SINGULAR MASS MATRICES AND HALF DEGREES OF FREEDOM: A GENERAL METHOD FOR SYSTEM REDUCTION

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Abstract. *The development of a mathematical model for a mechanical system is a common task on computational mechanics. Through mathematical description and numerical integration, it is possible to make a better understanding of a system's behavior.*
However, in some cases these models may lead to a system of equations which is not solvable for a wide range of reasons. One of the most common reasons why this system cannot be solved is because its inertia matrix is singular. One of the causes of this singularity is the presence of degrees of freedom without inertia, also known as half degrees of freedom. In this case, many authors have developed works on perturbation theory, or have developed formulations which are used only on the problem under investigation.
The main disadvantage of the use of the perturbation theory is that, once this problem is unsolvable in the form it is still presented, the solution brought by this method is a close solution for the problem, but it is not an exact solution. Furthermore, the eigenproblem for this system is also not exact.
On the other hand, developing a formulation for a specific problem does not allow a widely useful computational implementation of this method because if the system is slightly modified, this formulation has to be developed all over again. Considering a system with a considerably large amount of degrees of freedom, the use of this technique is not encouraged. For this reason, this work brings an outstanding solution for this problem. It shows a powerful formulation, which makes possible to turn a general singular system, with half degrees of freedom, into a smaller system, represented by fewer state variables, but this time exactly solvable, without any loss of generality. Besides, even though a system with half degrees of freedom has improper eigenvalues, the new system brought by this method has exactly the same eigenvalues of the previous one, except the improper ones.
This method is developed based on the fact that, the equation of motion of a degree of freedom without inertia is actually a constraint equation, which can be holonomic, in case it is purely algebraic, or nonholonomic otherwise. Both of those constraints eliminate some state variables from the mathematical description of the mechanical system, what makes of the transformation shown in this work also a reduction.
One of the advantages of this method is the fact that its formulation is valid for any system, since it has half degrees of freedom, and this fact makes it possible to be implemented in an easy computer routine.
Furthermore, regarding of FEM software, it is now possible to leave some degrees of freedom, or even nodes, with null values of inertia, because, after the assembly of the global matrices, if the mass matrix has a null column, it means that there is a half degree of freedom in the system and the method described in this work can be applied directly to the global matrices. This means that this technique can be implemented on any commercial software by simply adding a new step after the assembly of the global matrices. Changes on the other parts of this software are completely needless.
After that, an application of this method on a model a hydroelastic mount of an engine suspension is shown in this work, in which the model of the polymer in contact with oil leads to the presence of half degrees of freedom. Its eigenproblem is solved before and after the application of this method, showing that the proper eigenvalues of the system are not affected by this formulation.

Keywords: *singular mass matrix, system reduction, half degree of freedom, linear, nonlinear*

1. INTRODUCTION

The study of linearized systems is very important as a first step to the investigation of the dynamic behavior of several mechanical systems. Linear analysis procedures are widely used and generally based on the eigenproblem analysis or numerical integration. Both of these demand that the system is represented in a state space form.

The most common state space form is shown in Eq. (2), and it requires a nonsingular inertia matrix. There are many systems which can be represented with inertia matrices with null columns, which is the case of mechanical systems with half degrees of freedom (i.e. degrees of freedom without inertia). Hence, those systems cannot be represented in the usual state space form, because the first derivative of the array of state variables with respect to time cannot be isolated.

Although there are numerical integration routines that perform integration on second order systems, and also do not demand the inversion of the inertia matrix, those routines are not implemented on many platforms, and using the formulation shown in this work, it is possible to perform the same analyses using first order numerical integrators.

Furthermore, there are also routines for the solution of the generalized eigenproblem. These demand that the system is represented in such a state space form that does not require the inversion of the mass matrix. The consequence is that, if the mass matrix is singular, this routine will return the regular eigenvalues of the system, and also undefined eigenvalues, represented by the entities *NaN* (not a number) or $\pm$ *Inf* (positive or negative infinity). The formulation presented in this work performs such a reduction on the system that those undefined eigenvalues disappear, and the regular ones are not influenced.

