# ANALYSIS OF THE NAVIER-STOKES EQUATIONS IN TWO-DIMENSIONAL FLOW WITH PRIMITIVE VARIABLES FORMULATION VIA GITT

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***Abstract.*** *In this paper the Generalized Integral Transform Technique is employed to produce hybrid solutions for the velocity and pressure fields of a newtonian fluid in two dimensional flow. The problem is formulated by using primitive variables and the necessary mathematical manipulations were used to obtain the Poisson equation for the pressure field. The momentum equations in the axial direction of flow and Poisson are transformed to remove the transversal dependency. The resulting transformed fields are solved with the IMSL numerical subroutine, DBVPFD. The obtained results for the longitudinal velocity profile at the center of the channel are compared with the available data in the open literature for validation and model fitting. Even so, studies are carried out about the convergence of the solution for the velocity profile in the centerline as well as testing different values of the scale factor of axial coordinate for the choice of a factor which can fit perfectly for comparison with available data. Interest practical datas such as: friction factor and mean velocity are obtained along the duct for a entry condition into the parallel flow channel (v = 0).*

***Keywords****: Navier-Stokes, Poisson, Primitive Variables, Newtonian Fluid, GITT*

## 1. INTRODUCTION

The analysis of the flows is of fundamental importance in our lives and in a lot of various areas of engineering, and this refers on the knowledge of the exact sciences and nature, such as mathematics, physics and mechanical engineering, for the preparation of models to be submitted to simulations and tests. The derivation and mathematical development enables the deployment, simplified solutions and physical interpretations and conclusions. The Navier-Stokes equations has been widely used in mathematical modeling for many phenomena in fluid mechanics.

Using the Navier-Stokes and Poisson we can understand the physical phenomena and relate them to our everyday life. Therefore, we propose in this study develop a solution to the Navier-Stokes problem of hydrodynamics, a two-dimensional laminar flow of a newtonian fluid in circular duct with a formulation in primitive variables, with profiles of uniform velocity and pressure in the entrance.

Even with the large number of previous studies on flow analysis, the theme is still attracting interest from researchers primarily in the hydrodynamic entrance region where viscous effects are more pronounced. The entrance region requires more complex analysis represented by robust formulations with greater mathematical difficulties associated with obtaining the velocity fields and pressure.

The knowledge of the pressure field along the flow can help to monitor and control over the flow of fluids. An application example is the oil and gas, preventing and reducing environmental damage.

Over the years, we can observe the developmentof studies involving laminar flows of fluids in the solution of the Navier – Stokes or Boundary Layer, the first numerical methods are: Wang and LongwelL (1964), Friedmann et al (1968) and McDonald et al (1972) and applying the Generalized Integral Transform Technique (GITT) and the stream function formulation are: Paz et al (2007), Silva et al (2009), Silva et al (2004), Pereira et al (1998) among many others, and with the formulation in primitive variables, which is the formulation under study is still small, ie, there is little work in this area, we can cite Lima (2002), Lima et al (2006), and Veronese (2008).

The Generalized Integral Transform Technique (GITT) arose more than two decades standing out as a powerful tool that allows the solution of the complex problems with the work of Özisik & Murray (1984) from the ideas of Integral Transform Technique Classical, Mikhailov & Özisik (1974). The G.I.T.T. provides hybrid numerical-analytical solutions for problems of diffusion and convection-diffusion integral transformation which results in systems of ordinary differential equations coupled. Since then the application of G.I.T.T. has solved problems in more general classes, both linear and nonlinear. The most detailed and comprehensive study on G.I.T.T. was done by Cotta (1993).

The main idea is to transform a system of partial differential equations in an original infinite system of ordinary differential equations by expanding in eigenfunctions, which is truncated to a number of terms required for convergence. The solution is obtained analytically for problems that can be transformed into decoupled systems that can be resolved simply, or numerically for more complex problems.

This study aims to initially extend the application of the Generalized Integral Transform Technique (GITT) in the solution in terms of primitive variables of the Navier-Stokes equations for two-dimensional problem of flow in circular ducts with a newtonian fluid inside, taking into account the velocity and pressure with the Poisson equation.

## 2. THE PROBLEM

The physical model mathematical is the development of the use of a two-dimensional laminar flow of an incompressible newtonian fluid in a circular duct, shown in fig. 1 to solve the hydrodynamic problem is necessary to consider the following hypothesis: the effects of viscous dissipation are neglected, constant physical properties, impermeability and no-slip walls of the duct and steady, the longitudinal velocity (u) and transverse velocity (v).

