# The CQM-Based BEM: Implementation to Engineering Problems

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***Abstract.*** *In this work, the Boundary Element Method based on the Convolution Quadrature Method for the numerical solution of integral formulations in the time domain is described. In the present approach the basic integral equation of the BEM is numerically approximated by a quadrature formula, whose weights are calculated using the Laplace transform of the fundamental solution. An important feature of this method is its applicability to problems where the fundamental solution in time-domain does not exist, or where its computational implementation is very difficult. In terms of the numerical implementation, the proposed formulation only requires to define the size of the time-step, and this feature is one of the advantages of the actual formulation with respect to other numerical methods that operate directly in the Laplace transformed domain. When very fine meshes are required, the computation of the influent matrices is very time-consuming. This work shows how the matrices of this integral representation can be computed using the fast Fourier transform to reduce significantly the computational complexity of this task. Some numerical examples of typical problems of wave propagation and transient heat conduction are presented to demonstrate the versatility and reliability of the method.*

***Keywords:*** *Boundary Element Method, Convolution Quadrature Method, Wave propagation, Heat conduction*

## 1. INTRODUCTION

This work presents a formulation of a time-domain Boundary Element Method (TD-BEM) for the analysis of scalar wave propagation and heat transfer problems. TD-BEM approaches present convolution integrals with respect to the time variable. One of the recognized disadvantages of the classical TD-BEM approaches lie in the high computational cost concerning the calculation of the matrices, and also the evaluation of the time-dependent integrals due to the convolution performed from the initial time to the current time. In this work the TD-BEM employs the Convolution Quadrature Method (CQM) firstly described by Lubich (1988a,b) and Lubich and Schneider (1992). In the CQM, fundamental solutions in the Laplace transformed-domain are considered and a numerical approximation of the basic integral equations of the TD-BEM is worked out by a quadrature formula based on a linear multi-step method. The main advantage of the CQM-based BEM is that provides a direct procedure to obtain a stable TD-BEM approach, and that it can be applied to problems where the TD fundamental solution is not available or has a very difficult expression.

In the works (Gaul and Schanz, 1999; Schanz, 2001; Schanz *et al.*, 2005) can be found implementations of the CQM-based BEM to elastodynamics, viscoelasticity and poroelasticity problems. The works (Abreu *et al.*, 2003, 2006, 2009) applied the CQM-based BEM to scalar wave propagation problems, and applications for potential theory can be found in the paper (Vera-Tudela *et al.*, 2006). Furthermore, with the aim to accelerate and improve the computational efficiency, it was combined with the multipole method to formulate a CQM-based BEM for diffraction of waves problems (Saitoh *et al.*, 2004, 2007). All approaches shown that CQM-based BEM is very robust and suitable to such kind of problems.

The BEM has been already also used to solve transient heat conduction problems. In the literature, different BEM approaches have been used to address this topic. One approach is by using convolution schemes, where the TD fundamental solution is introduced to state a transient boundary integral equation model (Wrobel, 2002). Other approach is to define a time-stepping procedure: this approach requires domain integration which, in their turn, can be addressed using the dual reciprocity technique, triple reciprocity technique or similar ones (Tanaka and Chen, 2001; Kassab and Divo, 2006; Ochiai *et al.*, 2006). Other common alternative approaches to address transient heat problems consist of the use of the Laplace transform and its inverse (Rizzo and Shippy, 1970). In these approaches to recover the real TD solution, the result obtained in the Laplace domain is inverted by means of the inverse Laplace transform. However, special methods for numerical Laplace inversion scheme are required and the most commonly used is the Stehfest algorithm (Stehfest, 1970; Kassab and Divo, 2006).

In the present work, a CQM-based BEM is used for the TD solution of two-dimensional wave propagation and transient heat conduction problems. It is also presented how the CQM-based BEM can be implemented using fast Fourier transform (FFT) technique to reduce the computational cost of the numerical calculation of the matrices. In the following sections, the CQM-based BEM formulation is reviewed, the governing equations and their integral representations are introduced for each kind of problem. Numerical examples that show the effectiveness of the proposed formulation are presented.