There are other reduction methods, such as the one presented by Guyan (1964), which is widely used on finite elements dynamic simulations. Nevertheless, the focus of his method is to remove the degrees of freedom at which no forces are applied. As a consequence, it is shown that the eigenvalues of the system are not preserved.

As an extension of the work described above, Bhat and Bernstein (1996) shew that there is always an orthogonal transformation matrix that reduces the system with singular mass matrix, although in their work no general method for system reduction is proposed.

The objective of this work is to present a general formulation for the reduction of systems with null columns in the inertia matrix, the new state space form in which this system can be represented, and also apply this methodology on related problems.

2. FORMULATION

A dynamic LTI (*Linear time-invariant*) system, can be generally represented by inertia, stiffness and damping matrices, as it is shown in the following equation:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \tag{1}$$

In this case, the inertia, damping and stiffness matrices are represented by $[M]$, $[C]$ and $[K]$, respectively. This is a system of second order differential equations. In many cases, it is necessary to represent it in a state space form. One of the possible representations for this system, if the mass matrix is not singular, is the standard:

$$\left\{ \begin{array}{rcl} \{\dot{y}\} & = & [A]\{y\} + [B]\{u\} \\ \{z\} & = & [C]\{y\} + [D]\{u\} \end{array} \right. \tag{2}$$

Where:

$$\begin{array}{rcl} [A] & = & \left[\begin{array}{cc} [0_{n\times n}] & [I_{n\times n}] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{array} \right] \\ [B] & = & \left[\begin{array}{c} [0_{n\times n}] \\ [M]^{-1} \end{array} \right] \\ [C] & = & [I_{2n\times 2n}] \\ [D] & = & [0_{2n\times n}] \end{array}$$

From the theory of linear systems analysis, $[A]$ is called the system matrix, $[B]$ is the input matrix, $[C]$ is the output matrix and $[D]$ is the feed through matrix. The last two are arbitrary.

However, in some cases, it is possible that the system in study is not only represented by second order differential equations, but also by first order and sometimes by algebraic equations. If there are only first and second order differential equations, the mass matrix $[M]$ is singular. On the other hand, when there are algebraic equations, the matrix $[C]$ is also singular. This can be verified by the fact that, if there is a first order equation, one of the columns of the matrix $[M]$ is null. For algebraic equations, the same is valid for the matrix $[C]$.

Hence it is impossible to represent the system using the state space form proposed above because the mass matrix is singular. Furthermore, a single state variable is necessary to describe the physical variables which are differentiated only once. Besides, the algebraic equations show that one of the variables of the system can be calculated from the other state variables (which means it is a constraint equation), so it needs no state variable to be represented. Consequently, the presence of differential equations with order lower than two leads to a reduction of the system, when it is written in state space form.

It is possible to define N_i as the number of second order differential equations, N_c as the number of first order differential equations, and N_k as the number of algebraic equations. For this reason, the number of equations of this system is given by $N = N_i + N_c + N_k$. Consequently, the number of equation of the system in state space form is $N_s = 2N_i + N_c$.

Before proposing the methods for the reduction of the system, some definitions must be made in order to simplify the understanding of further steps.

$[I] \equiv$ Identity matrix.

$[I_c] \equiv$ Diagonal matrix with ones only in the positions where there is a first order differential equation in the system of equations of motion .The other terms are null.

$[I_k] \equiv$ Diagonal matrix with ones only in the positions where there is an algebraic equation in the system of equations of motion. The other terms are null.

$[J_i] \equiv$ Rectangular matrix $N \times N_i$ with the non-null columns of the matrix $[I] - [I_c] - [I_k]$.

$[J_c] \equiv$ Rectangular matrix $N \times N_c$ with the non-null columns of the matrix $[I_c]$.