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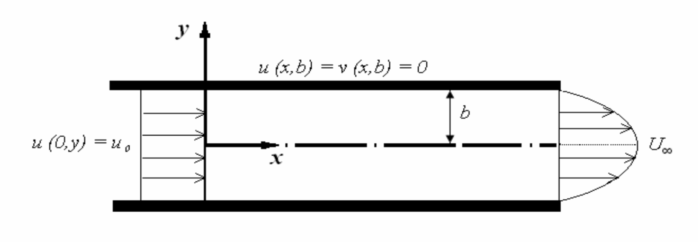


Figure 1. Defining the proposed problem.

## 3. MATHEMATICAL FORMULATION

The flow in a circular duct shown in fig. 1, is an application of the solution of the Navier-Stokes equation is a nonlinear partial differential 2nd order and formulated in primitive variables, whose general equations governing this problem are listed below:

Continuity Equation: $$\frac{\partial u(x,y)}{\partial x} + \frac{1}{y}\frac{\partial[y.v(x,y)]}{\partial y} = 0 \qquad (1)$$

Equation of Conservation of momentum in the x direction:

$$u(x,y)\frac{\partial u(x,y)}{\partial x} + v(x,y)\frac{\partial u(x,y)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\frac{1}{y}\frac{\partial}{\partial y}\left(y.\frac{\partial u(x,y)}{\partial y}\right) + \frac{\partial^2 u(x,y)}{\partial x^2}\right] \qquad (2)$$

Equation of Conservation of momentum in the y direction:

$$u(x,y)\frac{\partial v(x,y)}{\partial x} + v(x,y)\frac{\partial v(x,y)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left[\frac{\partial}{\partial y}\left(\frac{1}{y}.\frac{\partial(y.v(x,y))}{\partial y}\right) + \frac{\partial^2 v(x,y)}{\partial x^2}\right] \qquad (3)$$

The Poisson equation is determined from the mathematical manipulations in the equations of momentum in the directions x and y. Be μ is the dynamic viscosity newtonian and ρ is the density. These equations appear in the analysis of problems in physics, in engineering and chemistry.

Poisson Equation: $$\frac{\partial^2 p}{\partial x^2} + \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial p}{\partial y}\right) = 2\rho\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{v^2}{y^2}\right] \qquad (4)$$

For the construction of the problem solution is applied the Generalized Integral Transform Technique (GITT) to provide a hybrid solution, ie, an analytical-numerical solution of the equations of conservation of momentum in the x and y directions of the Poisson equation, with knowledge of the velocity fields, pressure along the channel examined. And the soft computing will be appropriate if the FORTRAN and in particular the DPVPFD subroutine of IMSL.

Initial and boundary conditions:

$u(x,y)=u_0,\ v(x,y)=0,\ p(x,y)=p_0$ , to x = 0 (5 a-c)

$\frac{\partial u(x,y)}{\partial y} = 0,\ \ v(x,y) = 0,\ \frac{\partial p(x,y)}{\partial y} = 0$ , to y =0 (5 d-f)

$u(x,y) = u_\infty(y),\ \ v(x,y) = 0,\ \frac{\partial p(x,y)}{\partial x} = \frac{\mu}{y}\frac{\partial}{\partial y}\left[y\frac{\partial u_\infty(y)}{\partial y}\right]$, to x > 0 (5 g-i)

$u(x,y) = 0,\ \ v(x,y) = 0,\ \frac{\partial p(x,y)}{\partial y} = \mu\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial(y.v(x,y))}{\partial y}\right]$, to y = b (5 j-l)

The dimensionless groups used:

$$X=\frac{x}{b},\ Y=\frac{y}{b},\quad U=\frac{u}{u_0},\ V=\frac{v}{u_0},\quad P=\frac{p}{\rho u_0^2}, \tag{6 a-e}$$

Being the Reynolds number defined based on the velocity of the duct entrance.

$$\mathrm{Re}=\frac{\rho u_0 b}{\mu}=\frac{u_0 b}{\nu},\ \text{ being }\ \nu=\frac{\mu}{\rho} \tag{7 a-b}$$

Application of dimensionless groups in the above equations has been the system of dimensionless equations in the domain 0 < y < 1 and x > 0:

Continuity Equation: $$\frac{\partial U}{\partial X}+\frac{1}{Y}\frac{\partial[Y.V]}{\partial Y}=0 \tag{8 a}$$