## 2. THE CONVOLUTION QUADRATURE METHOD (CQM)

Consider first the following equation:

$$y(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau \tag{1}$$

Equation (1) represents a convolution. In the work (Lubich, 1988a,b) has been showed that function $y$ can be approximated at points $n\Delta t$ as the following quadrature:

$$y(n\Delta t) \approx \sum_{k=0}^{n} \omega_{n-k}(\Delta t) g(k\Delta t)\,, \quad \text{with} \quad n = 0, 1, \ldots, N \tag{2}$$

where $N$ is the total number of time sampling and the weights $\omega_n$ are:

$$\omega_n(\Delta t) = \frac{1}{2\pi \mathrm{i}} \int_{C_R} \hat{f}\left(\frac{\gamma(z)}{\Delta t}\right) z^{-n-1}\mathrm{d}z \approx \frac{R^{-n}}{L} \sum_{\ell=0}^{L-1} \hat{f}\left(\frac{\gamma(R\, e^{\mathrm{i}\ell\frac{2\pi}{L}})}{\Delta t}\right) e^{-\mathrm{i}n\ell\frac{2\pi}{L}} \tag{3}$$

The function $\hat{f}$ is the Laplace transform of $f$ and $C_R = \{z \in \mathbb{C}; |z| = R\}$ is the contour employed to perform the integration where $R$ is the radius of a circle in the domain of analyticity of $\hat{f}\left(\frac{\gamma(z)}{\Delta t}\right)$. In Eq. (3) a polar coordinate system was adopted and the integral was approximated by the trapezoidal rule with $L$ steps equal to $2\pi/L$. The function $\gamma$ is the quotient of the characteristic polynomial generated by a linear multi-step method. Using a backward differentiation formula of order 2, the expression of $\gamma$ is given by:

$$\gamma(z) = 3/2 - 2z + (1/2)z^2 \tag{4}$$

Setting $L = N$ and $R^N = \sqrt{\varepsilon}$ in Eq. (3), the quadrature weights are computed within an error of order $O(\varepsilon)$, where $\varepsilon$ is the precision with which $\hat{f}$ is calculated.

## 3. GOVERNING EQUATIONS AND INTEGRAL REPRESENTATIONS

In this section are introduced the governing equations of the problems analyzed. These equations are the scalar wave equation that governs problems such as acoustic propagation, and the equation that governs energy transport processes such as heat conduction problems. In subsections are also introduced the integral representation of each differential equations.

### 3.1 Wave propagation problems

The scalar wave equation for a homogeneous and isotropic body $\Omega$ in the absence of body forces is given by (Morse and Feshbach, 1953):

$$c^2\, \nabla^2 u(X,t) - \frac{\partial^2 u(X,t)}{\partial t^2} = 0 \tag{5}$$

where $t \in \mathbb{R}$ and $X \in \mathbb{R}^2$ represent the time and space coordinates, respectively, $c$ is the wave propagation velocity and the function $u$ represents the potential. The initial conditions at time $t_0$ and the boundary conditions for $t > t_0$ are:

$$u(X,t_0) = u_0(X) \quad \text{and} \quad \frac{\partial u}{\partial t}(X,t_0) = v_0(X) \quad \text{in } \Omega \tag{6}$$

$$u(X,t) = \bar{u}(X,t) \quad \text{on } \Gamma_u \quad \text{and} \quad p(X,t) = \frac{\partial u}{\partial n}(X,t) = \bar{p}(X,t) \quad \text{on } \Gamma_p \tag{7}$$

Note that, in above equations $\Gamma_u \cup \Gamma_p = \Gamma$, where $\Gamma$ represents the boundary of the body $\Omega$. $p$ is the flux and $n$ is the outward unit normal to $\Gamma$. The wave propagation problem consists to solve Eq. (5) for the unknown potential $u$ when the material properties (incorporated in $c$), the initial conditions and boundary conditions are known. The time-dependent integral equation corresponding to problems governed by the 2D scalar wave equation with homogeneous initial conditions for any point source $\xi$ is (Brebbia *et al.*, 1984; Mansur, 1983; Dominguez, 1993):

$$c(\xi)u(\xi,t) = \int_\Gamma \int_{t_0}^{t^+} u^*(\boldsymbol{r}, t-\tau)p(X,\tau)\,\mathrm{d}\tau\mathrm{d}\Gamma - \int_\Gamma \int_{t_0}^{t^+} p^*(\boldsymbol{r}, t-\tau)u(X,\tau)\,\mathrm{d}\tau\mathrm{d}\Gamma \tag{8}$$

where $\boldsymbol{r} = \xi - X$, and $X$ represents the field point. The function $u^*$ is the TD fundamental solution and $p^*(\boldsymbol{r}, t-\tau) = \frac{\partial u^*}{\partial n}(\boldsymbol{r}, t-\tau)$.