$[J_k] \equiv$ Rectangular matrix $N \times N_k$ with the non-null columns of the matrix $[I_k]$.

$\{x\} \equiv$ Physical coordinates array.

$\{^I x\} \equiv$ Array containing only the physical coordinates which were differentiated twice with respect to time.

$\{^C x\} \equiv$ Array containing only the physical coordinates which were differentiated once with respect to time.

$\{^K x\} \equiv$ Array containing only the physical coordinates which were not differentiated with respect to time.

With the definitions above, it is now possible to propose the system reduction for the new state space representation. However, the reduction will be derived for three cases: systems with only first and second order differential equations, systems with only algebraic and second order differential equations and finally systems with both of the three kinds of equations.

2.1 Systems with 1^{st} and 2^{nd} order differential equations

In this case, a system with first and second order differential equations will be considered. For this reason, it is expected that one or more of the columns of the mass matrix is null, because some of the physical variables that were differentiated twice will never be used in this set of equations. Furthermore, if there are first order differential equations in this system, it is expected that one or more columns of the mass matrix are also null. Those facts make of the matrix $[M]$ a singular matrix.

Making use of the transformation matrices J_i and J_c, and the matrix I_c, it is possible to write the following equalities:

$$([I] - [I_c])\{x\} = [J_i]\{^I x\} \tag{3}$$

$$[I_c]\{x\} = [J_c]\{^C x\} \tag{4}$$

The array of physical coordinates $\{x\}$ is composed by the arrays $\{^I x\}$ and $\{^C x\}$ as it can be seen in Eqs. (3) and (4):

$$\{x\} = ([I] - [I_c])\{x\} + [I_c]\{x\} = [J_i]\{^I x\} + [J_c]\{^C x\} \tag{5}$$

Concerning the null columns of the mass matrix, the following equality is held:

$$[M]\{\ddot{x}\} = [M]([I] - [I_c])\{\ddot{x}\} = [M][J_i]\{^I \ddot{x}\} \tag{6}$$

The array $\{^I x\}$ contains all the physical coordinates that were differentiated twice with respect to time, so $\{^I \ddot{x}\}$ exists.

It is known that, in this system, all the physical variables are differentiated at least once. Hence, the array $\{\dot{x}\}$ can be written by simply differentiating the Eq. (5) with respect to time, as it follows:

$$\{\dot{x}\} = [J_i]\{^I \dot{x}\} + [J_c]\{^C \dot{x}\} \tag{7}$$

Replacing Eqs. (5), (6) and (7) on Eq. (2) leads to:

$$[M][J_i]\{^I \ddot{x}\} + [C][J_i]\{^I \dot{x}\} + [C][J_c]\{^C \dot{x}\} + [K][J_i]\{^I x\} + [K][J_c]\{^C x\} = \{F(t)\} \tag{8}$$

It is important to notice that left multiplying the Eq. (8) by $[J_i]^T$ means selecting all the second order differential equations, while left multiplying the Eq. (8) by $[J_c]^T$ means selecting all the first order differential equations. The array $\{^C \dot{x}\}$ cannot be considered part of the new array of state variables, because differentiating this array would lead to an array that would contain $\{^C \ddot{x}\}$, which does not exist.

Left multiplying the Eq. (8) by $[J_c]^T$, makes possible to isolate the term $\{^C \dot{x}\}$ as:

$$\{^C \dot{x}\} = \left([J_c]^T[C][J_c]\right)^{-1} [J_c]^T \left(\{F(t)\} - [C][J_i]\{^I \dot{x}\} - [K][J_i]\{^I x\} - [K][J_c]\{^C x\} - [M][J_i]\{^I \ddot{x}\}\right) \tag{9}$$

Now, left multiplying the Eq. (8) by $[J_i]^T$ (i.e. selecting all the second order differential equations) makes possible to derive an expression for $\{^I \ddot{x}\}$:

$$\{^I \ddot{x}\} = \left([J_i]^T[M][J_i]\right)^{-1} [J_i]^T \left(\{F(t)\} - [C][J_i]\{^I \dot{x}\} - [C][J_c]\{^C \dot{x}\} - [K][J_i]\{^I x\} - [K][J_c]\{^C x\}\right) \tag{10}$$