Equation of Conservation of momentum in the X direction:

$$U\frac{\partial U}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial X}+\frac{1}{\mathrm{Re}}\left\{\frac{1}{Y}\frac{\partial}{\partial Y}\left[Y\frac{\partial U}{\partial Y}\right]+\frac{\partial^2 U}{\partial X^2}\right\} \tag{8 b}$$

Equation of Conservation of momentum in the Y direction:

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial Y}\left[\frac{1}{Y}\frac{\partial(Y.V)}{\partial Y}\right]+\frac{\partial^2 V}{\partial X^2}\right\} \tag{8 c}$$

Poisson Equation:

$$\frac{\partial^2 P}{\partial X^2}+\frac{1}{Y}\frac{\partial}{\partial Y}\left(Y\frac{\partial P}{\partial Y}\right)=2\left[\frac{\partial U}{\partial X}\frac{\partial V}{\partial Y}-\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}-\frac{V^2}{Y^2}\right] \tag{8 d}$$

The initial and boundary conditions become:

$$U(X,Y)=1;\quad V(X,Y)=0;\quad P(X,Y)=P_0,\qquad \text{in } X=0 \tag{9 a-c}$$

$$\frac{\partial U}{\partial Y}(X,Y)=0;\quad V(X,Y)=0;\quad \frac{\partial P(X,Y)}{\partial Y}=0;\qquad \text{in } Y=0 \tag{9 d-f}$$

$$U(X,Y)=U_\infty(Y);\quad V(X,Y)=0;\quad \frac{\partial P(X,Y)}{\partial X}=\frac{1}{\mathrm{Re}}\left\{\frac{1}{Y}\frac{\partial}{\partial Y}\left[\frac{Y\partial U_\infty(Y)}{\partial Y}\right]\right\},\quad \text{in } X\to\infty \tag{9 g-i}$$

$$U(X,Y)=0;\quad V(X,Y)=0;\quad \frac{\partial P(X,Y)}{\partial Y}=\frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial Y}\left[\frac{1}{Y}\frac{\partial(YV)}{\partial Y}\right]\right\},\quad \text{in } Y=1 \tag{9 j-l}$$

To properly implement GITT and improve computational performance, ie improving the convergence, it is necessary to make a homogenization in the boundary conditions in the chosen direction by using filters which means the separation of potential as velocity, velocity field development , which is a function of X and Y, and fully developed velocity field, which is only a function of Y.

Filter for the velocity field:

$$U(X,Y)=U^*(X,Y)+U_\infty(Y),\ \text{ where:} \tag{10 a}$$

$U^*(X,Y)$: is the filtered developing velocity profile to be evaluated;

$U_\infty(Y)$: is the fully developed velocity profile

$$U_\infty(Y)=2(1-Y^2) \tag{10 b}$$

Filter to the pressure field:

$$P(X,Y)=P^*(X,Y)+P_F(X,Y),\ \text{ where:} \tag{10 c}$$

$P^*(X,Y)$: the potential pressure field development;

$P_F(X,Y)$: the filter that satisfies the equation of conservation of momentum in the y direction in the duct wall, ie when y = 1 and has analytical solution given by:

$$P_F(X,Y)=P_0+\frac{1}{\mathrm{Re}}\left[\frac{1}{Y}\frac{\partial(Y.V(X,Y))}{\partial Y}\right]\ \text{ or }\ P_F(X,Y)=P_0-\frac{1}{\mathrm{Re}}\left[\frac{\partial U^*(X,Y)}{\partial X}\right] \tag{10 d-e}$$

Replacing the filters in the velocity and pressure in the above equations is obtained:

Continuity Equation: $$\frac{\partial U^*}{\partial X} + \frac{1}{Y}\frac{\partial [Y.V]}{\partial Y} = 0 \tag{11 a}$$

Equation of Conservation of momentum in the X direction:

$$V\frac{\partial U^*}{\partial Y} + V\frac{\partial U_\infty}{\partial Y} + U^*\frac{\partial U^*}{\partial X} + U_\infty\frac{\partial U^*}{\partial X} = -\frac{\partial P^*}{\partial X} - \frac{\partial P_F}{\partial X} + \frac{1}{\mathrm{Re}}\left\{\frac{1}{Y}\frac{\partial}{\partial Y}\left[Y\frac{\partial U^*}{\partial Y}\right] + \frac{1}{Y}\frac{\partial}{\partial Y}\left[Y\frac{\partial U_\infty}{\partial Y}\right] + \frac{\partial^2 U^*}{\partial X^2}\right\} \tag{11 b}$$