### 3.2 Heat conduction problems

The heat equation that describes the evolution of the temperature $T$ within a homogeneous and isotropic material of constant thermal conductivity $K$, specific heat $c_e$, mass density $\rho$ and in the absence of sources of heat is given by (Özişik, 1993; Carslaw and Jaeger, 1988; Crank, 1975):

$$\frac{\partial T(X,t)}{\partial t} - \kappa \nabla^2 T(X,t) = 0 \tag{9}$$

where $t \in \mathbb{R}$, $X \in \mathbb{R}^2$ and $\kappa$ is the thermal diffusivity given by $\kappa = K/(\rho c_e)$. Note that, in this model the material properties $K$, $c_e$ and $\rho$ do not depend on the time. The initial condition at time $t_0$ and the boundary conditions for $t > t_0$ are:

$$T(X,t_0) = T_0(X) \qquad \text{in } \Omega \tag{10}$$

$$T(X,t) = \bar{T}(X,t) \quad \text{on } \Gamma_T \qquad \text{and} \qquad q(X,t) = \bar{q}(X,t) \quad \text{on } \Gamma_q \tag{11}$$

where $q(X,t) = -k\frac{\partial T}{\partial n}(X,t)$ and $\Gamma_T \cup \Gamma_q = \Gamma$. The heat conduction problem consists to solve Eq. (9) for the unknown variable $T$ when the material properties, the initial condition and boundary conditions are known. The time-dependent integral equation corresponding to problems governed by the 2D heat equation with homogeneous initial condition for any point source $\xi$ is (Wrobel, 2002):

$$c(\xi)T(\xi,t) = \int_\Gamma \int_{t_0}^{t_f} T^*(\boldsymbol{r}, t-\tau) q(X,\tau)\, \mathrm{d}\tau \mathrm{d}\Gamma - \int_\Gamma \int_{t_0}^{t_f} q^*(\boldsymbol{r}, t-\tau) T(X,\tau)\, \mathrm{d}\tau \mathrm{d}\Gamma \tag{12}$$

The function $T^*$ is the TD fundamental solution and $q^*(\boldsymbol{r}, t-\tau) = -\kappa\frac{\partial T^*}{\partial n}(\boldsymbol{r}, t-\tau)$. In Eqs. (8) and (12) the coefficients $c(\xi) = 1$ at any interior point $\xi$. If $\xi \in \Gamma$, these coefficients depend on the local geometry of $\Gamma$ at $\xi$, see (Brebbia *et al.*, 1984; Wrobel, 2002; Dominguez, 1993).

## 4. THE CQM-BASED BEM FORMULATION

To solve the boundary integral equations given by Eqs. (8) and (12) it is required both space and time discretizations. The BEM represents $\Gamma$ and the boundary values of $u$ and $p$ for acoustics problems, $T$ and $q$ for heat conduction problems, by using piece-wise polynomial functions. For this purpose the boundary is divided into $J$ elements $\Gamma_j$ $(j = 1, ...J)$. In this work lineal elements were used for the spatial discretization. The time discretization consists in dividing the time interval of analysis into $N$ time steps of equal size $\Delta t$. A discrete version of Eqs. (8) and (12) using the CQM introduced in Section 2 for a point source $\xi$ and at time $t_n = t_0 + n\Delta t$ $(n = 0, \ldots, N)$ are given, respectively, by:

$$\boldsymbol{c}(\xi_i)\boldsymbol{u}(\xi_i, t_n) = \sum_{j=1}^{J}\sum_{m=0}^{n} \boldsymbol{g_a}(\xi_i, \Delta t)_{n-m}^{j} \boldsymbol{p}_m^j - \sum_{j=1}^{J}\sum_{m=0}^{n} \boldsymbol{h_a}(\xi_i, \Delta t)_{n-m}^{j} \boldsymbol{u}_m^j \tag{13}$$

$$\boldsymbol{c}(\xi_i)\boldsymbol{T}(\xi_i, t_n) = \sum_{j=1}^{J}\sum_{m=0}^{n} \boldsymbol{g_T}(\xi_i, \Delta t)_{n-m}^{j} \boldsymbol{q}_m^j - \sum_{j=1}^{J}\sum_{m=0}^{n} \boldsymbol{h_T}(\xi_i, \Delta t)_{n-m}^{j} \boldsymbol{T}_m^j \tag{14}$$