Notice that $\{^I\ddot{x}\}$ still depends on $\{^C\dot{x}\}$ and vice versa. Making use of the Eqs. (9) and (10) the following expressions are derived:

$$\{^I\ddot{x}\} = [\gamma](\{F(t)\} - [C][J_i]\{^I\dot{x}\} - [K][J_i]\{^Ix\} - [K][J_c]\{^Cx\}) \quad (11)$$

$$\{^C\dot{x}\} = [\alpha](\{F(t)\} - [C][J_i]\{^I\dot{x}\} - [K][J_i]\{^Ix\} - [K][J_c]\{^Cx\}) \quad (12)$$

where:

$$[\gamma] = \left([J_i]^T\left([I] - [C][J_c]\left([J_c]^T[C][J_c]\right)^{-1}[J_c]^T\right)[M][J_i]\right)^{-1}[J_i]^T\left([I] - [C][J_c]\left([J_c]^T[C][J_c]\right)^{-1}[J_c]^T\right)$$

$$[\alpha] = \left([J_c]^T[C][J_c]\right)^{-1}[J_c]^T([I] - [M][J_i][\gamma])$$

Finally, using the Eqs. (11) and (12), and the equality $\{^I\dot{x}\} = \{^I\dot{x}\}$ this system can be represented in the form shown in Eq. (2), where the matrices $[A]$ and $[B]$ are given by:

$$[A]_{N_s\times N_s} = \begin{bmatrix} [0]_{N_i\times N_i} & [0]_{N_i\times N_c} & [I]_{N_i\times N_i} \\ -[\alpha][K][J_i] & -[\alpha][K][J_c] & -[\alpha][C][J_i] \\ -[\gamma][K][J_i] & -[\gamma][K][J_c] & -[\gamma][C][J_i] \end{bmatrix}$$

$$[B]_{N_s\times N} = \begin{bmatrix} [0]_{N_i\times N} \\ [\alpha] \\ [\gamma] \end{bmatrix}$$

since the array of state variables is given by:

$$\{y\} = \left\{ \begin{array}{c} \{^Ix\} \\ \{^Cx\} \\ \{^I\dot{x}\} \end{array} \right\}$$

2.2 Systems with algebraic and 2^{nd} order differential equations

In this situation a system with algebraic and second order differential equations will be considered. Notice that the following formulation also stands for conservative systems with singular mass matrices.

Concerning this system, the number of first order differential equations is zero. This means that the matrices $[I_c]$ and $[J_c]$ are empty. Consequently, the array of physical coordinates $\{x\}$ is only composed by $\{^Ix\}$ and $\{^Kx\}$. For this reason, the following equality holds for this case:

$$\{x\} = ([I] - [I_k])\{x\} + [I_k]\{x\} = [J_i]\{^Ix\} + [J_k]\{^Kx\} \quad (13)$$

Now, concerning the null columns of the mass and damping matrices, the following relations are also held:

$$[M]\{\ddot{x}\} = [M]([I] - [I_k])\{\ddot{x}\} = [M][J_i]\{^I\ddot{x}\} \quad (14)$$

$$[C]\{\dot{x}\} = [C]([I] - [I_k])\{\dot{x}\} = [C][J_i]\{^I\dot{x}\} \quad (15)$$

Replacing the terms from Eqs. (13), (14) and (15) in Eq. (2), it follows that:

$$[M][J_i]\{^I\ddot{x}\} + [C][J_i]\{^I\dot{x}\} + [K][J_i]\{^Ix\} + [K][J_k]\{^Kx\} = \{F(t)\} \quad (16)$$