Equation of Conservation of momentum in the Y direction:

$$V\frac{\partial V}{\partial Y} + U^*\frac{\partial V}{\partial X} + U_\infty\frac{\partial V}{\partial X} = -\frac{\partial P^*}{\partial Y} - \frac{\partial P_F}{\partial Y} + \frac{1}{\mathrm{Re}}\left\{\frac{\partial}{\partial Y}\left[\frac{1}{Y}\frac{\partial}{\partial Y}(Y.V)\right] + \frac{\partial^2 V}{\partial X^2}\right\} \tag{11 c}$$

Poisson Equation:

$$\frac{1}{Y}\frac{\partial}{\partial Y}\left[Y\frac{\partial P^*}{\partial Y}\right] + \frac{1}{Y}\frac{\partial}{\partial Y}\left[Y\frac{\partial P_F}{\partial Y}\right] + \frac{\partial^2 P^*}{\partial X^2} + \frac{\partial^2 P_F}{\partial X^2} = 2\left[\frac{\partial V}{\partial Y}\frac{\partial U^*}{\partial X} - \frac{\partial U^*}{\partial Y}\frac{\partial V}{\partial X} - \frac{\partial U_\infty}{\partial Y}\frac{\partial V}{\partial X} - \frac{V^2}{Y^2}\right] \tag{11 d}$$

The initial conditions and boundary after filtering becomes:

$$U^*(X,Y) = 1 - U_\infty(Y); \quad V(X,Y) = 0; \quad P^*(X,Y) = 0, \quad \text{to } X = 0 \tag{12 a-c}$$

$$U^*(X,Y) = 0; \; V(X,Y) = 0; \; \frac{\partial P^*(X,Y)}{\partial X} = \frac{1}{\mathrm{Re}}\left\{\frac{1}{Y}\frac{\partial}{\partial Y}\left[\frac{Y\partial U_\infty(Y)}{\partial Y}\right]\right\}, \quad \text{to } X \to \infty \tag{12 d-f}$$

$$\frac{\partial U^*(X,Y)}{\partial Y} = 0; \; V(X,Y) = 0; \; \frac{\partial P^*(X,Y)}{\partial Y} = 0, \quad \text{to } Y = 0 \tag{12 g-i}$$

$$U^*(X,Y) = V(X,Y) = 0; \; \frac{\partial P^*(X,Y)}{\partial Y} = 0, \; \text{to } Y = 1 \tag{12 j-l}$$

## 4 - APPLICATION OF GENERALIZED INTEGRAL TRANSFORM TECHNIQUE (GITT)

Determination of the Eigenvalue Problems

### 4.1 - Auxiliary Problem for the Velocity Field:

$$\frac{1}{Y}\frac{d}{dY}\left[Y\frac{d\phi_i(Y)}{dY}\right] + \mu_i^2\phi_i(Y) = 0; \qquad 0 < Y < 1 \tag{13 a}$$

Boundary conditions for the problem:

$$\phi_i(1) = 0 \quad \text{and} \quad \frac{d\phi_i(0)}{dY} = 0 \tag{13 b-c}$$

The auxiliary problem for the velocity field and pressure is a problem of Sturm-Liouville and has analytical solution given by OZISIK, (1993).

Eigenfunctions: $\phi_i(Y) = J_0(\mu_i, Y)$; $i = 1, 2, 3, ...$ (13 d)

Normalization integral: $N_i = \int_0^1 Y\phi_i^2(Y)dY$ (13 e)

The eigenvalues, $\mu_i$ 's, are the roots of transcendental equations:

$J_0(\mu_i) = 0$; $i$ = 1, 2, 3, ... (13 f)

Normalized eigenfunctions: $\tilde{\phi}_i(Y) = \dfrac{\phi_i(Y)}{\sqrt{N_i}}$ (13 g)

The eigenfunctions, $\phi_i$ , have the following orthogonality property to velocity:

$$\int_0^1 Y\tilde{\phi}_i(Y)\tilde{\phi}_j(Y)dY = \begin{cases} 0, & se \;\; i \neq j \\ 1, & se \;\; i = j \end{cases} \qquad (13\text{ h})$$

**4.2 - Auxiliary Problem for the Field Pressure:**

$$\frac{1}{Y}\frac{d}{dY}\left[Y\frac{d\psi_i(Y)}{dY}\right] + \beta_i^2\psi_i(Y) = 0 \qquad 0 < Y < 1 \qquad (14\text{ a})$$