Note that, a notation that aims to distinguish through the subscripts of acoustic problems "$a$" from the heat problems "$T$" was introduced. The CQM quadrature weights of Eq. (13) and Eq. (14) are:

$$\boldsymbol{g_a}(\xi_i, \Delta t)_n^j = \frac{R^{-n}}{L}\sum_{\ell=0}^{L-1} \int_{\Gamma_j} \hat{u}^*(\boldsymbol{r}, s_\ell) \boldsymbol{N}^j(X) \mathrm{d}\Gamma\, e^{-\frac{2\pi \mathrm{i}}{L} n\ell} \tag{15}$$

$$\boldsymbol{h_a}(\xi_i, \Delta t)_n^j = \frac{R^{-n}}{L}\sum_{\ell=0}^{L-1} \int_{\Gamma_j} \hat{p}^*(\boldsymbol{r}, s_\ell) \boldsymbol{N}^j(X) \mathrm{d}\Gamma\, e^{-\frac{2\pi \mathrm{i}}{L} n\ell} \tag{16}$$

$$\boldsymbol{g_T}(\xi_i, \Delta t)_n^j = \frac{R^{-n}}{L}\sum_{\ell=0}^{L-1} \int_{\Gamma_j} \hat{T}^*(\boldsymbol{r}, s_\ell) \boldsymbol{N}^j(X) \mathrm{d}\Gamma\, e^{-\frac{2\pi \mathrm{i}}{L} n\ell} \tag{17}$$

$$\boldsymbol{h_T}(\xi_i, \Delta t)_n^j = \frac{R^{-n}}{L} \sum_{\ell=0}^{L-1} \int_{\Gamma_j} \hat{q}^*(\boldsymbol{r}, s_\ell) \boldsymbol{N}^j(X) \mathrm{d}\Gamma \, e^{-\frac{2\pi \mathrm{i}}{L} n\ell} \tag{18}$$

In the previous expressions, $\boldsymbol{N}^j(X)$ represents the matrix of interpolation functions used in the spatial discretization. The parameter $s$ of the Laplace transform is numerically evaluated as $s_\ell = \gamma(Re^{2\pi \mathrm{i}/L\ell})/\Delta t$, and $\gamma$ was introduced in Eq. (4). The terms $\hat{u}^*(\boldsymbol{r}, \cdot)$ and $\hat{p}^*(\boldsymbol{r}, \cdot)$ are the Laplace transform of $u^*(\boldsymbol{r}, \cdot)$ and $p^*(\boldsymbol{r}, \cdot)$. Moreover, $\hat{T}^*(\boldsymbol{r}, \cdot)$ and $\hat{q}^*(\boldsymbol{r}, \cdot)$ are the Laplace transform of $T^*(\boldsymbol{r}, \cdot)$ and $q^*(\boldsymbol{r}, \cdot)$. The expressions of these fundamental solutions in the transformed domain are given by (Brebbia *et al.*, 1984; Wrobel, 2002):

$$\hat{u}^*(\boldsymbol{r}, s) = 2K_0(s\,r) \quad \text{and} \quad \hat{p}^*(\boldsymbol{r}, s) = -2sK_1(s\,r)\frac{\partial r}{\partial n} \tag{19}$$

$$\hat{T}^*(\boldsymbol{r}, s) = \frac{1}{2\pi\kappa} K_0\left(\sqrt{\frac{s}{\kappa}}\, r\right) \quad \text{and} \quad \hat{q}^*(\boldsymbol{r}, s) = -\frac{1}{2\pi}\sqrt{\frac{s}{\kappa}}\, K_1\left(\sqrt{\frac{s}{\kappa}}\, r\right)\frac{\partial r}{\partial n} \tag{20}$$

where $K_\nu$ is the modified Bessel function of the second kind and order $\nu$.