In this case it is known that the physical variables in $\{^Kx\}$ are represented by a linear combination of other state variables which already represent the physical variables in $\{^Ix\}$. In this case, it is necessary to derive an expression for $\{^Kx\}$, and this is made by simply selecting the algebraic equations from the system (i.e. left multiplying the equation (16) by $[J_k]^T$). Then, it follows that:

$$\{^Kx\} = \left([J_k]^T[K][J_k]\right)^{-1}[J_k]^T(\{F(t)\} - [C][J_i]\{^I\dot{x}\} - [K][J_i]\{^Ix\} - [M][J_i]\{^I\ddot{x}\}) \quad (17)$$

And also, it is necessary to derive the expression for $\{^I\ddot{x}\}$, which is made by left multiplying the Eq. (16) by $[J_i]^T$. Then:

$$\{^I\ddot{x}\} = \left([J_i]^T[M][J_i]\right)^{-1}[J_i]^T(\{F(t)\} - [C][J_i]\{^I\dot{x}\} - [K][J_i]\{^Ix\} - [K][J_k]\{^Kx\}) \quad (18)$$

Notice that $\{^I\ddot{x}\}$ still depends on $\{^Kx\}$. Replacing the array $\{^Kx\}$ in the Eq. (18) by the expression given by the Eq. (17), the expression for $\{^I\ddot{x}\}$ becomes:

$$\{^I\ddot{x}\} = [\gamma](\{F(t)\} - [C][J_i]\{^I\dot{x}\} - [K][J_i]\{^Ix\}) \quad (19)$$

where:

$$[\gamma] = \left([J_i]^T\left([I] - [K][J_k]\left([J_k]^T[K][J_k]\right)^{-1}[J_k]^T\right)[M][J_i]\right)^{-1}[J_i]^T\left([I] - [K][J_k]\left([J_k]^T[K][J_k]\right)^{-1}[J_k]^T\right) \quad (20)$$

Finally, making use of the Eq. (20) and the equality $\{^I\dot{x}\} = \{^I\dot{x}\}$ it is possible to represent this system in the form shown in Eq. (2), where the matrices A and B are given by:

$$[A]_{N_s\times N_s} = \begin{bmatrix} [0]_{N_i\times N_i} & [I]_{N_i\times N_i} \\ -[\gamma][K][J_i] & -[\gamma][C][J_i] \end{bmatrix}$$

$$[B]_{N_s\times N} = \begin{bmatrix} [0]_{N_i\times N_i} \\ [\gamma] \end{bmatrix}$$

2.3 Systems with algebraic, 1^{st} order and 2^{nd} order differential equations

In this case, a general system of equations will be considered, in which there are, from second order differential equations to algebraic equations. Hence, both of the matrices $[I_c]$, $[I_k]$, $[J_i]$, $[J_c]$ and $[J_k]$ exist, and it is possible to define that:

$$[J_i]\{^Ix\} = ([I] - [I_c] - [I_k])\{x\} \quad (21)$$

$$[J_c]\{^Cx\} = [I_c]\{x\} \quad (22)$$

$$[J_k]\{^Kx\} = [I_k]\{x\} \quad (23)$$

Adding the three equations above, it follows that:

$$\begin{aligned}\{x\} &= ([I] - [I_c] - [I_k])\{x\} + [I_c]\{x\} + [I_k]\{x\} \\ &= [J_i]\{^Ix\} + [J_c]\{^Cx\} + [J_k]\{^Kx\}\end{aligned} \quad (24)$$

Concerning the null columns in the mass matrix it is possible to write that:

$$[M]\{\ddot{x}\} = [M]([I] - [I_c] - [I_k])\{\ddot{x}\} = [M][J_i]\{^I\ddot{x}\} \quad (25)$$

Now, concerning the null columns in the damping matrix, the following equality is written:

$$\begin{aligned}[C]\{\dot{x}\} &= [C]([I] - [I_k])\{\dot{x}\} \\ &= [C]([I] - [I_c] - [I_k])\{\dot{x}\} + [I_c]\{\dot{x}\} \\ &= [C][J_i]\{^I\dot{x}\} + [C][J_c]\{^C\dot{x}\}\end{aligned} \quad (26)$$