Boundary conditions for the problem:

$$\frac{d\psi_i(0)}{dY} = 0 \quad \text{and} \quad \frac{d\psi_i(1)}{dY} = 0 \qquad (14\text{ b-c})$$

Eigencondition: $J_1(\beta_i) = 0$ (14 d)

Eigenfunctions: $\psi_i(Y) = J_0(\beta_i, Y)$; $i$ = 1, 2, 3, ... (14 e)

The eigenvalues, $\beta_i$ 's , are the roots of transcendental equations above:

Normalization integral: $M_i = \int_0^1 Y\psi_i^2(Y)dY$ (14 f)

Normalized eigenfunctions: $\tilde{\psi}_i(R) = \dfrac{\psi_i(R)}{M_i^{1/2}}$ (14 g)

The eigenfunctions, $\psi_i$ , have the following orthogonality property for the pressure:

$$\int_0^1 Y\tilde{\psi}_i(Y)\tilde{\psi}_j(Y)dY = \begin{cases} 0, & se \;\; i \neq j \\ 1, & se \;\; i = j \end{cases} \qquad (14\text{ h})$$

**4.3 - Determination of the Inverse-Transform Pairs**

Field Velocity:

Transform: $\overline{U}_i(X) = \int_0^1 Y^d\tilde{\phi}_i(Y)U^*(X,Y)dY$ and Reverse: $U^*(X,Y) = \sum_{i=1}^{\infty}\tilde{\phi}_i(Y)\overline{U}_i(X)$ (15 a-b)

Field Pressure:

Transform: $\overline{P}_i(X) = \int_0^1 Y^d\tilde{\psi}_i(Y)P^*(X,Y)dY$ and Reverse: $P^*(X,Y) = \sum_{i=1}^{\infty}\tilde{\psi}_i(Y)\overline{P}_i(X)$ (16 a-b)

Calculation of average velocity and transverse velocity:

$$U_m = 2\sum_{i=1}^{\infty}F_i(0)\overline{U}_i(X) + 1 \quad F_i(0) = \int_0^1 Y\tilde{\phi}_i(Y)dY \qquad (17\text{ a-b})$$

$$V(X,Y) = \frac{1}{Y}\sum_{i=1}^{\infty}\overline{F}_i(Y)\frac{d\overline{U}_i(X)}{dX} \quad \overline{F}_i(Y) = \int_{\overline{Y}}^1 Y\tilde{\phi}_i(Y)dY \qquad (17\text{ c-d})$$

## 5 - INTEGRAL TRANSFORMATION SYSTEM OF EQUATIONS

The process of integral transformation of the system of partial differential equations formed by equations of momentum in x direction and Poisson in an ordinary differential system is derived using the following operators.

First apply the operator $\int_0^1 Y\tilde{\phi}_i(Y)dY$ in Eq. (11 b), then applies the property of orthogonality Eq. (13 h), the formulas of the inverse Eq. (15 b) and Eq. (16 b), the transverse velocity Eq. (17 c) and the auxiliary problem for the field velocity Eq. (13 a), then:

$$\frac{d^2\overline{U}_i(X)}{dX^2} = \frac{\mathrm{Re}}{2}\left\{\sum_{k=1}^{\infty}\left[\sum_{j=1}^{\infty} AB_{ijk}\overline{U}_j(X) + AB_{ik\infty}\right]\frac{d\overline{U}_k}{dX} + \sum_{n=1}^{\infty}\frac{C_{in}d\overline{P}_n(X)}{dX}\right\} - \frac{D_{i\infty}}{2} + \frac{\mu_i^2\overline{U}_i(X)}{2} \tag{18}$$

Continuing to apply to the operator $\int_0^1 Y\tilde{\psi}_i(Y)dY$ in the Eq. (11 d), and replaces the orthogonality property Eq. (14 h), the inverse formulas of the Eq. (15 b) and Eq. (16 b), the transverse velocity Eq. (17 c) and the auxiliary problem for the pressure field Eq. (14 a), thus:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[\delta_{in} - \frac{1}{2}\sum_{m=1}^{\infty} G_{im}C_{mn}\right]\frac{d^2\overline{P}_n(X)}{dX^2} = \beta_i^2\overline{P}_i(X) - \frac{1}{\mathrm{Re}}\sum_{k=1}^{\infty} Q_{ik}\mu_k^2\frac{d\overline{U}_k(X)}{dX} + \frac{1}{2}\sum_{m=1}^{\infty} G_{im}A_m + \\
&2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} E_{ijk}\frac{d\overline{U}_j(X)}{dX}\frac{d\overline{U}_k(X)}{dX} - 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} F_{ijk}\overline{U}_j(X)\frac{d^2\overline{U}_k(X)}{dX^2} - 2\sum_{k=1}^{\infty} F_{ik\infty}\frac{d^2\overline{U}_k(X)}{dX^2} - \\
&2\sum_{k=1}^{\infty} E_{ik}\frac{d\overline{U}_k(X)}{dX}\frac{d\overline{U}_k(X)}{dX}
\end{aligned} \tag{19}$$