The vectors $\boldsymbol{u}_m^j$ and $\boldsymbol{p}_m^j$ of Eq. (13) represent, respectively, the prescribed or unknown nodal values of potential $u$ and flux $p$ defined at each element $j$ of the boundary and at the time step $m$ ($m = 0, 1, \ldots, N$), i.e.:

$$\boldsymbol{u}_m^j = \boldsymbol{u}^j(t_m) = \boldsymbol{u}^j(t_0 + m\Delta t) \tag{21}$$

$$\boldsymbol{p}_m^j = \boldsymbol{p}^j(t_m) = \boldsymbol{p}^j(t_0 + m\Delta t) \tag{22}$$

The vectors $\boldsymbol{T}_m^j$ and $\boldsymbol{q}_m^j$ of Eq. (14) which represent, respectively, the prescribed or unknown nodal values of temperature $T$ and heat flux $q$ defined at each element $j$ of the boundary and at the time step $m$ are given by:

$$\boldsymbol{T}_m^j = \boldsymbol{T}^j(t_m) = \boldsymbol{T}^j(t_0 + m\Delta t) \tag{23}$$

$$\boldsymbol{q}_m^j = \boldsymbol{q}^j(t_m) = \boldsymbol{q}^j(t_0 + m\Delta t) \tag{24}$$

Equations (13) and (14) can be rewritten in matrix form as follows:

$$\boldsymbol{c}\boldsymbol{u}^n = \sum_{m=0}^{n} \boldsymbol{G_a}^{n-m}\boldsymbol{p}^m - \sum_{m=0}^{n} \boldsymbol{H_a}^{n-m}\boldsymbol{u}^m \tag{25}$$

$$\boldsymbol{c}\boldsymbol{T}^n = \sum_{m=0}^{n} \boldsymbol{G_T}^{n-m}\boldsymbol{q}^m - \sum_{m=0}^{n} \boldsymbol{H_T}^{n-m}\boldsymbol{T}^m \tag{26}$$

The responses on the boundary and at interior points are calculated from Eq. (25), where $\boldsymbol{G_a}$ and $\boldsymbol{H_a}$ are the BEM influence matrices for acoustic problems. For heat conduction problems, the responses are calculated from Eq. (26) and $\boldsymbol{G_T}$ and $\boldsymbol{H_T}$ are the corresponding BEM influence matrices. In Eqs. (25) and (26) $\boldsymbol{c}$ is the diagonal matrix containing the coefficients $c(\xi)$. The indices $n$ and $m$ correspond to the discrete times $t_n = t_0 + n\Delta t$ and $t_m = t_0 + m\Delta t$, respectively. To compute the responses on the boundary, the boundary conditions have to be imposed into Eqs. (25) and (26). Afterwards, the following general expressions are obtained:

$$\boldsymbol{A_a}^0\boldsymbol{y}^n = \boldsymbol{f}^n + \sum_{m=0}^{n-1} \left(\boldsymbol{G_a}^{n-m}\boldsymbol{p}^m - \boldsymbol{H_a}^{n-m}\boldsymbol{u}^m\right) \tag{27}$$

$$\boldsymbol{A_T}^0\boldsymbol{y}^n = \boldsymbol{f}^n + \sum_{m=0}^{n-1} \left(\boldsymbol{G_T}^{n-m}\boldsymbol{q}^m - \boldsymbol{H_T}^{n-m}\boldsymbol{T}^m\right) \tag{28}$$

For acoustic problems, i.e in Eqs. (27), $\boldsymbol{A_a}^0$ stores the columns of $(\boldsymbol{c} + \boldsymbol{H_a}^0)$ corresponding to the unknown values of $\boldsymbol{u}$ and the columns of $\boldsymbol{G_a}$ corresponding to the unknowns values of $\boldsymbol{p}$. The unknown and known values of $\boldsymbol{u}$ and $\boldsymbol{p}$ at time $t_n$ are stored, respectively, in the vectors $\boldsymbol{y}^n$ and $\boldsymbol{f}^n$. The components of the system given by in Eqs. (28) have the same meaning for heat conduction problems.

It is important to notice that the quadrature weights must be obtained efficiently using the FFT algorithm (Brigham, 1988) if the quadrature weights from Eq. (15) to Eq. (18) were rewritten as:

$$\hat{\boldsymbol{u}}_\ell^{*j} = \int_{\Gamma_j} \hat{u}^*(\boldsymbol{r}, s_\ell)\boldsymbol{N}^j(X)\mathrm{d}\Gamma \quad \text{and} \quad \hat{\boldsymbol{p}}_\ell^{*j} = \int_{\Gamma_j} \hat{p}^*(\boldsymbol{r}, s_\ell)\boldsymbol{N}^j(X)\mathrm{d}\Gamma \tag{29}$$

$$\hat{\boldsymbol{T}}_\ell^{*j} = \int_{\Gamma_j} \hat{T}^*(\boldsymbol{r}, s_\ell)\boldsymbol{N}^j(X)\mathrm{d}\Gamma \quad \text{and} \quad \hat{\boldsymbol{q}}_\ell^{*j} = \int_{\Gamma_j} \hat{q}^*(\boldsymbol{r}, s_\ell)\boldsymbol{N}^j(X)\mathrm{d}\Gamma \tag{30}$$