Then, replacing the Eqs. (24), (25) and (26) in Eq. (2), it is possible to find that:

$$[M][J_i]\{^I\ddot{x}\} + [C][J_i]\{^I\dot{x}\} + [C][J_c]\{^C\dot{x}\} + [K][J_i]\{^Ix\} + [K][J_c]\{^Cx\} + [K][J_k]\{^Kx\} = \{F(t)\} \quad (27)$$

First, it is necessary to find an expression for $\{^Kx\}$, because these variables are calculated as a linear combination of the states used to represent the response of the variables in $\{^Ix\}$, $\{^Cx\}$ and $\{^I\dot{x}\}$. This can be done by left multiplying the Eq. (27) by $[J_k]^T$. In this case, the expression for $\{^Kx\}$ is:

$$\{^Kx\} = \left([J_k]^T[K][J_k]\right)^{-1}[J_k]^T(\{F(t)\} - [K][J_i]\{^Ix\} - [K][J_c]\{^Cx\} - [C][J_i]\{^I\dot{x}\} - [C][J_c]\{^C\dot{x}\} - [M][J_i]\{^I\ddot{x}\}) \quad (28)$$

In this case, the array of state space variables is given by:

$$\{y\} = \left\{ \begin{array}{c} \{^I x\} \\ \{^C x\} \\ \{^I \dot{x}\} \end{array} \right\}$$

As the system must be represented in the state space form shown in Eq. (2), and the equality $\{^I \dot{x}\} = \{^I \dot{x}\}$ still holds for this case, $\{^K x\}$ should be eliminated from the equations. It is now necessary to find expressions to calculate $\{^C \dot{x}\}$ and $\{^I \ddot{x}\}$. The expression for $\{^C \dot{x}\}$ can be derived by left multiplying the Eq. (27) by $[J_c]^T$. Performing this step, and replacing the term $\{^K x\}$ by the term given by Eq. (28) it is possible to write:

$$\{^C \dot{x}\} = [\delta](\{F(t)\} - [M][J_i]\{^I \ddot{x}\} - [C][J_i]\{^I \dot{x}\} - [K][J_i]\{^I x\} - [K][J_c]\{^C x\}) \tag{29}$$

where:

$$[\delta] = \left([J_c]^T \left([I] - [K][J_k]\left([J_k]^T[K][J_k]\right)^{-1}[J_k]^T\right)[C][J_c]\right)^{-1}[J_c]^T\left([I] - [K][J_k]\left([J_k]^T[K][J_k]\right)^{-1}[J_k]^T\right)$$

Now, replacing Eq. (29) on Eq. (28) it follows that:

$$\{^K \dot{x}\} = [\beta](\{F(t)\} - [M][J_i]\{^I \ddot{x}\} - [C][J_i]\{^I \dot{x}\} - [K][J_i]\{^I x\} - [K][J_c]\{^C x\}) \tag{30}$$

where:

$$[\beta] = \left([J_k]^T[K][J_k]\right)^{-1}[J_k]^T([I] - [C][J_c][\delta])$$

Notice that $\{^C \dot{x}\}$ still depends on $\{^I \ddot{x}\}$. The expression for this term can be derived. by simply left multiplying the Eq. (27) by $[J_i]^T$ and replacing the terms found on Eqs. (29) and (30) as it is shown below:

$$\{^I \ddot{x}\} = [\gamma](\{F(t)\} - [C][J_i]\{^I \dot{x}\} - [K][J_i]\{^I x\} - [K][J_c]\{^C x\}) \tag{31}$$

where:

$$[\gamma] = \left([J_i]^T([I] - [C][J_c][\delta] - [K][J_k][\beta])[M][J_i]\right)^{-1}[J_i]^T([I] - [C][J_c][\delta] - [K][J_k][\beta])$$

