be identified. It is note not all expansions coeficients are related to Taylor series, but in the case of the normal strain expansions. The shear strain ones show terms called parasitc shear which should be selectively eliminated. Otherwise the element will have a false resistance to deformation causing locking.

The efficiency of a laminated composite plate finite element using the strain gradient notation is easily confirm when the numerical results are compared with analytical solutions obtained from the work of Reddy (2004).

4.1 Case # 1

In this section, a isotropic plate is solved using the three-noded element described. The problem is solved first with meshes containing the spurious terms, and then after their elimination. Solutions are compared to show the effects of the spurious terms.

The plate is subjected to a uniforme load as shown in Fig. 2 . The mechanical properties are $E = 75$ GPa and $\nu_{12} = 0.3$ and dimensions of laminae are the following: $a = b = 1$ m. To show the strength of the element, various side-to-thickness ratio are adopted: $a/h = 10$, $a/h = 20$, $a/h = 50$ and $a/h = 100$.

To compute numerical results for maximum transverse displacement, the following nondimensionalizations are used:

$$\overline{w} = w_0 \left(\frac{E_2 h^3}{a^4 q_0}\right) \quad (27)$$

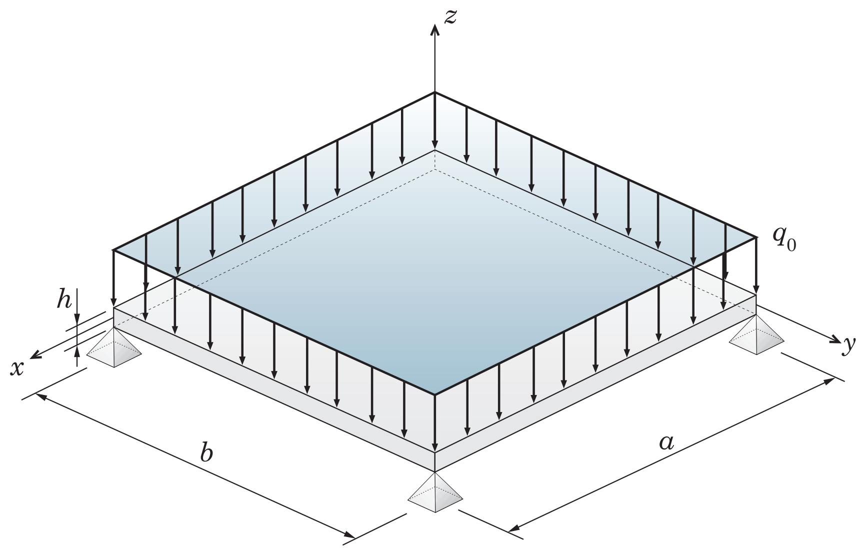


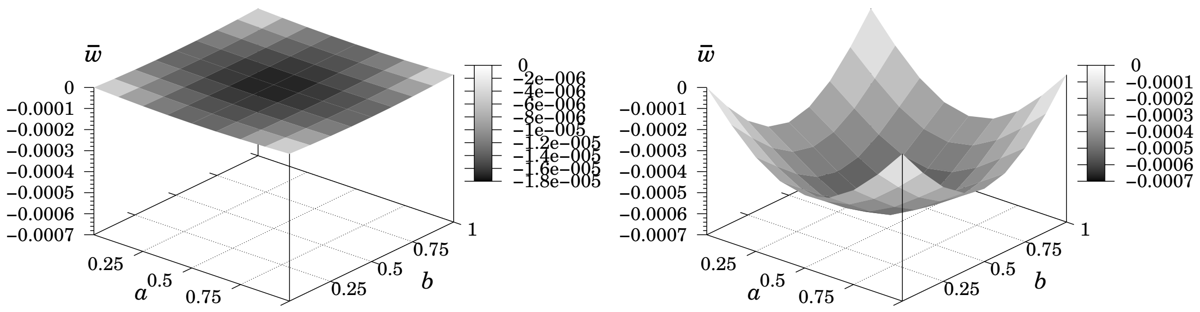
Figure 2. Geometry and boudary conditions of case 1.

The problem is solved using two uniform meshes; namely, 4×4 and 8×8. Table 1 shows the values for the different thicknesses of the plate to the non-dimensinalized transverse displacement. The meshes are run first with the elements containing parasitic shear terms (values in parentheses), and then with the elements corrected for parasitic shear.

Table 1. Non-dimensionalized transverse displacement, $\bar{w}$, at the center of the plate.

a/h	Analytical solution	Meshes 4×4	Meshes 8×8
10	6.613E-04	5.863E-04 (1.815E-06)	6.076E-04 (2.986E-06)
20	6.613E-04	6.029E-04 (4.402E-06)	6.138E-04 (5.942E-06)
50	6.613E-04	6.004E-04 (4.883E-06)	6.297E-04 (1.767E-05)
100	6.613E-04	6.260E-04 (6.506E-05)	6.407E-04 (6.418E-06)

The table clearly depict the stiffening effects of the spurious terms at the meshes. This effect can be more easily seen from Fig. 3(a) to 3(b). From results shown in Tab. 1, the percent error in the strain gradient element corrected, side-to-thickness ratio $a/h = 100$ (to emphasize the shear locking) and 8×8 is 3.12% against 99.03% in the element containing spurious terms.



(a) 8×8 with parasitic shear

(b) 8×8 without parasitic shear

Figure 3. Non-dimensionalized transverse displacement, $\bar{w}$, at the center of the plate for side-to-thickness ratio $a/h = 100$: 8×8 mesh.