Thus, for example, to obtain $(\boldsymbol{g_a})_n$ and $(\boldsymbol{h_a})_n$ it is enough to calculate the FFT transform of $\hat{\boldsymbol{u}}_\ell^*$ and $\hat{\boldsymbol{p}}_\ell^*$ and multiply the results by the factor $R^{-n}/L$. Using the FFT algorithms one quadrature weight can be obtained with a number of operations the order $N\log(N)$, keeping in mind that $N = L$. In addition, the element integrals should be performed before the FFT in order to reduce the number of times that the FFT routine is called. Same approach can be followed to obtain the other quadrature weights.

## 5. NUMERICAL EXAMPLES

Next, to validate the formulation presented in this work two numerical examples are shown. The numerical responses of wave propagation and heat conduction problems for two geometrical configurations were compared to the analytical responses (Morse and Feshbach, 1953; Graff, 1975; Carslaw and Jaeger, 1988). In the treatment of wave propagation problem will be made reference to the dimensionless parameter $\beta = c\,\Delta t/l$ that gives a measure of the time-step length to be adopted to perform the numerical analysis ($l$ is the smaller boundary element length). In all the numerical examples was taken $L = N$ and $\varepsilon = 10^{-4}$. The acoustic and heat sources are not present (homogeneous equations). Zero initial conditions were fixed at $t_0 = 0.0$, i.e., $u_0(X) = 0.0$, $v_0(X) = 0.0$ and $T_0(X) = 0.0$. For the CQM procedure it was taken $\varepsilon = 10^{-4}$ and the expresion of the function $\gamma(z)$ is given by Eq. (4).

### 5.1 Wave propagation example

This first example consists of a one-dimensional rod of longitudinal elasticity modulus $E$ under a Heaviside-type forcing function applied at $t = 0$ as $\bar{p} = \frac{p}{E}H(t-0)$, where $H$ is the Heaviside function. This boundary condition is kept constant from time $t = 0$ onwards according to Fig. 1. A boundary element mesh of $J = 24$ linear elements was used. A time interval from $t_0 = 0.0$ to $t_f = 4.15$ was analyzed. The time discretization consists of 128 time-steps of $\Delta t = 0.032$, therefore, the parameter $\beta = 0.3$.

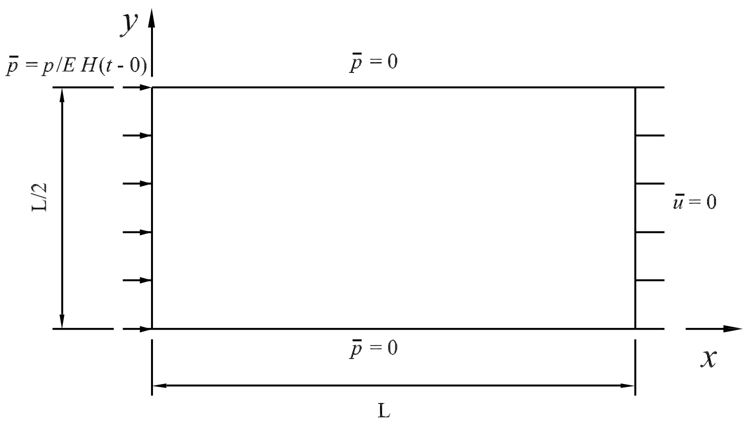


Figure 1. One-dimensional rod analysis: geometry definitions and boundary conditions.

Figure 2 presents the results of potential time-histories at boundary node $(0, L/4)$ and at interior points $(L/4, L/4)$, $(L/2, L/4)$ and $(3L/4, L/4)$ carried out with $\beta = 0.3$. Flux time-history is depicted in Fig. 3 at boundary node $(L, L/4)$ for the same value of $\beta$.