Finally, replacing the term $\{^I \ddot{x}\}$ in Eq. (29) by the term found in Eq. (31), it is written that:

$$\{^C \dot{x}\} = [\alpha](\{F(t)\} - [C][J_i]\{^I \dot{x}\} - [K][J_i]\{^I x\} - [K][J_c]\{^C x\}) \tag{32}$$

where:

$$[\alpha] = [\delta]([I] - [M][J_i][\gamma])$$

Consequently, using the Eqs. (31) and (32) and the equality $\{^I \dot{x}\} = \{^I \dot{x}\}$ it is possible to represent this system in the form shown in Eq. (2), where the matrices $[A]$ and $[B]$ are given by:

$$[A]_{N_s \times N_s} = \left[\begin{array}{ccc} [0]_{N_i \times N_i} & [0]_{N_i \times N_c} & [I]_{N_i \times N_i} \\ -[\alpha][K][J_i] & -[\alpha][K][J_c] & -[\alpha][C][J_i] \\ -[\gamma][K][J_i] & -[\gamma][K][J_c] & -[\gamma][C][J_i] \end{array} \right]$$

$$[B]_{N_s \times N} = \left[\begin{array}{c} [0]_{N_i \times N} \\ [\alpha] \\ [\gamma] \end{array} \right]$$

and the array of state variables is given by:

$$\{y\} = \left\{ \begin{array}{c} \{^I x\} \\ \{^C x\} \\ \{^I \dot{x}\} \end{array} \right\}$$

3. APPLICATION

This procedure can be applied to any mathematical model with half degrees of freedom. A model of a hydroelastic mount of an engine suspension is shown in the book of Morello *et al.* (2011). A cross section of this mount is shown on Fig. 1. In this illustration, (a) is the primary rubber, (b) is the set of holes through which the fluid laminates from the first to the second chamber, (c) is the secondary rubber, which pulls the oil back to the first chamber, and (d) is the lower chamber, which is filled with air. This air can escape through the large holes of the lower steel case.

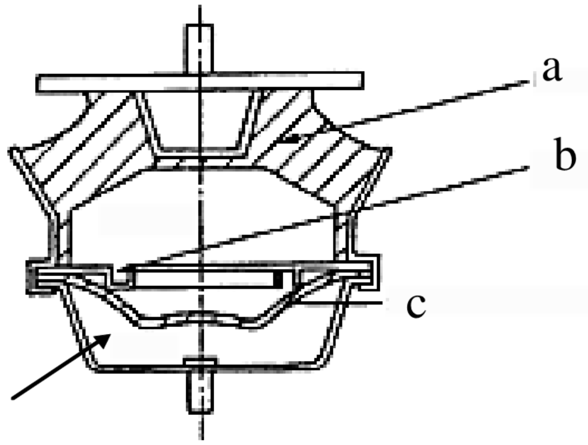

Figure 1. Cross section of the hydroelastic engine mount.

This mount can be idealized as the model shown in Fig. 2. In this case, K_r represents the stiffness of the primary rubber, while C_{rs} and K_{r_s} represent its viscoelastic behavior in a simplified approach. The stiffness K_v represents the volumetric stiffness of the oil, C_h is the damping coefficient due to the lamination of the oil passing through the holes and K_s is the stiffness of the secondary rubber. The variables y, y_1 and y_2 represent respectively the vertical displacements of the upper part of the cushion, the node between the damper C_{rs} and the spring K_{rs}, and the node between the spring K_v and the parallel spring damping mount represented by C_h and K_s.