The equation written in matrix form is: $$\frac{d^2\overline{P}_n(X)}{dX^2} = (P_{mat})^{-1}.\left(G_i + \frac{1}{2}\sum_{m=1}^{\infty} G_{im}A_m\right) \tag{20}$$

Where the coefficients of equations (18) and (20) are:

$$AB_{ijk} = \int_0^1 \tilde{\phi}_i(Y)\frac{d\tilde{\phi}_j(Y)}{dY}\overline{F}_k(Y)dY + \int_0^1 Y\tilde{\phi}_i(Y)\tilde{\phi}_j(Y)\tilde{\phi}_k(Y)dY \tag{21 a}$$

$$AB_{ik\infty} = \int_0^1 \tilde{\phi}_i(Y)\frac{dU_\infty(Y)}{dY}\overline{F}_k(Y)dY + \int_0^1 Y\tilde{\phi}_i(Y)\tilde{\phi}_j(Y)U_\infty(Y)dY \tag{21 b}$$

$$C_{in} = \int_0^1 Y\tilde{\phi}_i(Y)\tilde{\psi}_n(Y)dY \tag{21 c}$$

$$\delta_{ij} = \int_0^1 Y\tilde{\phi}_i(Y)\tilde{\phi}_j(Y)dY \tag{21 d}$$

$$D_{i\infty} = \int_0^1 \tilde{\phi}_i(Y)\frac{d}{dY}\left[Y\frac{dU_\infty(Y)}{dY}\right]dY \tag{21 e}$$

$$Pmat = \delta_{in} - \frac{1}{2}\sum_{m=1}^{\infty} G_{im}C_{mn} \tag{21 f}$$

$$\begin{aligned}
G_i = {}&\beta_i^2\overline{P}_i(X) - \frac{1}{\mathrm{Re}}\sum_{k=1}^{\infty} Q_{ik}\mu_k^2\frac{d\overline{U}_k(X)}{dX} + 2\sum_{k=1}^{\infty}\left[\sum_{j=1}^{\infty} E_{ijk}\frac{d\overline{U}_j(X)}{dX} - E_{ik}\frac{d\overline{U}_k(X)}{dX}\right]\frac{d\overline{U}_k(X)}{dX} \\
&- 2\sum_{k=1}^{\infty}\left[\sum_{j=1}^{\infty} F_{ijk}\overline{U}_j(X) + F_{ik\infty}\right]\frac{d^2\overline{U}_k(X)}{dX^2}
\end{aligned} \tag{21 g}$$

$$A_m = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} AB_{mjk} \frac{d\overline{U}_j(X)}{dX}\frac{d\overline{U}_k(X)}{dX} + \sum_{k=1}^{\infty}\left[\sum_{j=1}^{\infty} AB_{mjk}\overline{U}_j(X) + AB_{mk\infty}\right]\frac{d^2\overline{U}_k(X)}{dX^2} + \frac{\mu_m^2}{\mathrm{Re}}\frac{d\overline{U}_m(X)}{dX} \tag{21 h}$$

$$Q_{ik} = \int_0^1 Y\tilde{\psi}_i(Y)\tilde{\phi}_k(Y)dY \tag{21 i}$$

$$G_{im} = \int_0^1 Y\tilde{\psi}_i(Y)\tilde{\phi}_m(Y)dY \tag{21 j}$$

$$E_{ik} = \int_0^1 \frac{1}{Y^3}\tilde{\psi}_i(Y)\left[\overline{F}_k(Y)\right]^2 dY \tag{21 k}$$

$$E_{ijk} = \int_0^1 Y\tilde{\psi}_i(Y)\tilde{\phi}_j(Y)\frac{d}{dY}\left[\frac{\overline{F}_k(Y)}{Y}\right]dY \tag{21 l}$$

$$F_{ijk} = \int_0^1 \tilde{\psi}_i(Y)\overline{F}_k(Y)\frac{d\tilde{\phi}_j(Y)}{dY}dY \tag{21 m}$$

$$F_{ik\infty} = \int_0^1 \tilde{\psi}_i(Y)\overline{F}_k(Y)\frac{dU_\infty(Y)}{dY}dY \tag{21 n}$$