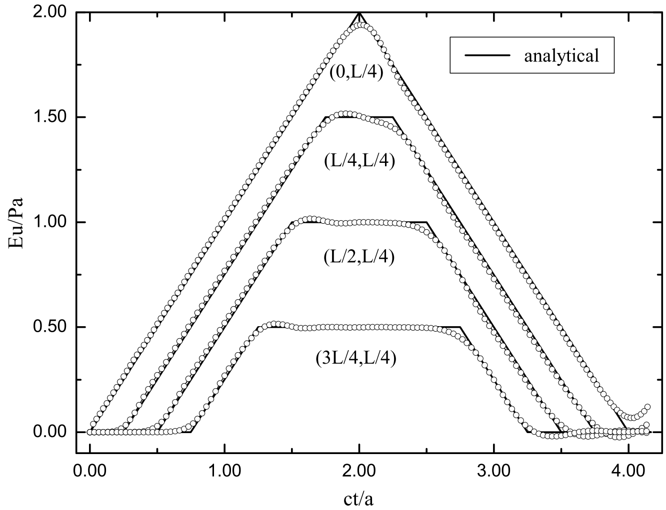


Figure 2. Potential at boundary nodes and interior points for the one-dimensional rod analysis. Circles: numerical results.

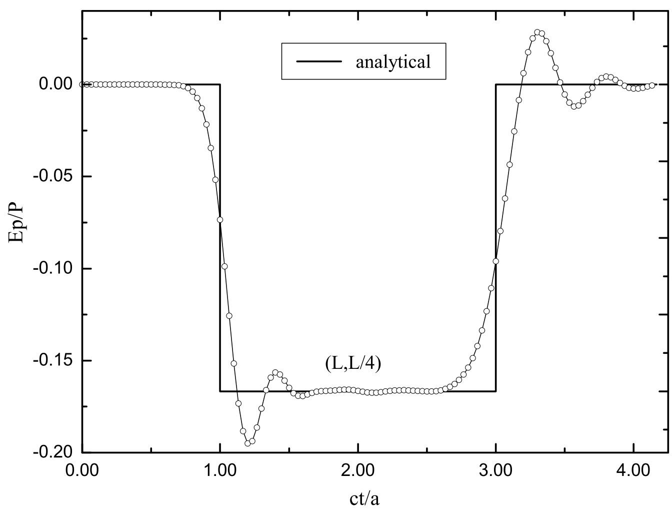


Figure 3. Flux at boundary node $(L, L/4)$ for the one-dimensional rod analysis. Circles: numerical results.

From the calculated responses one can observe that Fig. 2 and 3 show that all the numerical responses are accurate when compared with the exact solution. Moreover, it is important to point out that Fig. 3 displays typical numerical oscillations for boundary flux around the discontinuities for the CQM-based BEM numerical results. In general, this oscillatory tendency concentrates around the response discontinuity.

### 5.2 Heat conduction example

In this example is considered the transient heat conduction analysis in a circular region of radius $R = 10.0$. The geometry of this region is depicted in Fig. 4.

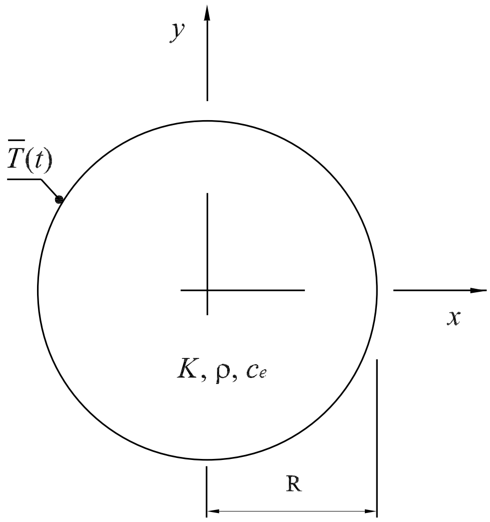


Figure 4. Circular region: geometry and boundary condition.

The thermal diffusivity is $\kappa = 4.0$ and a boundary mesh of $J = 48$ linear elements was used. A time interval from $t_0 = 0.0$ to $t_f = 20.0$ was analyzed. The time discretization consists of 2048 time-steps of $\Delta t = 0.00977$. Dirichlet harmonic boundary condition was prescribed. The expression of this boundary condition is given by:

$$T(r,t)|_{r=10} = \bar{T}(t) = 10\left[1 - \cos(\frac{\pi}{2}\,t)\right] \tag{31}$$

Figure 5 shows the evolution of the temperature at two interior points of coordinates $(0.0, 0.0)$ and $(8.0, 0.0)$. Figure 6 shows the temperature at time $t = 10.0$ on the horizontal diameter of the circular region with the Dirichlet harmonic boundary condition prescribed.