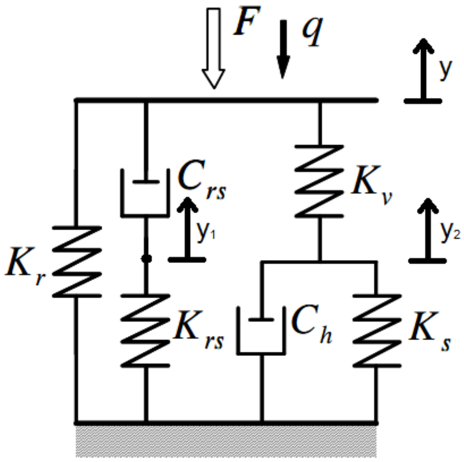

Figure 2. Simplified model of the hydroelastic engine mount.

In order to test the formulation presented in this work a mass m is added to the top of this cushion, and this model is studied. The equations of motion for this problem are shown in matricial form:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} C_{rs} & -C_{rs} & 0 \\ -C_{rs} & C_{rs} & 0 \\ 0 & 0 & C_h \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} K_r + K_v & 0 & -K_v \\ 0 & K_{rs} & 0 \\ -K_v & 0 & K_v + K_s \end{bmatrix} \begin{Bmatrix} y \\ y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{33}$$

Through these matrices, it is possible to see that there are only first and second order differential equations, and for this reason the formulations derived in section 2.1. The following data were used for this simulation:

$m = 20kg$

$K_r = 5000N/m$
$K_{rs} = 1000N/m$
$K_v = 500N/m$
$K_s = 500N/m$
$C_{rs} = 100N.s/m$
$C_h = 300N.s/m$

The matrices $[M]$, $[C]$ and $[K]$ are respectively:

$$[M] = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; [C] = \begin{bmatrix} 100 & -100 & 0 \\ -100 & 100 & 0 \\ 0 & 0 & 300 \end{bmatrix}; [K] = \begin{bmatrix} 5500 & 0 & -500 \\ 0 & 1000 & 0 \\ -500 & 0 & 1000 \end{bmatrix};$$

Using the formulation, the matrices $[J_i]$, $[J_c]$, $[\alpha]$ and $[\gamma]$ become:

$$[J_i] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; [J_c] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; [\alpha] = \begin{bmatrix} 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{1}{300} \end{bmatrix}; [\gamma] = \begin{bmatrix} \frac{1}{20} & \frac{1}{20} & 0 \end{bmatrix}$$

And then the matrices $[A]$ and $[B]$ are calculated:

$$[A] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -10 & 0 & 1 \\ \frac{5}{3} & 0 & \frac{-1}{300} & 0 \\ -275 & -50 & 25 & 0 \end{bmatrix}; [B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{1}{300} \\ \frac{1}{20} & \frac{1}{20} & 0 \end{bmatrix}$$

Using the matrices $[M]$, $[C]$ and $[K]$, the eigenvalues of the original system are obtained using the generalized eigenproblem formulation, through the routine `eig()` in Matlab 7.1® environment,the results were -0.6897 ± 17.7059, -8.7797, -3.1742 and two `NaN`.

Now, using the matrix $[A]$, which was obtained through the formulation presented in this work, it is possible to obtain the eigenvalues of this system by simply calculating the eigenvalues of $[A]$. Using the function `eig()` from Matlab 7.1®, the results are -0.6897 ± 17.7059,-8.7797 and -3.1742.

Notice that the eigenvalues `NaN` disappeared, while the other ones remained constant. The difference between the eigenvalues calculated in both ways is smaller than 10^{-13}, so, they are considered to be equal. This proves that, although the system is represented in a mathematically different way, its physical behavior is not warped through the use of this formulation.

4. CONCLUSION

The technique presented in this work can be applied in systems with algebraic, first order and/or second order differential equations without changing its physical behavior. Besides, this method can be easily adapted for nonlinear systems. The only changes will be on the criteria for the assembly of the matrices $[I_c]$ and $[I_k]$. It was applied successfully to the model of the hydroelastic cushion, and made it possible to find the system's eigenvalues without `NaN`s and also to integrate it numerically using first order integrators such as Runge-Kutta's. Moreover, it can be implemented on any software without changing its current routines for the assembly of the global matrices, once it is applied after this process.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

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7. Responsibility notice

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