Applying the integral transform in the initial and boundary conditions:

Velocity:

$$\overline{U}_i(X) = \int_0^1 Y(1-U_\infty(Y))\tilde{\phi}_i(Y)dY, \qquad \text{in } X = 0 \tag{22 a}$$

$$\overline{U}_i(X) = 0, \qquad \text{in } X \to \infty \tag{22 b}$$

Pressure:

$$\overline{P}_i(X) = 0, \qquad \text{in } X = 0 \tag{22 c}$$

$$\frac{d\overline{P}_i(X)}{dX} = \frac{1}{\mathrm{Re}}\int_0^1 \tilde{\psi}_i(Y)\frac{\partial}{\partial Y}\left[Y\frac{dU_\infty(Y)}{dY}\right]dY, \qquad \text{in } X \to \infty \tag{22 d}$$

## 6. RESULTS AND DISCUSSION

The program developed for solving the system of ordinary differential equations with the transformed potential was built in Fortran language and implemented on a micro computer with Pentium Dual-Core 1.87 GHz with 2 GB of RAM and run on Windows Vista. The code is focused on the use of the IMSL Library subroutine through DBVPFD, tolerance used was 10-4, to determine the error in the automatic evaluation of velocity fields and pressure. The tables represent the convergence of the longitudinal velocity at the center of the channel (Y = 0) and average velocity for circular ducts with the same Reynolds number and different values of the contraction of scale.

Table 1 - Convergence of longitudinal velocity at the center of the channel U (X, 0) for Re = 20, entry conditions and U = 1, V = 0. Shrinkage factor of scale: C = 1,2 and $y_{00}$ = 0,2.

| **N/x** | **0,1000** | **0,2500** | **0,3000** | **0,5000** | **0,7000** | **0,7500** | **1,0000** |
|---|---|---|---|---|---|---|---|
| **10** | 0,9644 | 1,0065 | 1,0264 | 1,1136 | 1,2071 | 1,2343 | 1,4130 |
| **20** | 0,9710 | 1,0283 | 1,0535 | 1,1547 | 1,2594 | 1,2902 | 1,5035 |
| **30** | 0,9679 | 1,0281 | 1,0526 | 1,1431 | 1,2186 | 1,2376 | 1,3553 |
| **32** | 0,9777 | 1,0479 | 1,0763 | 1,1868 | 1,2990 | 1,3321 | 1,5696 |
| **40** | 0,9819 | 1,0582 | 1,0881 | 1,2034 | 1,3205 | 1,3552 | 1,6101 |
| **50** | 0,9865 | 1,0684 | 1,0996 | 1,2193 | 1,3402 | 1,3761 | 1,6452 |
| **50/ $E_{ik}$ = 0** | 1,0144 | 1,0858 | 1,1097 | 1,2092 | 1,3384 | 1,3811 | 1,7180 |

Tabela 2 – Convergence of longitudinal pressure at the center of the channel U (X, 0) for Re = 20, entry conditions and U = 1, V = 0. Shrinkage factor of scale: C = 1,2 and $y_{00}$ = 0,2.

| **N/x** | **0,1000** | **0,2500** | **0,3000** | **0,5000** | **0,7000** | **0,7500** | **1,0000** |
|---|---|---|---|---|---|---|---|
| **10** | 0,1486 | 0,1495 | 0,1499 | 0,1499 | 0,1489 | 0,1480 | 0,1468 |
| **20** | 0,3682 | 0,3651 | 0,3606 | 0,3552 | 0,3344 | 0,3183 | 0,3015 |
| **30** | 0,5573 | 0,5459 | 0,5306 | 0,5126 | 0,4715 | 0,4499 | 0,4282 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **32** | 0,5899 | 0,5876 | 0,5850 | 0,5822 | 0,5726 | 0,5653 | 0,5574 |
| **40** | 0,7260 | 0,7143 | 0,7001 | 0,6841 | 0,6414 | 0,6151 | 0,5882 |
| **50** | 0,8902 | 0,8735 | 0,8540 | 0,8323 | 0,7844 | 0,7527 | 0,7203 |
| **50/ $E_{ik}$ = 0** | 0,8756 | 0,8580 | 0,8376 | 0,8150 | 0,7656 | 0,7330 | 0,6998 |