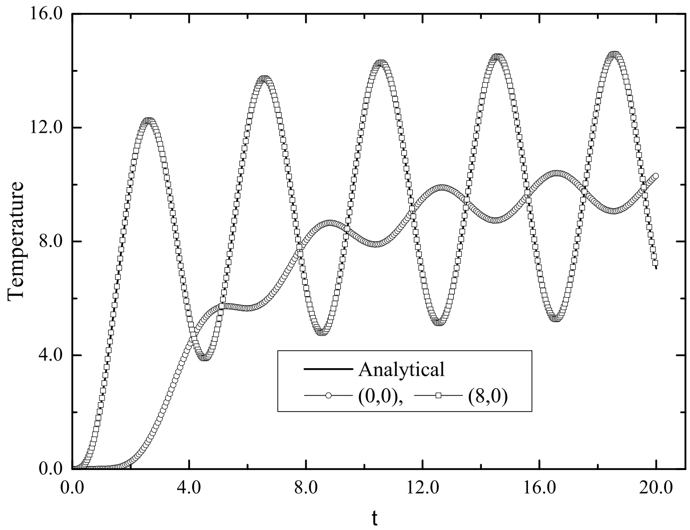


Figure 5. Evolution of the temperature at two interior points of the circular region.

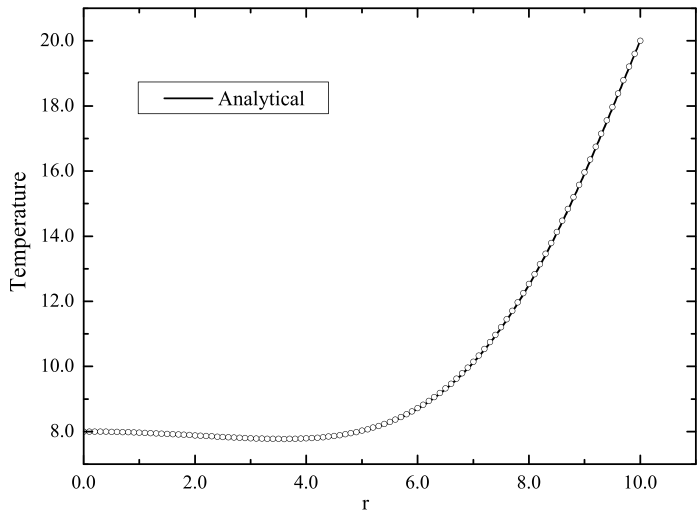


Figure 6. Temperature at time $t = 10.0$ on the horizontal diameter of the circular region. Circles: numerical results.

Usually, in what concerns numerical accuracy $\Delta t$ can be as small as the machine accuracy allows. However, the authors experience demonstrated that excessively reduced of $\Delta t$ for the wave propagation problems may not be accurate if the backward difference formula of second order (see Eq. (4)) is employed. In this case, a backward difference formula of first order should be employed.

For a fixed boundary mesh the behavior of the numerical responses for heat conduction problems are accurate when both backward difference formula of second and first order are employed independent of the reduction of $\Delta t$.

## 6. CONCLUSION

In this work a CQM-based BEM formulation was presented to the computation of numerical responses of two-dimensional problems governed by the scalar wave equation and the heat equation. The CQM-based BEM is an attractive method for the time discretization of convolution integrals of the BEM in the time-domain. Despite this formulation is a time-stepping procedure, the fundamental solutions are evaluated in the Laplace domain instead in the time-domain, and requires only to know *a priori* the time-step size $\Delta t$ . This represents an interesting advantage regarding the techniques that need an adequate selection of several parameters to perform the numerical Laplace inversion to obtain accurate results. Furthermore, when the fundamental solutions BEM either are not available in the time-domain or have difficult expressions to evaluate in numerical terms, the CQM-based BEM formulation overcomes this difficulty as alternative method of time procedure.

Regarding to the computation cost of the numerical implementation, an important conclusion is achieved: when computing the influent matrices of the BEM the FFT algorithm can be used to reduce the number of operations. However, the cost of the convolution to solve the boundary problem and also the cost of storage is still high due to the FFT cannot be used and a complete storage of the influent matrices is required.

The examples analyzed shown that the proposed formulation is accurate and exhibits a stable behavior with respect to the parameter $\Delta t$, thus the CQM-based BEM is in general well suited for solving transient problems of wave propagation and heat conduction problems.

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## 8. Responsibility notice

The authors are the only responsible for the printed material included in this paper