4.2 Case # 2

The second problem is a square simply supported plate subjected to a uniform load ($q_o = 10$ N/m^2) - Fig. 4. It is composed of four laminae with lamination scheme ($0^o/90^o/90^o/0^o$). The mechanical properties and dimensions of the laminae are the following: $E_1 = 175$ GPa, $E_2 = 7$ GPa, $G_{12} = 3.5$ GPa, $G_{13} = 3.5$ GPa, $G_{23} = 1.4$ GPa, $\nu_{12} = 0.25$, $h_1 = h_2 = h_3 = h_4 = 2.5$ mm, $a = 1$ m, $b = 1$ m. To show the effects of locking, a thin plate with side-to-thickness ratio $a/h = 100$ was chosen.

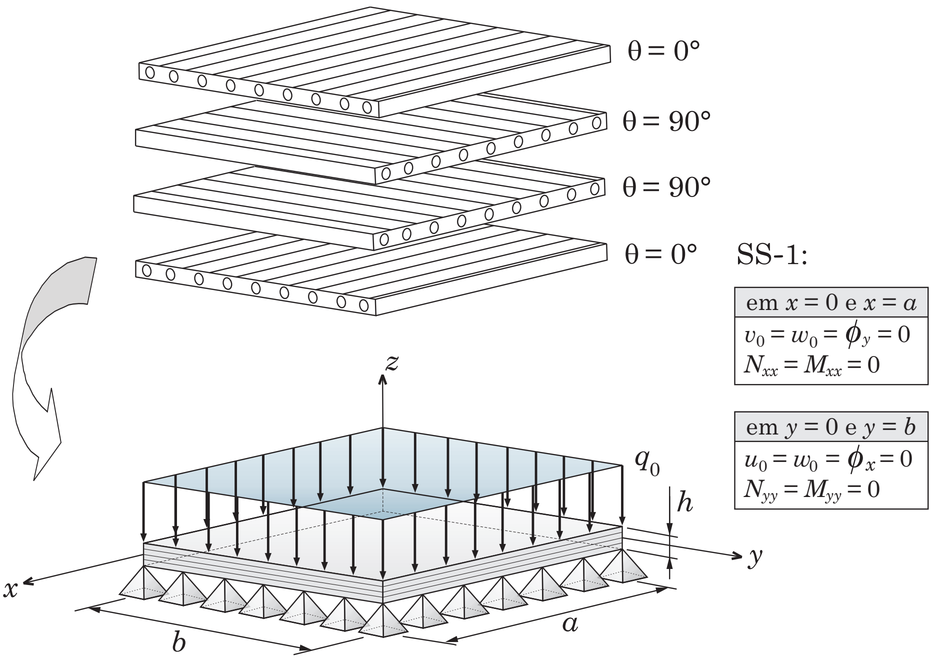


Figure 4. Geometry, lamination scheme and boudary conditions of case 2.

The problem is solved using two uniform meshes; namely, 4×4 and 8×8. The meshes are run first with the elements containing parasitic shear terms, and then with the elements corrected for parasitic shear. The results are depicted in Tab. 2.

Table 2. Non-dimensionalized transverse displacement, $\bar{w}$, at the center of the laminated plate (0/90/90/0).

	Analytical	Meshes	
a/h	solution	4×4	8×8
100	6.843E-02	6.145E-02 (4.506E-04)	6.429E-02 (3.837E-03)

As shown, the percent error in the strain gradient element with parasitc shear and 8×8 is 94.39%, against 6.05% in the element corrected.

5. CONCLUSIONS

In this paper, a brief approach about the shear locking problem in the three-noded laminated composite plate finite element was made. Also, it was shown how this problem affects the numerical results good behavior and that it happens due to presence of spurious terms. Furthermore, the convergence is attained quicker with the mesh refinement, but not always this convergence occurs in the case of a locked model.

It can be concluded the strain gradient notation is reliable and allows modeling a locking-free element with a physical interpretable notation since the first steps of the formulation.

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7. REFERENCES

Abdalla, J.E. and Dow, J.O., 1994. "An error analysis approach for laminated composite plate finite element models". *Computers and Structures*, Vol. 52, No. 4, pp. 611–616.

Belo, I.M., 2006. *Análise eficiente de compósitos laminados planos utilizando-se a formulação de elementos finitos corrigida a priori sem os efeitos do travamento*. Master's thesis, Pontifícia Universidade Católica do Paraná.

Belo, I.M., Abdalla, J.E. and Machado, R.D., 2005. "Shear locking analysis in a serendipity laminated composite plate model". In *XXVI Iberian Latin-American Congress on Computational Methods in Engineering CILAMCE*.

Bonet, J. and Wood, R.D., 2008. *Nonlinear continuum mechanics for finite element analysis*. Cambridge University Press, 2nd edition.

Chiaverini, V., 1986. *Tecnologia Mecânica*. São Paulo.

Dow, J.O., Ho, T.H. and Cabiness, H.D., 1985. "A generalized finite element evaluation procedure". *Journal of Structural Engineering, ASCE*, Vol. 111, No. ST2, pp. 435–452.

Felippa, C. and Haugen, B., 2005. "A unified formulation of small-strain corotational finite elements: I. theory". *Comput. Methods Appl. Mech. Engrg. 194*, Vol. 21-24, pp. 2285–2335.

Hughes, T.J.R., 2000. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Dover Publications, New York, USA.

Jones, R.M., 1975. *Mechanics of Composite Materials*. Hemisphere Publishing Corporation, New York.

Norton, R.L., 2004. *Projeto de máquinas: uma abordagem integrada*. Porto Alegre.

Reddy, J.N., 1997. *Mechanics of Laminated Composite Plates: Theory and Analysis*. CRC Press, Boca Raton.

Reddy, J.N., 2004. *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*. CRC Press, Boca Raton.

Vinson, J.R. and Sierakowski, R.L., 2002. "The behavior of structures composed of composite materials". *Kluwer Academic Publishers*.

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