Table 3 - Convergence of the average velocity in the center of of the channel $U_c/U_m$ for circular duct with Re = 20, entry conditions and U = 1, V = 0. Shrinkage factor of scale: C = 1,2 and $y_{00}$ = 0,2.

| **N/x** | **0,1000** | **0,2500** | **0,3000** | **0,5000** | **0,7000** | **0,7500** | **1,0000** |
|---|---|---|---|---|---|---|---|
| **10** | 1,0004 | 1,0568 | 1,0818 | 1,1855 | 1,2892 | 1,3182 | 1,5035 |
| **20** | 1,0072 | 1,0785 | 1,1087 | 1,2264 | 1,3409 | 1,3734 | 1,5935 |
| **30** | 1,0034 | 1,0770 | 1,1064 | 1,2130 | 1,2980 | 1,3184 | 1,4408 |
| **32** | 1,0135 | 1,0977 | 1,1310 | 1,2580 | 1,3801 | 1,4149 | 1,6594 |
| **40** | 1,0175 | 1,1077 | 1,1064 | 1,2130 | 1,2980 | 1,3184 | 1,6999 |
| **50** | 1,0218 | 1,1175 | 1,1537 | 1,2899 | 1,4205 | 1,4581 | 1,7349 |
| **50/ $E_{ik}$ = 0** | 1,0504 | 1,1355 | 1,1642 | 1,2793 | 1,4185 | 1,4631 | 1,8095 |

Table 4 - Convergence of longitudinal velocity at the center of the channel U (X, 0) for a newtonian fluid flowing in a circular duct. Re = 20 and C = 1.2, entry conditions and U = 1 V = 0.

| **Re/x** | **Referências** | **.1000** | **.2500** | **.3000** | **.5000** | **.7000** | **.7500** |
|---|---|---|---|---|---|---|---|
| **20** | Presente Trabalho | 0,9819 | 1,0582 | 1,0881 | 1,2034 | 1,3205 | 1,3552 |
| **40*** | SILVA *et al.* (2009) | 1,0170 | 1,0570 | 1,0770 | 1,1800 | 1,3000 | 1,3300 |
| **40*** | FRIEDMANN (1968) | 1,0080 | 1,0484 | 1,0740 | 1,1738 | 1,3100 | 1,3263 |

* $Re_h$ = 2Re

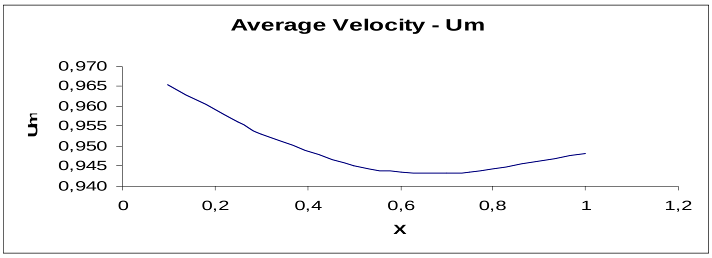


Figure 2 - Average Velocity of development along the axial position for Re = 20 and NU = NP = 50

Tables 1 e 2 represent the convergence of velocity and pressure longitudinal center of the channel (Y = 0), Tab. 3 represent the relationship between velocity and average velocity in the center $U_c/U_m$ for circular ducts with the same Reynolds number and different values of the contraction of scale.

Table 4 shows that the results show good agreement and compared with literature data provided by Silva *et al* (2009a) and Friedman (1968a). The formulation was used in the cited references and the current function of this work is primitive variables for 40 terms.

Figure 2 shows the behavior of the average velocity at different axial positoins with reynolds number equal to 20 and using 50 terms, watching a small fluctuation and the results indicate that the calues are very close.

## 7. CONCLUSIONS

The results obtained in solving the Navier-Stokes equations in terms of primitive variables, with entry conditions U = 1, V = 0 and P = $P_0$, it appears that at the beginning of the channel values are lower than those of the reference but then the results are very close in terms of stream function formulation of reference works.

In our study we used filters that were crucial for both the velocity and pressure, as well as homogenize the boundary conditions also accelerates the convergence.

We conclude that the Generalized Integral Transform Technique (GITT) used for analysis in a circular duct of the laminar flow of newtonian fluid was considered satisfactory with a good agreement with data available in literature. Using the formulation in primitive variables for the solution of the Navier-Stokes and Poisson showed that the mathematical model is very complicated which has hampered its computational implementation. The subroutine of the IMSL Library DBVPFD was used to solve the system of equations and obtained results consistent with the references.

## 8. ACKNOWLEDGEMENTS

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## 10. RESPONSIBILITY NOTICE

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